DEPARTMENT OF PHILOSOPHY
FACULTY OF ARTS
NATONAL OPEN UNIVERSITY OF NIGERIA

Course Guide for PHL. 431: Further Logic

Course Code **PHL. 431**
Course Title **Further Logic**

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Course Guide

Introduction

Welcome to **PHL 431: Further Logic.** PHL 431 is a three-credit unit course with a minimum duration of one semester. The course is expected to deepen the learner's skills in formal proof of validity and sharpen their understanding of mathematical logic. It begins with preliminary introduction to the logic of quantification; this involves analyzing the internal structures of simple propositions into their subject and predicate terms, exposition of the device for quantification, that is, the device for symbolizing propositions that contain words like "All" and "Some". It then discusses proof of validity using the rules of quantification, inference and replacement. The course will also expose the students to Conditional Proof, Indirect Proof, Proofs of Invalidity. It will also deal with truth tree analysis; this involves rule of inference in truth trees, the trees test, problems of adequacy of the tree test and deduction trees, and the application of truth tree; to first order logic

Course Objectives

By the end of the course you will be able to:

- analyze the internal structures of simple propositions into their subject and predicate terms
- symbolize propositions that contains quantification
- state the rules of quantification
- acquire in-depth knowledge of predicate calculus
- construct formal proof of validity using quantification rules
- construct Conditional Proofs of Validity
- construct Indirect Proofs
- construct Proofs of Invalidity
- do truth tree analysis of propositions and arguments
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Working Through the Course

To complete this course of study successfully, you are expected to read the study units, do all the assignments, open the links and read, participate in discussion forums, read the recommended books and other materials provided, prepare your portfolios, and participate in the online facilitation. Each study unit has introduction, intended learning outcomes, the main content, conclusion, summary and references/further readings. The introduction will tell you the expectations in the study unit. Read and note the intended learning outcomes (ILOs). The intended learning outcomes tell you what you should be able to do at the completion of each study unit. So, you can evaluate your learning at the end of each unit to ensure you have achieved the intended learning outcomes. To meet the intended learning outcomes, knowledge is presented in texts and links arranged into modules and units. Click on the links as may be directed, but where you are reading the text offline, you will have to copy and paste the link address into a browser. You can print or download the text and save in your computer or external drive. The conclusion gives you the theme of the knowledge you are taking away from the unit. Unit summaries are presented in downloadable audios and videos.

There are two main forms of assessment—the formative and the summative. The formative assessment will help you monitor your learning. This is presented as in-text questions, discussion forums and self-Assessment Exercises. The summative assessments would be used by the university to evaluate your academic performance. This will be given as Pen-on-Paper (POP) which serves as continuous assessment and final examinations. A minimum of two or a maximum of three computer-based tests will be given with only one final examination at the end of the semester. You are required to take all the computer-based tests and the final examination.

Study Units

There are 10 study units in this course divided into Three modules. The modules and units are presented as follows:-

Module 1

Unit 1 Quantificational Logic
Unit 2 Symbolization in Quantificational Logic
Unit 3 Symbolizing Relational Propositions
Unit 4 Properties of Relations

Module 2

Unit 1 Formal Proof of Validity in Quantificational Logic
Unit 2 Conditional Proof
Unit 3 Indirect Proof

Module 3

Unit 1 Truth Tree Tests of Propositions
Unit 2 Truth Tree Test of Validity in Propositional Logic
Unit 3 Proving Invalidity in Predicate Logic

References and Further Reading

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MODULE 1

Unit 1 Quantificational Logic

Contents

1.1 Introduction

This study unit introduces the learner to the methods for analyzing the internal structures of propositions into smaller parts, not simple propositions but ***"terms"*** of which they are composed. It equips us with the technique to separate the subject and the predicate of a proposition and symbolize them separately. It also develops the practice of replacing specific subject terms i.e. individual names (constants) with that of individual variables.

1.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. analyze simple propositions by showing their subject and predicate terms
2. obtain a proposition from a propositional function by substituting a constant for a variable
3. express the key ideas of universality and particularity
4. solve the problem of cross-reference between propositions which cannot be handled in propositional logic.
5. symbolize universal propositions accurately
6. symbolize particular propositions accurately

1.3. Quantificational Logic.

Our techniques as they stand in PHL. 301 are only effective with respect to arguments whose validity depend on how simple propositions are connected to one another to form compounds or propositions that are more complex. Yet Inferences, which are made in terms of propositional logic, which we dealt with in PHL. 301 do not exhaust all possible inferences. There are, indeed, many simple and logically valid inferences, which the techniques we developed in PHL. 301 are inadequate to take care of. The inadequacy of symbolic logic as thus far developed in PHL.301 stems from the fact that propositions may also be built up, not out of other propositions but out of elements that are not themselves propositions; “there are operators which form propositions, not out of other propositions, but out of names” (Prior 1963:72).

The point of emphasis is that the techniques of propositional logic is only adequate in dealing with compound propositions and not just inadequate, but cannot indeed deal with the internal structure of the individual propositions which make up the compound proposition. This point is illustrated by the arguments below:

1. If it is raining then it is wet outside
   It is raining
   Therefore, it is wet outside

2. All students are hardworking
   Some beautiful girls are students
   Therefore, some beautiful girls are hardworking

Argument 1 can be symbolized as:

$$R \supset W$$
$$R \; / \therefore \; W$$

and this correctly and aptly captures the argument form of argument 1; this is because the correct symbolization of the argument depends upon the logical (connectives) operators. It is also clear that the validity of the argument depends upon the logical operators, hence the argument’s validity can be established by the Rule of Modus Ponens.

On the other hand, argument 2 cannot be correctly symbolized given the technique we used in argument 1. This is because to correctly symbolize argument 2 we need to take into cognizance the inner or internal logical structures of the non-compound propositions it contains. If this is not done, we would ordinarily symbolize it as

$$S$$
$$B \; / \therefore \; H$$

which symbolization is not only incorrect but also renders the argument invalid when in fact the argument is indeed valid. This problem arises because our techniques as thus far developed deals with the structures of compound propositions relative only to their component propositions, propositions remain the smallest unit of analysis, that is, unanalyzable unit. Yet argument 2 requires a technique that will make it possible for us to correctly display its inner or internal structure and consequently determine its validity or otherwise invalidity.

In effect, to overcome the inadequacy of our technique we need to give an exposition of or introduce the methods for analyzing the internal structures of propositions into smaller parts, not simple propositions but ***“terms”*** of which they are composed. “Logically valid inferences depend for their validity on the structures of the propositions concerned but the relevant structures may either be the broad outward structures as in propositional logic or the finer substructures” which shall be our concern in this chapter. While argument 1 falls into propositional logic, argument 2 does not. Those inferences whose validity depends on the finer substructures as argument 2 is called ***Predicate Logic*** or ***Quantificational Logic***.

1.3.1 Analysis of Propositions into Terms

The simplest statements of quantificational logic are, nevertheless, propositions, but instead of regarding propositions as unanalyzed units and letting a single letter stand for it, quantification logic enables us to separate the subject and the predicate of a proposition and symbolize them separately. Little wonder Quantificational logic is also called Predicate logic (Purtill (1976) argues that it would be much more apt to call it ***Subject – Predicate Logic***).

The point here is that whereas in propositional Logic, a single capital letter standing alone is a well-formed formula (WFF); in predicate Logic, it is not. A single capital letter standing alone is not a WFF nor does it represent a proposition, in fact, the main preliminary impetus in predicate logic is to devise methods for describing and symbolizing simple (non-compound) statements by reference to their inner logical structures.

In this connection, symbolization in Quantificational Logic would usually begin with introducing two kinds of *constants: constants, which stand for subject* (or individual) terms (this will be the names or designations of whatever we choose to regard as individuals. This will include such things as persons, (say Socrates, Nnamdi, Tamuno, Minimah, Akpan, Folunsho etc) objects, (say Tables, Cars, Televisions, Houses etc), places (say shop, University, hotel, office, etc), times, propositions and actions); and *constants which stand for predicate terms*.

The first type of constants, that is, constants which stand for subject terms are the small letters of the Roman alphabets from "***a***" to "***w***". In this respect, the small letters of the alphabets "a" through "w" are stand-ins for specific subject terms and, the convention is to use the first letter of the initial key word of the subject term to denote the subject term, or as logicians prefer to call it *"individuals"*.

Thus in the following propositions:

Amarachi is beautiful
Nkem is hardworking
Policemen are sadists
The streets are flooded

the small letters "a", "n", "p" and "s" respectively denote Amarachi, Nkem, Policemen and Streets, the subject terms of the propositions, and are called *individual constants.*

The second type of constants, that is, constants which stand for predicate terms, are capital letters of the Roman alphabet ***A*** through ***Z***. The point here is that we use the capital letters of the alphabets "A to Z" as stand-ins for specific predicate terms. Thus in statements 1-4, "B", "H", "S" and "F" would stand for the predicates. "Beautiful", Hardworking", "Sadists", and "Flooded" respectively.

Applying this procedure of using the small letters of the alphabets "a" through "w" as stand-ins for specific subject terms and the capital letters "A" through "Z" as stand-ins for specific predicate terms to symbolize any simple statement (proposition), we adopt the rather confusing but deeply entrenched convention of writing the predicate term symbol or stand-in to the left of the symbol for its subject term. Thus, the simple proposition "Amarachi is beautiful" is symbolized as *Ba*. Nothing in this procedure permits us to write the subject term stand-in to the left of the predicate term stand-in notwithstanding that the subject term normally comes before the predicate term. As such *aB* is not a wff.

Following this procedure therefore "Nkem is hardworking" becomes

Hn

"Policemen are Sadists" would be symbolized as

Sp

"The street are flooded" as

Fs

"Students are demonstrating" as

Ds

"University Education is over subscribed" as

Su

"Ukaegbu is well-behaved" as

Wu

Ba, Hn, Sp, Fs, Ds, Su, Wu etc are propositions, they assert something that could be definitely true or definitely false about individual names that are specified. It is possible and indeed logicians have developed the practice of replacing specific subject terms i.e. individual names (constants) with that of individual variables. This convention allows us to instead of attaching a small letter that refers to a specific subject term to the predicate letter standing in for the predicate term, attach a small letter that is an individual variable, that is, a place maker for any subject term or individual whatsoever. As individual variables, the small letters x, y, z, are usually used and in this respect we usually begin with x. Thus instead of "Hs" for "students are hardworking", we could have any of Hx, Hy or Hz. Any of these Hx, Hy or Hz could stand for "students are hardworking", "Nwankwo is hardworking" etc. the specific subject term symbolization like

Ha for Agu is hardworking
Hn for Nwagu is hardworking
Hi for Ijoma is hardworking
Hu for Ukaegbu is hardworking

are respectively and separately either true or false; but Hx is neither true nor false, nor any of either Hy or Hz true or false. x, y, z being variables are not propositions and as such no truth values can be attached to them. In this sense, expressions of the form Hx, that is, expressions where an individual variable is attached to the Predicate letter standing-in for a predicate term, are called *Propositional functions.* When the individual variable in a propositional function is replaced by an individual constant, the propositional function becomes a proposition. Succinctly put, therefore, propositional functions are expressions that contain individual variables and becomes propositions when their individual variables are replaced by individual constants. Accordingly the following expressions:

Hx
Hy
Hz
Bx
Fx
Dx
Sx
Wx

are propositional functions, and could become propositions as in

Ha
Hn
Hi
Ba
Fs
Ds
Su
Wu

The point is that Hx, Hy, Hz, Bx, Fx, Dx, Sx, Wx say nothing until some x is specified, become a definite (specific) proposition with a definite truth-value. When a proposition is thus specified, we say it is a substitution instance of the propositional function from which it results by the substitution of an individual constant for the individual variable in the propositional function. Accordingly Ha, Hn, Hi, Ba, Fs, Ds, Su and Wu are substitution instances of Hx, Hy, Hz, Bx, Fx, Dx, Sx and Wx. The technical name for this process of obtaining a proposition from a propositional function by substituting a constant for a variable is *instantiation*.

Having taken note of how to analyze a simple proposition in predicate logic using stand-ins for the subject and predicate terms we now add the operators on propositions already discussed in propositional logic symbolization procedure; these are: •, v, ⊃, ≡, and ( ), [ ], { }. Thus to symbolize

1. Students are not hardworking

we have

~Hs

This expression is a denial and it is notable that the denial symbol ~ is placed to the left of the predicate letter; ~Hs, in this respect is a wff. H ~ s is ill-formed just as Hs~ is also ill-formed.

2. Agu is hardworking and Ezinne is beautiful

   Ha •Be

3. If it is raining outside then the streets are flooded

   Or ⊃ Fs

4. Either Ijoma qualifies for the National Art Exhibition or he will secure a scholarship for further studie

   Ai v Si

5. Moralists are humanists if and only if Sadists are moralists

   Hm ≡ ~Ms

6. If Agu is hardworking and Ezinne is beautiful then either Agu wins the National Merit award or Ezinne qualifies to represent Abia State in the beauty pageant

   (Ha • Be) ⊃ (Na v Pe)

7. If moralists are kind-hearted and teachers are sympathetic then either politicians are corrupt or the Naval team is not honest

   (Km • St) ⊃ (Cp v ~ Hn)

8. Ugo is not qualified to contest the elections unless she is cleared of bankruptcy and either the masses revolt against the injustice or the party primaries will be marred by mass disqualification.

~Eu v [Bu • (Im v Dp)]

9. Kama is not hardworking if Nkechi is ahead of him and either Chima is already a graduate or Peter has been awarded a scholarship

[An • (Gc v Sp) ⊃ ~ Hk

10. Uchechi is indeed a prodigy if it is confirmed that Ijoma walked at 11 months and Ngozi either was assisted to crawl at 4 months or the testimony given by her mother is misleading.

[Wi • (Cn v Mt)] ⊃ Pu

1.3.2 Quantification

In addition to analyzing the internal structures of simple propositions such that the subject and predicate terms are displayed, the idea of *quantification* is one other feature of quantificational logic, which both underscores the distinction between quantification logic and propositional logic, and shows the more powerful and encompassing character of, quantification logic. Indeed, the real advantage of quantificational logic over propositional logic lies in the point that the former has a device for symbolizing propositions that contain words like ”All” and “Some”, that is propositions that have quantity or contain quantifiers; this device is called ***quantification.***

To handle this idea of quantification, Quantificational Logic introduces two operators called ***quantifiers***, which stands in front of propositional functions, somewhat as the denial symbol: “~” stands in front of a simple proposition or a compound proposition. The two quantifiers are (1) a subject variable within parentheses, for example (x), (y) or (z). The subject variables within parentheses may also be used with an upside A in front of it, giving something like (∀x), (∀y), (∀z), and it is this latter one that we shall use in this book; (2) a subject variable within parentheses with a backward E in front of it, for example (∃x), (∃y), (∃z). When a quantifier stands in front of a left parenthesis its scope extends to the corresponding right parenthesis just as in the case of ~.

These operators: (∀x) and (∃x), or (∀y) and (∀y), or (∃z) and (∃y) have two important characteristics. First, they can be used to express the key ideas of universality and particularity; and second; they can be used to solve the problem of cross-reference between propositions which cannot be handled in propositional logic. In this connection, the quantifier (∀x), called the ***Universal quantifier*** is used to quantify universal propositions. Thus, we can symbolize universal propositions (that is, propositions quantified by the terms “All”, “Every”, “Anyone”) with the following predicates: Beautiful, hardworking, Sadist, Moralists and Democrats as follows:

(∀x) Bx
(∀x) Hx
(∀x) Sx
(∀x) Mx
(∀x) Dx

In terms of particularity, propositions that contain “some”, that is, refer to an indefinite part of a class, are called particular propositions, and the quantifier (∃x) called ***Existential Quantifier*** is used to quantify such particular propositions. Thus, we can symbolize particular propositions with the same predicates used above, as follows:

(∃x) Bx

$(\exists x)$ Hx
$(\exists x)$ Sx
$(\exists x)$ Mx
$(\exists x)$ Dx

In reading the expressions quantified by the Universal quantifiers, we use the phrase "Given any individual that individual is – ". Thus the following:

$(\forall x)$ Bx would read "given any individual, that individual is beautiful"

$(\forall x)$ Hx = "given any individual that individual is hardworking"

$(\forall x)$ Sx = "given any individual that individual is a Sadist"

$(\forall x)$ Mx = "given any individual that individual is a Moralist"

$(\forall x)$ Dx = "given any individual that individual is a democrat"

On the other hand, in reading expressions quantified by the Existential quantifier, we say "There is an individual such that, that individual is – ". In this regard

$(\exists x)$ Bx would read "there is an individual such that, that individual is a beautiful"

$(\exists x)$ Hx "there is an individual such that, that individual is hardworking"

$(\exists x)$ Sx "there is an individual such that, that individual is a sadist"

$(\exists x)$ Mx "there is an individual such that, that individual is a moralist"

$(\exists x)$ Dx "there is an individual such that, that individual is a democrat"

It is obvious that all our quantified expressions above are affirmative propositions, and there is no gain saying that not all propositions are affirmative. To say the least, the denial (or negation) of an affirmative proposition is a negative proposition. This thus invites the necessity to consider negative propositions and here the denial of a universal affirmative proposition would yield a universal negative $(\forall x) \sim$ which would read "given any individual, that individual is not –". Thus while $(\forall x)$ Bx symbolize "All x are beautiful", $(\forall x) \sim$ Bx symbolizes "Nothing is beautiful".

Similarly, the denial of an Existential affirmative proposition would yield an Existential negative $(\exists x) \sim$ which would read "there is an individual such that, that individual is not –". In this sense while $(\exists x)$ Bx symbolizes "Some x is beautiful", $(\exists x) \sim$Bx symbolizes "Some x is not beautiful".

Following the traditional classification of (categorical) propositions into *Universal Affirmative, Universal Negative, Particular Affirmative* and *Particular Negative* the following symbolizations will hold

1. Universal Affirmative

All humans are mortal $(\forall x)$ [Hx $\supset$ Mx)

which reads as "Given any individual if that individual is human then that individual is mortal"?

2. Universal Negative

No humans are mortal $(\underline{\forall x})$ [Hx $\supset \sim$ Mx]

which reads as "Given any individual if that individual is human then that individual is not mortal"

3. *Particular Affirmative*

Some humans are mortal $(\exists x)\ [Hx \bullet Mx]$

which reads as “There is at least one individual such that, that individual is human and he/she is mortal.

4. *Particular Negative*

Some humans are not mortal $(\exists x)\ [Hx \bullet {\sim}Mx]$

which reads as “There is at least one individual such that, that individual is not human and he/she is not mortal

1.3.3 Relationship between Universal & Existential Quantification

Our rendering of quantificational logic in the schema of traditional logic’s classification of standard form categorical propositions into four above does not discuss the relationship of equivalence between Universal and Existential quantifiers. Thus, when we addressed the question of negative propositions and purported the denial of a Universal Affirmative proposition to be a Universal Negative, our treatment of negation was purely in terms of the quality and not quantity. There is, therefore, in this connection a need to look at the relations between Universal and existential quantification within the context and presentation of traditional logic.

Traditional logic emphasizes four types of propositions with the familiar example of:

1. All humans are mortal
2. No humans are mortal
3. Some humans are mortal
4. Some humans are not mortal

These four propositions are usually classified as:

1. Universal Affirmative
2. Universal Negative
3. Particular Affirmative, and
4. Particular Negative

and are abbreviated as ***A, E, I*** and ***O*** propositions respectively. The abbreviations *A, E ,I* and *O* are said to be derived from the Latin words **Affirmo** and ***nEgO*** meaning respectively, “I affirm” and “I deny”. The first vowels in each of <u>*Affirmo*</u> and <u>*nEgO*</u> symbolizes Universal propositions while the second vowel refers to Particular propositions.

In analyzing the relationship, which holds between *A, E, I*, and *O* propositions, a theory called the ***square of opposition*** was introduced (first by Aristotle). Usually presented such that the *A, E, O propositions, and I* are arranged in a square as follows:

The square is constructed to exemplify in a systematic way the relations that hold between any two of the standard four categorical propositions.

In symbolizing Universal and Existential quantifiers we had

$(\forall x)\ [Hx \supset Mx]$

symbolizing Universal Affirmative propositions

$(\forall x)\ [Hx \supset \sim Mx]$

symbolizing Universal Negative propositions

$(\exists x)\ [Hx \bullet Mx]$

symbolizing Particular Affirmative propositions

and

$(\exists x)\ [Hx \bullet \sim Mx]$

symbolizing particular Negative propositions.

If we use the letter "M" to stand for any predicate whatsoever, the relations between Universal and Existential quantification comes out clearly as follows:

1. $[(\forall x)\ Mx] \equiv [\sim(\exists x) \sim Mx]$
2. $[(\exists x)\ Mx] \equiv [\sim(\forall x) \sim Mx]$
3. $[(\forall x) \sim Mx] \equiv [\sim(\exists x)\ Mx]$
4. $[(\exists x) \sim Mx] \equiv [\sim(\forall x)\ Mx]$

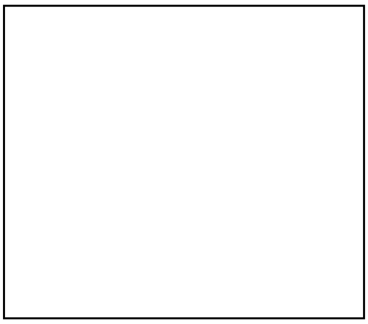
In No.1, "Everything is m(ortal)" means the same thing as, that is, is equivalent to "it is false that there is something which is not m(ortal)".

No. 2 "Something is "M" means the same thing as, that is, is equivalent to "it is false that everything is not M"

No. 3 "Nothing is "M" means the same thing as, that is, is equivalent to "it is false that there is something which is M".

No. 4 "Something is not M" is equivalent to "it is false that everything is M".

This relationship between Universal and Existential quantification could be superimposed on the traditional square of opposition to produce the picture below; if we use H and M for any predicate whatsoever



1.4 Conclusion

Quantificational Logic analyzes the internal structures of propositions into subject and predicate terms and symbolize them separately. It has a device for symbolizing propositions that contain words like "All" and "Some", that is propositions that have quantity or contain quantifiers; this device is called ***quantification.***

1.5 Summary

This unit has introduced you to the technique used to separate the subject and the predicate of a proposition and symbolize them separately. It also developed the practice of replacing specific subject terms i.e. individual names (constants) with that of individual variables. To handle this idea of quantification, the unit introduces two operators called ***quantifiers***. These operators: $(\forall x)$ and $(\exists x)$, or $(\forall y)$ and $(\exists y)$, or $(\forall z)$ and $(\exists z)$ are used to express the key ideas of universality and particularity; the quantifier $(\forall x)$, called the

Universal quantifier is used to quantify universal propositions and the quantifier (∃x) called ***Existential Quantifier*** is used to quantify particular propositions. It also discusses relationship of equivalence between Universal and Existential quantifiers.

1.6 Glossary

Constants The Logical device, which operates on propositions to yield other propositions and are such that they have a fixed specific meaning.

Existential Generalization (E.G.) A rule of inference in the theory of quantification which permits the valid substitution of a propositional function with the existential quantification of that propositional function, that is from ψ***v***, we can validly infer ***(∃x)*** ψ***x***.

Existential Instantiation (E.I.) A rule of inference admissible in quantification theory which permits one to validly infer from the existential quantification of a propositional function the truth of its substitution instance with respect to any individual constant that does not occur earlier in that context; that is, from ***(∃x)*** ψ***x*,** we can infer ψ***v***.

Existential quantifier A symbol ***(∃)*** in modern quantificational theory which indicates that any propositional function (x,y,z) immediately following it has some true substitution instance;***"(∃x) Fx, (∃y) Fy, (∃z) Fz"*** mean respectively that (1) "There exists an x that has F", (2) "there exists a y that has F", and (3) "there exists a z that has F".

Formal Proof of Validity – The deduction of the conclusion of an argument from its premisses by a sequence of statement forms each of which is either a premiss of the given argument, or follows from the preceding statement form of the sequence by one of the rules of inference.

Generalization In quantification theory, the process or forming a propositional function by placing a universal quantifier (e.g ∀***x***) or an existential quantifier (e.g ∃***x***) before it.

Individual Constant – A symbol (conventionally a lower case letter ***a*** through ***w*)** used in logical notation to denote an individual.

Individual Variable A symbol (conventionally the lower case letters "x", "y" and "z") which serves as a place maker for an individual constant or any subject term whatsoever.

Instantiation A process in quantification theory of substituting an individual constant for an individual variables, thereby converting a propositional function into a proposition.

Particular Proposition A proposition that refers to some but not to all the members of a class. Two variants of this proposition are distinguished namely Particular Affirmative and Particular Negative. Particular Affirmative Propositions (called ***I – propositions***) say that some S is P while Particular Negative propositions (called ***O – propositions***) say that some S is not – P. Both in traditional and Modern logic, particular propositions are understood to have existential import; in the theory of quantification, they are symbolized using the Existential Quantifier, (∃x).

Propositional Function A term used in quantification theory to refer to an expression from which a proposition may result either by ***instantiation*** or by ***generalization***. A propositional function is *instantiated* when the individual variables within it are replaced by individual constants (e.g Hx is instantiated as Ha). A propositional function is *generalized* when either the universal or the Existential quantifier is introduced to precede it (e.g Hx is generalized as either (∀x) Hx or as (∃x)Hx.

Quantification – Literally specification as to quantity. A method for describing and symbolizing non compound statements by reference to their inner logical structure in such
a manner that those saying something about everything (all), and those saying something about at least one thing (some) are distinguished.

Quantifier – A concern with the notion of quantity. See Existential, Quantifier and Universal Quantifier.

Universal Instantiation (U.I.) A rule of inference in quantificationallogic that permits the valid inference of any substitution instance of a propositional function from the universal quantification of that propositional function.

Universal Generalization (U.G.) A rule inference in quantificational logic that permits the valid inference of a generalized, or universally quantified expression from an expression that is given as true of any arbitrarily selected individual.

Universal proposition A proposition that refers to all the members of a class. Two types of this proposition are usually distinguished namely; *Universal Affirmative propositions* (traditionally called A-proposition) which says that All S is P; and *Universal Negative propositions* (traditionally called E-proposition) which say No S is P.

Universal Quantifier Any of the symbols $(\forall x)$, $(\forall y)$ and $(\forall z)$, in quantificational logic, used before a propositional function which asserts that the predicate following it is true of everything.

1.7 Check your Progress

Exercises 1. 7. 1

Symbolize the following simple propositions showing their subject and predicate terms

1. Nigeria is a rich country
2. Abia is God's own state
3. Nigeria is endowed with plenty of oil deposits
4. Nguzu is the headquarters of Afikpo South L.G.A.
5. The military leader is a dictator
6. The traffic is heavy
7. Urediya is a good mother
8. Adaku is exceptionally brilliant
9. God is love
10. Shehu is not happy

Exercise 1.7.2

Symbolize the following expressions

1. If Briggs is elected president and he is sworn-in by the vice chancellor then Briggs is not known to be a clown.
2. If Ngeribara is born in River State then he is of the Niger-Delta stock.
3. Chibuikem is both a moralist and a democrat or if he is neither of the two then Amarachi is influential.
4. Nkechi will win the beauty pageant if Blessing withdraws and Ezinne elects to step down for Ijeoma and Adanna.
5. If it is raining outside and the roads are wet then students will not come to school unless the streets are not flooded and the traffic wardens are mobilized.
6. Nigeria will qualify for the next world cup finals and play against Brazil if and only if, if Argentina wins the preliminaries both Paraguay and Spain will not petition to FIFA.
7. Either if Akpan is born in Calabar then Savage is French or if Akpan is not born in Calabar then Savage is a Cross River State citizen.
8. If Agu is happy but Ijay is not then either Agu is not really ambitious or both Agu and Ijay are yet to know of the scholarship award.
9. Either Ukaegbu is more than 1.8 metres tall or if Amarachi wins the beauty pageant then neither Nneka nor Chibuike will participate in the selection process.
10. If moralists are humanists if and only if Sadists are atheists then either moralists do not read Marxist Literature or the progressives simply are good natured.

Exercise 1.7.3

Symbolize the following expressions:

1. There are no atheists
2. Nothing Exists
3. Something is not in question
4. Anyone can contest
5. Everybody resigned
6. Some things are white
7. It is not true that everybody is mortal
8. It is not true that some police officers are not corrupt
9. Only taxpayers can context elections
10. No teachers are well paid
11. Every democrat is progressive
12. Some politicians are greedy
13. Some democrats are corrupt
14. Some corrupt priests are not trustworthy
15. All teachers are not well paid.

Exercise 1.7.4

Symbolize each of the following expressions and identify those that are negations of the others and state the logical equivalence of each of them.

1. Everything is good
2. Everybody is a moralist
3. Some things are not good
4. It is not true that everything is good
5. Some things are commendable
6. Everything is not commendable
7. There are no moralists
8. Moralists exists
9. Nothing exists
10. It is not true that moralists exist

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Unit 2 Symbolization in Quantificational Logic

Contents

2.1 Introduction

This study unit introduces the learner to the techniques for symbolization in quantificational logic. It emphasizes that quantifiers have scope and that unless we determine the scope of a quantifier, we might indeed be unable to determine whether an expression is a proposition or a propositional function. It also brings to sharp focus the idea of free and bound occurrences of variables in symbolization in quantificational logic;

2.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. conduct symbolization in Quantificational logic.
2. translate expressions into the logical notation of propositional functions and quantifiers
3. identify quantifiers that have the entire formulae as their scope and those quantifiers that do not have the entire formulae as their scope
4. identify propositions that contain free or bound occurrences of variables and thereby correctly symbolize expressions in quantificational logic.
5. symbolize normal form Universal propositions as always having as their major connective the horse shoe symbol, that is, they express a relation of implication
6. symbolize Existential propositions as expressing conjunction relationships, hence have the dot “• “ symbol as its major operator.
7. identify formulae that are propositional function rather than proposition

2. 3 Symbolization in Quantificational Logic

With the machinery we developed in Unit 1, we can conduct symbolization in Quantificational logic. However, before we proceed, certain important points need to be emphasized.

The point is remarkable that quantifiers, like the tilde “~”, have scope; in this connection unless we determine the scope of a quantifier, we might be unable to really determine whether an expression is a proposition or a propositional function

If we take the following quantified expressions:

(1) $(\exists x)\ [Hx \bullet Mx]$

and

(2) $(\exists x)\ Hx \bullet Mx$

in symbolization I, the scope of the quantifier $\exists x$ is $Hx \bullet Mx$, that is, the quantifier has the entire formula as its scope, the brackets in grouping $Hx \bullet Mx$ together defines the scope of the quantifier. But this is not the case with symbolization 2. The scope of the quantifier $\exists x$ in symbolization 2 is Hx; Mx is outside its scope. The point on relief here is that if there is a bracket following a quantifier, the scope of the quantifier covers all variables within the bracket that is represented by the quantifier. As such if no bracket follows a quantifier, then the scope of the quantifier is restricted to the formula immediately following the quantifier. The point is also notable that if the variable accompanying a predicate differs from the quantifier variable then it is outside the scope of the quantifier. Thus in expressions 3, 4, 5 and 6 below

(3) $(\exists x)$ $(Hx \bullet Mx)$

(4) $(\forall x)$ $[(Hx \vee Rx) \supset (Mx \equiv Bx]$

(5) $(\forall x)$ $[(Hx \bullet Rx) \supset (My \vee By)$

(6) $(\exists x)$ $[(Hx \bullet My) \bullet Rx]$

whereas in expressions 3 and 4 the quantifiers have the entire formulae as their scope in 5 and 6 the quantifiers do not have the entire formulae as their scope.

As a matter of procedure, any occurrence of a variable that is within the scope of the quantifier is said to be a *bound occurrence* of that variable; and any variable outside the scope of a quantifier is said to be a *free occurrence* of that variable. Thus in the expression:

(7) $(\forall x)$ $[(Hx \supset My)$

as in 5 and 6 every x is considered a bound occurrence while y is a free occurrence in the expression.

It needs be emphasized here that the idea of free and bound occurrences of variables is so crucial in doing symbolization in quantificational logic; whether or not a proposition is correctly symbolized depends on whether it contains free or bound occurrences of variables. For, any well-formed formula (wff) to represent a proposition in predicate logic it must not contain free occurrence of any variable. The point is that once there is a free occurrence of any variable in any wff, that wff cannot represent a proposition. If a wff, therefore, contains one free occurrence of a variable, then that formula is taken as a propositional function rather than a proposition. In other words, any proposition that contains any free occurrence of a variable has no truth since propositional functions do not have truth values.

The meaning of this is that if a symbolization is to represent a proposition, all the variables that occur in the quantified symbolization must be bound variables. For example in the following symbolizations

(1) $(\forall x)$ $[(Hx \bullet Mx) \supset Rx]$

(2) $(\forall x)$ $(Hx \bullet Mx) \supset Rx$

(3) $(\forall x)$ $[(Hx \bullet Mx) \supset Ry]$

(4) $(\exists x)$ $Hx \bullet Mx$

(5) $(\exists y)$ $[(Hy \bullet My) \equiv Ry]$

(6) $\exists x$ $[(Hx \vee Ry) \bullet Mx]$

symbolization (1) has no free occurrence of any variable. The Universal quantifier $(\forall x)$ ranges over the entire x within the formula. All occurrences of x are bound occurrence. Accordingly, the expression is a proposition.

Symbolization (2) has a free occurrence of a variable. The scope of the quantifier is restricted to the antecedent, that is, the quantifier ranges over only the variables grouped together by the brackets. The variable x, which occurs in the consequent, is a free occurrence of x. The result is that symbolization (2) is not a proposition but a propositional function.

Symbolization (3) has a free occurrence of a variable. The quantifier $\forall x$ is restricted to the occurrence of x and not any other variable; "y" is in the formula which means that "y" is not bound by the quantifier. The meaning of this is that symbolization (3) is a propositional function and not a proposition.

Symbolization (4) has a free occurrence of a variable. As already explained, Mx is outside the scope of the quantifier ($\exists x$) because if no bracket follows a quantifier, the scope of the quantifier is restricted to the wff immediately following the quantifier. So, as with 2 and 3, symbolization (4) is a propositional function, not a proposition.

Symbolization (5) has no free occurrence of any variable. The scope of the quantifier ($\exists y$) ranges over the entire formula; every occurrence of y is a bound occurrence of y. Therefore, symbolization (5) is a proposition.

Finally, symbolization (6) has a free occurrence of a variable. The quantifier ($\exists x$) ranges over the entire formula but $\exists x$ is restricted to the occurrence of x and not any other variable; "y" is in the formula, which means that "y" is not bound by the quantifier. The implication is that symbolization (6) is not a proposition, but a propositional function.

Another very significant remark is that in the normal form Universal propositions always have as their major connective the horse shoe symbol, that is, they express a relation of implication, while Existential propositions express conjunction relationships, hence have the dot "$\bullet$ " symbol as its major operator.

Let us now symbolize some quantified expressions:

1. "All students are either hardworking or cheats"

is symbolized as

$(\forall x)\ [Sx \supset (Hx \vee Cx)]$

and it reads "given any individual if that individual is a student then either he is hardworking or a cheat"

2. "Only tax-payers and unemployed adults will vote in the election" is symbolized as

$(\forall x)\ [Ex \supset (Tx \vee Ux)]$

and it reads "given any individual, if that individual will vote in the election then either he is a tax-payer or an unemployed adult. The temptation is to symbolize the expression as

$(\forall x)\ [(Tx \bullet Ux) \supset Ex]$

which would read "given any individual, if that individual is a tax-payer and an unemployed adult then he will vote in the election. This it is obvious is not what the expression says. The expression "Only tax-payers and unemployed adults will vote" translates into "All who will vote in the election are tax payers and unemployed adults". And it would also be wrong to symbolize the expression as

$(\forall x)\ [Ex \supset (Tx \bullet Ux)]$

which would read "given any individual if that individual will vote in the election then he is both a tax-payer and an unemployed adult. To say that only tax-payers and unemployed adults will vote in the election is to say that anybody who will vote in the election will be either a tax-payer or an unemployed adult, not that any one who will vote has to simultaneously be a tax-payer and an unemployed adult.

3. "No student can escape the examination" is symbolized as

$(\forall x)\ [Sx \supset {\sim}Ex]$

which reads "given any individual if that individual is a student then he/she cannot escape the examination"

4. “All that glitters is not gold” is symbolized as

$(\exists x)$ [Gx • ~ Ox]

which reads “there is at least one individual such that it glitters and it is not gold”. Here although “all” appears as the quantifier the tempting symbolization

$(\forall x)$ [Gx ⊃ ~Ox]

is palpably incorrect because it would read “given any individual if it glitters then it is not gold”; and this is not what the expression: “all that glitters are not gold” means. It rather means “some things that glitter are not gold” which means that there is at least one thing which is not gold and it glitters. Were one to symbolize it as

$(\exists x)$ [Ox • ~ Gx]

which would read “there is at least one individual such that it is a gold and it does not glitter” it would it be wrong for the expression would mean that “there is one thing which is a gold and it does not glitter”, and this is not equivalent to there is at least one thing which is not gold and it glitters”.

(5) $(\forall x)$ [Lx ⊃ (~Px v Hx)]

which reads “given any individual if that individual is a leader then either he is not popular or he is humane” which means the same thing as given any leader he is not popular unless he is human”

(6) “Some students are successful if and only if they are hardworking” is symbolized as

$(\exists x)$ [Sx • (Ux ≡ Hx)]

which reads “there is at least one individual such that, the individual is a student and that he is successful if and only if he is hardworking”.

(7) “Some moralists are humanitarians or agnostics” is symbolized as

$(\exists x)$ [Mx • (Hx v Ax)]

which reads “there is at least one individual such that, that individual is a student and he/she is either a humanitarian or an agnostic.

2.4 Conclusion

The scope of a quantifier determines whether an expression is a proposition or a propositional function. The scope of the quantifier covers all variables within the bracket that is represented by the quantifier and if no bracket follows a quantifier, then the scope of the quantifier is restricted to the formula immediately following the quantifier. For, any well-formed formula (wff) to represent a proposition in predicate logic it must not contain free occurrence of any variable. If a wff, therefore, contains one free occurrence of a variable, then that formula is taken as a propositional function rather than a proposition.

2.5 Summary

This unit has introduced you to the technique used to determine whether an expression is a proposition or a propositional function. It also demonstrates how to represent a proposition in predicate logic for it to be a well-formed formula. The unit also demonstrates that in the normal form Universal propositions always have as their major connective the horse shoe “⊃” symbol, that is, they express a relation of implication, while Existential propositions express conjunction relationships, hence have the dot “• “ symbol as its major operator.

2.6 Glossary

Bound variable Any variable that occurs within the scope of a quantifier containing the same variable. Thus in the expression: ***(∀x) [Hx ⊃ Mx]*** every occurrence of "x" is a bound variable.

Free Variable Any variable that occurs outside the scope of a quantifier. Thus in the expressions (∀x) [Hx ⊃ My] and (∃x) [(Hx•My) •Rx], every occurrence of y is a free occurrence of the variable "y".

Wff Well-formed formula, i.e string of symbols of a formal language correctly constructed with respect to its formation rules.

2.7 Check your Progress

Exercises

Translate each of the following expressions into the logical notation of propositional functions and quantifiers, in each case use letters of a key word.

1. Students are not always hardworking.
2. No student who is a mediocre will win a scholarship if unbiased assessors are engaged.
3. It is not true that Nigeria will win the next edition of the Nation's cup if and only if Cameroon is disqualified and South Africa withdraws at the early stages.
4. Any philosopher is sound if he is hardworking and reads a lot.
5. Only academics and students are both marginalized and ignored by government.
6. No student is a genius unless he is naturally endowed.
7. Any girl that is humble has been well trained
8. Every successful politician is either corrupt, selfish or ambitious.
9. Not any groom attended the marriage seminar.
10. Not every lecturer is sound who is hardworking.
11. Some girls are religious only if faced with crisis.
12. Only arrogant and beautiful girls will contest the beauty pageant.
13. Politicians and police officers are either acclaimed corrupt or they are just mischievous.
14. Not all students who score good grades are both hardworking and naturally endowed.
15. To excel in life, one must work hard if one is a student or undergo real tutelage if one ventures into a trade.

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Unit 3 Symbolizing Relational Propositions

Contents

3.1 Introduction

This study unit introduces the learner to the techniques for symbolizing relational propositions. The analyses of the internal structure of relational propositions is important in predicate (quantificational) logic. This is because without displaying the relationship that hold between individuals in a proposition we are sure to muddle up both the symbolization of such propositions and logical inferences involving such relational propositions.

3.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. analyze the internal structure of certain propositions of quantification logic called *relational propositions*
2. understand the types of relationship that hold between propositions and name of the predicate that holds such relations.
3. form propositional functions with relational predicates.
4. identify which predicate letter followed by any finite series of variables are well-formed formula.
5. identify which predicate letter followed by a mixture of names and variables are well-formed formula
6. symbolize relational propositions
7. use quantifiers in symbolizing relational propositions.

3.3 Symbolizing Relational Propositions

In addition to traditional predicate propositions, quantificational logic also deal with propositions that contain two or more proper names which obviously are truth-functional compounds of singular propositions that have different subject terms. For example, the proposition "Agu and Ijoma are brothers" is not a conjunction or a truth function of "Agu is a brother" and "Ijoma is a brother"; the proposition does not purport to assert that Agu and Ijoma both have certain attribute(s), it rather asserts that they stand in a certain relation. The point, therefore, is that certain propositions of quantification logic deal with *relations* hence called *relational propositions*.

The analyses of the internal structure of such relational propositions is indeed pursued in predicate (quantificational) logic, because without displaying the relationship that hold between individuals in a proposition we are sure to muddle up both the symbolization of such propositions and logical inferences involving such relational propositions.

A relational proposition, in this context, is simply a proposition, which is such that given any predicate, for example "L" some or all individuals share the attribute of that predicate with some other individual(s). If the proposition is such that the relationship is between two individuals, it is called a *binary* or dyadic relation; the predicate that holds such relation is called *dyadic predicate*. If we consider the following propositions:

1. Ijoma is older than Ukaegbu
2. Nkem likes Chijioke
3. Adaku is the sister of Esther
4. Nnanna is married to Comfort

we discover that example 1 talks of a relationship of age between ijoma and Ukaegbu and the predicate there is "Older", examples 2-4 talks of other forms of relationship between two individuals; 2 of "likeness", 3 "of sister" and 4 "of marriage".

Relations in which three individuals relate are called *triadic* or *ternary* relations, and the predicate that holds such relationship called *triadic* or *ternary* predicate. Propositions such as

1. Briggs introduced Maduba to Chima.
2. Kalu transferred his property to his children.
3. Okafor likes Ugonma more than Nneka
4. Nguzu is between Ohafia and Abiriba

are examples of triadic relations.

Relations in which four individuals relate are called *tetradic* or *quaternary* relations. Examples of propositions in which such relationship hold are

1. Uche, Ikechi, Chima and Akpan studied together.
2. Nnanna sent Agu to Chima from Umuahia
3. Kalu transferred his property to his children in a will.
4. Peter bought the car from Amadi in Lagos.

There is no doubt that these types of relations enter into arguments in various ways, there is therefore need for us to first tackle the problem of symbolizing relational propositions.

Beginning with our examples of dyadic relations, using "i" for Ijoma and "u" for Ukaegbu and "O" (the predicate) for older, "Ijoma is older than Ukaegbu" symbolizes as

Oiu

Using "L" for the predicate and "n" and "c" for Nkem and Chijioke respectively 'Nkem likes Chijioke" is symbolized as

Lnc

Using "S" for sister and "a" and "e" for Adaku and Esther, "Adaku is the sister of Esther" is symbolized as

Sae

In "Nnanna is married to Comfort", if we use "n" and "c" for the names and "M" for the predicate, we get

Mnc

Our examples of triadic relations would be symbolized as follows:

Briggs introduced Maduba to Chima

Ibmn

"Introduced" is the triadic predicate in this three person relation and is translated into capital "I" and names into "b", "m" and "c"

Kalu transferred his property to his children

Tkpc

The triadic predicate ‘transferred” is translated into capital “T” and the names “Kalu”, “property” and “children” in “k”, “p” and “c” respectively.

Okafor likes Ugonma more than Nneka

The triadic predicate “likes” is translated into capital “L” and the names “Okafor”, “Ugonma” and “Nneka” into “o” “u” and “n” respectively.

Nguzu is between Ohafia and Abiriba

Bnoa

The triadic predicate “between” is translated into capital “B’ and the names “Nguzu”, “Ohafia” and “Abiriba” into “n” “o” and “a” respectively.

The examples given in defining tetradic relations would be symbolized as in below:

Uche, Ikechi, Chioma and Akpan studied together

Suica

The tetradic predicate “studied” is translated into capital “S” and the names “Uche” “Ikechi” “Chioma” and “Akpan” into “u”, “i”, “c” and “a” respectively.

Nnanna sent Agu to Chima from Umuahia

Snacu

The tetradic predicate “sent” is translated into capital “S” and the names Nnanna, Agu, Chima and Umuahia into “n” “a” “c” and “u” respectively.

Kalu transferred his property to his children in a will.

Tkpcw

The tetradic predicate “transferred” is translated into capital “T” and the names kalu, property, children and will into “k”, “p”, “c” and “w” respectively.

Peter bought the car from Amadi in Lagos

Bpcal

The tetradic predicate “bought” is translated into capital B and the names Peter, car Amadi and Lagos into “p”, “c”, “a” and “l” respectively.

It is clear from the foregoing that in symbolizing relational propositions the rule is that a predicate letter followed by any finite series of names is a well-formed formula. Thus

$\underbrace{L}_{\text{Predicate}}\overbrace{n\ c}^{\text{names}}$

$\underbrace{\mathbf{I}}_{\text{predicate}}\overbrace{b\ m\ c}^{\text{names}}$

$\underbrace{\mathbf{T}}_{\text{predicate}}\overbrace{k\ p\ c\ w}^{\text{names}}$

are all wff’s

It is notable to indicate here that we can form propositional functions with relational predicates. In this connection, what is required is to replace one or all the individual names with individual variables. Accordingly, just as a propositional function of a variable, for example, “x” is a human” could be translated into Hx, so a propositional function of a dyadic relation such as “x” is a friend of ‘y’ is symbolized as Fxy. In the same vein the propositional function “x introduced y to z” is symbolized as Ixyz’ and the propositional function “x sent y to z from w” is symbolized as Sxyzw. As a matter of fact the following

Fxy

Ixyz

Sxyzw

are wff's, so are

T r r k x

T x y y z

T x b y a

In this later set, we see that there is mixture of names and variables. The import of this that just as we have the rule that a predicate letter followed by any finite series of names is a wff, two other rules follow

(1) A predicate letter followed by any finite series of variables is a wff.

(2) A predicate letter followed by a mixture of names and variables is a wff.

Having mastered these rules governing the symbolization of relational propositions we can symbolize some relational expressions.

1. Either Tamuno is not a friend of Adaku or Tamuno is not a friend of Ogonnaya

$\sim Fta \vee \sim Fto$

or

$\sim Fxy \quad \vee \quad Fxz$

In the first symbolization we used the names "t" "a" and "o" for Tamuno, Adaku and Ogonnaya respectively and capital "F" for the dyadic predicate; while in the second we used variables x,y and z for Tamuno, Adaku and Ogonnaya respectively.

2. If Ijoma, Kelechi and Okorie are friends then either Nwankwo marries Chinyere or Edak loves Archibong

$Fiko \supset (Mnc \vee Lea)$

or

$Fxyz \supset (Mwc \vee Lea)$

3. If Chima loves Ola and Dele is taller than Musa then either Chima is not richer than Peter or Ola does not love Peter.

$(Lco \bullet Tdm) \supset (\sim Rcp \vee \sim Lop)$

or

$(Lxo \bullet Tym) \supset (\sim Rxz \vee \sim Loz)$

In the symbolizations in 1 – 3 above whereas in the first names are used, the alternate symbolization is normally a mixture of names and variables.

It becomes indeed significant at this point to introduce the use of quantifiers in symbolizing relational propositions. In this connection, the dyadic relational proposition: "Somebody is a friend to Tunde" which reads "there exists an individual such that, that individual is a friend to Tunde" is symbolized as

$(\exists x)\ Fxt$

Tunde is a friend to someone is symbolized as

$(\exists x)\ Ftx$

Everybody is a friend of Tunde is symbolized as

$(\forall x)\ Fxt$

Tunde is a friend of everyone is symbolized as

$$(\forall x)\ Ftx$$

We notice that in the above symbolizations that in symbolizing "Tunde is the friend of everyone or somebody", the name of Tunde is placed before the variable while "Everyone or somebody is a friend of Tunde" is symbolized such that the variable is placed before the name. This means that any one of

$$(\exists x)\ Fxt$$
$$(\exists y)\ Fyt$$
$$(\exists z)\ Fzt$$

is a correct symbolization of "somebody is a friend of Tunde", while any one of

$$(\exists x)\ Ftx$$
$$(\exists y)\ Fty$$
$$(\exists z)\ Ftz$$

is a correct symbolization of "Tunde is a friend of someone".

Similarly,

$$(\forall x)\ Fxt$$
$$(\forall y)\ Fyt$$
$$(\forall z)\ Fzt$$

is a correct symbolization of "Everybody is a friend of Tunde" while any one of

$$(\forall x)\ Ftx$$
$$(\forall y)\ Fty$$
$$(\forall z)\ Ftz$$

is a correct symbolization of "Tunde is a friend of everybody".

Everyone is not a friend of Tunde would be symbolized by any one of

$$\sim (\forall x)\ Fxt$$
$$\sim (\forall y)\ Fyt$$
$$\sim (\forall z)\ Fzt$$

while "Tunde is not a friend of everyone" is symbolized by any one of

$$\sim (\forall x)\ Ftx$$
$$\sim (\forall y)\ Fty$$
$$\sim (\forall z)\ Ftz$$

However, when several quantifiers or even more than one quantifier occurs in a single relational proposition, symbolization becomes more complicated. The simplest propositions of this kind are

1. Everyone is a friend to everyone

   $(\forall x)\ (\forall y)\ Fxy$
2. Someone is a friend to someone.

   $(\exists x)\ (\exists y)\ Fxy$
3. Nobody is a friend to anybody

   $(\forall x)\ (\forall y)\ \sim Fxy$

If we consider other completely general relational propositions that are complex, the complication in symbolizing such propositions became more obvious.

4. "If someone is taller than everybody then everybody attracts someone".

   $(\exists x)\ (\forall y)\ Txy \supset (\forall x)\ (\exists y)\ Axy$

This expression reads as "if there exist an x and for any y, x is taller than y, then for any x and for at least one y, x attracts y".

5. "Either everyone loves someone or someone does not love someone

   $(\forall x)\ (\exists y)\ Lxy\ \ v\ \ (\exists x)\ (\exists y)\ \sim Lxy$

6. "If everyone likes John then John likes himself or someone does not like John"

   $(\forall x)\ Lxj\ \supset (Ljj\ \ v\ \ (\exists y)\ \sim Lyj)$

7. "If everyone is loved by someone then someone is loved by everyone"

   $(\forall x)\ (\exists y)\ Lxy\ \supset (\exists x)\ (\forall y)\ Lxy$

Because of the apparent complicated character of relational proposition, the best way to translate relational propositions into logical symbolism is by adopting a kind of stepwise process.

Thus to symbolize the proposition:

8. "Any sound academic can outsmart some politician", as a first step we may write

$(\forall x)$ {(x is a sound academic) $\supset$ (x can outsmart some politician)

Next, the consequent of the conditional between the braces

X can outsmart some politician

is symbolized as a generalization or quantified expression:

$(\exists y)$ [(y is a politician) $\bullet$ (x can outsmart y)]

Using the abbreviations, "Sx", "Px", and "Oxy" for is "x is a sound academic", "x is a politician", and "x can beat y", the expression in 8 is symbolized as follows

$(\forall x)\ [Sx \supset (\exists y)\ (Py \bullet Oxy)$

Applying this same stepwise process the expression:

9. "Some politicians can outsmart all sound academics"

would first translate into

$(\exists x)$ [(x is a politician) $\bullet$ (x can outsmart all sound academics)]

then into

$(\exists x)$ {(x is a politician) $\bullet$ $(\forall y)$ [(y is a sound academic) $\supset$ (x can outsmart y)]}

and finally into

$(\exists x)\ [Px\ \bullet (\forall x)\ (Sy \supset Oxy)]$

We can also apply this same method to more complex cases, where more than one relation is involved. For example in the expression:

10. Anyone who transfers everything to everyone is sure to offend somebody

the first step is to paraphrase it as

$(\forall x)$ {[x is a person) $\bullet$ (x transfers everything to everyone)] $\supset$ [x offends somebody]}

The second conjunct of the antecedent

x transfers everything to everyone

may be further paraphrased first as

$(\forall y)$ [(y is a person) $\supset$ (x transfer everything to y)]

and then as

$(\forall y)$ [(y is a person) $\supset$ $(\forall z)$ (x transfers z to y)]

The consequent in our first paraphrase

x offends somebody

has its structure clearly by being expressed as

$(\exists u)$ [(u is a person) $\bullet$ (x offends u)]

The original proposition can now be expressed instead as

$(\forall x)$ {{(x is a person) • $(\forall y)$ [(y is a person) ⊃ $(\forall z)$
(x transfers z to y)]} ⊃ $(\exists u)$[(u is a person) • (x offends u)]}

Then using the abbreviations Px, Txyz, Oxy for "x is a person", "x transfers y to z", and "x offends y", our expression in 10 is expressed as

$(\forall x)$ {{Px • $(\forall y)$ [Py ⊃ $(\forall z)$ Txzy]} ⊃ $(\exists u)$ (Pu • Pxu)}

3.4 Conclusion

Analyses of the internal structure of relational propositions in predicate (quantificational) logic helps in displaying the relationship that hold between individuals in a proposition, which ultimately facilitates the symbolization of relational propositions and logical inferences involving such relational propositions.

3.5 Summary

This unit introduced the learner to the analyses of the internal structure of certain propositions of quantification logic called *relational propositions.* It gives an exposition of the types of relationship that hold between propositions and name of the predicate that holds such relations and how to form propositional functions with relational predicates. It further demonstrates which predicate letter followed by any finite series of variables are well-formed formula, and which predicate letter followed by a mixture of names and variables are well-formed formula. It generally facilitates how to symbolize relational propositions and how to use the quantifiers in symbolizing relational propositions.

3.6 Glossary

Dyadic relation One of the types of a relational propositions which expresses a relation between two individuals such as 'Amarachi is the sister of Uchechi", "Agu loves Chinyere" etc It is also called binary relation.

Quaternary Relation. This is also called ***tetradic relation***; it is a relation in which four individuals relate.

Relational Proposition. A proposition which in such that given any predicate some or all individuals share the attribute of the predicate with some other individuals.

Triadic Relation A relation in which three individuals relate; it is also called **ternary relation**.

3.7 Check your Progress

Exercise 3.7.1

Using the appropriate relational predicates and individual names, symbolize each of the following relational propositions.

(1) Chika is a Nigerian citizen

(2) Chika is not the brother of Udodirim

(3) Okoro is not happy if and only if either Archibong is not happy and Briggs is a friend of Boma or it is not true that Tamuno is not a friend of Archibong.

(4) Either if Uche is born in Umuahia and Okorie speaks French then Kalu is not an indigene of Rivers State or if Amaka is not born in Port-Harcourt then Chima is not a friend to Uche.

(5) If Emeka is taller than Okoroafor then either Amadi is not a brother to Chinedu or both Okoroafor and Ukaegbu are not parties to the argument.

(6) If Chijioke is taller than Ngozi and Ngozi loves Diri then if Okey is taller than Chijioke, Ngozi does not love Diri.

(7) Agu is taller than Chibuzo if and only if Obinna is stronger than Ikechi then Agu is not stronger than Chibuzo.

(8) If Nwoye, Alice, Okoronkwo and Udo studied together then either Okoronkwo and Alice will pass the examination or if all four of them fail then Ijoma, Esther and Ogonnaya will win the scholarship.

(9) It is not true that if Chukwu is a brother to both Ugo and Onyema then either Ewa, Archibong and Florence are citizens of Cross River State or Ewa is not a friend to both Ugo and Florence.

(10) Okoro is taller than Chinwe and Muoneke if either Ukaegbu, Ijoma and Agu are brothers or Chinwe is not really friend to Muoneke.

Exercise 3.7.2

Using the appropriate relational predicates and individual variables, symbolize each of the above relational propositions.

Exercise 3.7.3

Using the appropriate Quantifiers, relational predicates and individual variables, symbolize the following relational propositions.

1. If Everyone is a French citizen then either nobody is born in the English speaking world or someone is not a French citizen
2. Either nobody is a friend to Tunde or if somebody is a friend to Tunde then it is not the case that nobody is a friend to everybody.
3. If three friends are traveling together then either somebody is a friend to all three or the three friends cannot travel together.
4. Everyone loves John if and only if either someone is not a friend to Boma or no one loves Chima and Uchechi.
5. If everyone loves Amarachi and Uchechi then either somebody is not truthful or someone does not love Amarachi but loves Uchechi.
6. If both Chinedu and Joseph are brothers to Ezinne then if everyone is a friend to Ezinne the everyone is a friend to Chinedu and Joseph.
7. If everyone is happy then either someone who loves Ukanna is a friend to everyone or if Oji is loved by everybody then Enyinnaya is not a friend to someone.
8. If any two brothers are quarrelling and someone is a friend to both of then either someone intervenes or nobody is a friend to both of them.
9. Both if Kalu is not a friend to everyone then someone is not happy with Nkechi and either everyone likes Ijeoma or no one loves both Nkechi and Kalu.
10. If Ukaegbu hates anyone who is insulting then if everyone who likes Ukaegbu, either everyone hates anyone who is insulting or Ukaegbu is not liked by somebody.

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Unit 4 Properties of Relations

Contents

4.1 Introduction

This study unit introduces the learner to Properties of Relations. These properties are normally grouped under *Symmetrical, Transitive* and *Reflexive*. Symmetrical relations are further analyzed into *asymmetrical* and non-symmetrical while Transitive properties are also analyzed into *Transitive, Intransitive* and *Non-Transitive* relations while Reflexive properties may be further characterized as *Reflexive, Irreflexive* or *non-reflexive.*

4.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. analyze the properties of relations into three broad categories and nine sub categories
2. symbolize the various categories of the properties of relations.
3. distinguish between the various types of relations
4. use the appropriate relational Predicate and individual variables, in symbolizing which relational proposition is *Symmetrical, Asymmetrical and Non-symmetrical*
5. use the appropriate relational Predicate and individual variables, in symbolizing which relational proposition is *Transitive, Intransitive* and Non-Transitive
6. use the appropriate relational Predicate and individual variables, in symbolizing which relational proposition is *Reflexive, Irreflexive and Non-reflexive*

4.3 Properties of Relations

In everyday usage, relational propositions such “is as strong as”, “is stronger than”, “is married to” are common. An analysis of such phrases indicates some properties of relations themselves. These properties are normally grouped under *Symmetrical, Transitive* and *Reflexive.*

A relation is, said to be *symmetrical* if when one thing bears a particular relation to a second, the second bear it also to the first. In this sense, a relation of the form Mxy and Myx where “M” stands for “is married to” is symmetrical

If x is married to y is true then y is married to x must be true. That is, if (∀x), (∀y) Mxy is true then (∀x) (∀y) Myx must also be true; which could be expressed as

(∀x) (∀y) (Mxy ⊃ Myx)

The relations “is standing with”, “has the same height as”, “weighs the same as “ are all symmetrical.

If, however, when one thing bears a relation to a second then the second *cannot* bear it to the first, such a relation is said to be *asymmetrical*. In this sense, any relation of the form Txy and Tyx, where “T” stands for “taller than” is asymmetrical; if x is taller than y then y cannot be taller than x. Accordingly if (∀x) (∀y) Txy is true then (∀x) (∀y) Tyx must be false, and this could be expressed as

(∀x) (∀y (Txy ⊃ ~ Tyx)

The relations “is the mother of” “is bigger than” are other examples of asymmetrical relation. The point here is that given any x and y, if x is (for example) the mother of y, then it is not the case that y is the mother of x”.

A relation, which is neither symmetrical nor asymmetrical, is said to be *non-symmetrical.* In this case, a non-symmetrical relation arises when we cannot definitely say whether the relation Lxy is Lyx or ~Lyx, that is, where “L” stands for “looking at”, if x is looking at y, we cannot for sure say whether y is also looking at x or not looking at x.

Other examples of non-symmetrical relation include “x loves y”, “x is the sister of y”, “x admires y”. If “x loves y” it does not follow that “y loves x” nor does it follow that “y does not love x”. Similarly, if “x is the sister of y”, it does not follow that y is a sister to x (y could possibly be a brother instead) nor does it follow that y is not a sister to x. So also so, if “x admires y”, it does not follow that “y admires x” nor does it follow that y does not admire x.

Relations may also be characterized as *Reflexive, Irreflexive* or *non-reflexive.*

A relation is defined as *reflexive* if a thing has that relation to itself. In this sense, for any relation between x and y, if x shares that relation to itself, we have a reflexive relation. The formula for a reflexive relation accordingly is

$(\forall x)\ Sxx$

where “S” stands for “as stronger as”

Here if x is as strong as y, it means that x is as strong as itself. Similarly, the relation “is identical with” is reflexive since everything must be identical with itself.

A distinction, however, is usually made between relations that are *totally reflexive* and those that are just reflexive. A relation of the form “Rxx” is totally reflexive, but a relation of the form “Rxy” is not totally reflexive. The relation of love is one example of not totally reflexive relations; when we say that “x loves y”, it does not necessarily follow that “y loves x”, or that “everybody love themselves”.

If, on the other hand, nothing (or no individual) bears that relation to itself, we say it is an irreflexive relation. If, for example, x is taller than y, x cannot be taller than (itself) x. Thus in an irreflexive relation the formula is

$(\forall x)\ {\sim}Txx$

where “T” stands for “taller than”.

Other examples of irreflexive relation are “faster than” “is a parent of” etc.

Relations that are neither reflexive nor irreflexive are said to be *non-reflexive relations.* In this type of relation, some individuals bear that relation to themselves while others do not. For example “x loves y” or “x takes care of y” are non-reflexive because it is possible for x to love y and at the same time x will not love (itself) x”. Just as x can love y and also love itself (x). Similarly, if x takes care of y, it is possible that x takes care of (itself) x or x does not take care of (itself) x.

Finally, relations may be characterized as *Transitive, Intransitive* and *Non-Transitive*. A relation is said to be *transitive* if it is such that if one thing bears it to a second, and the second bears it to a third, then the first must bear it to third. Thus in a transitive relation the formula is

$(\forall x)\ (\forall y)\ (\forall z)\ [(Sxy \bullet Syx) \supset Sxz]$

where “S” stands for stronger than.

In this respect, if x is stronger than y and y is stronger than z then x is stronger than z.

A relation is called *intransitive* if when one thing bears that relation to a second, and the second to a third, then the first cannot bear it to the third. An intransitive relation is thus presented symbolically as

$(\forall x)\ (\forall y)\ (\forall z)\ [(Gxy \bullet Gyz) \supset {\sim}\ Gxz]$

where “G” stands for grandmother

In this reading if x is the grandmother of y and y is the grandmother of z, x cannot be the grandmother of z. Similarly, if x is the father of y and y is the father of z, x cannot be the father of z, that is,

$(\forall x)\ (\forall y)\ (\forall z)\ [(Fxy \bullet Fyz) \supset {\sim}\ Fxz]$

where “F” stands for father.

Other examples of intransitive relation include “is two years older than”, “is mother of” etc.

Relations that are neither transitive nor intransitive are classified as *non-transitive relations*. In this type of relation, for any three individuals x,y and z, the relation that holds between x and y on one hand, and y and z on the other hand, does hold between x and z at times. In this sense, if x loves y on one hand, and y loves z on the other hand, x at times may love z also and at other times may not love z. Other non-transitive relations include hate, look at, admire etc.

4.4 Conclusion

Analyses of relational propositions yield relational properties that are grouped broadly into Symmetrical, Transitive and Reflexive and further subcategorized into symmetrical, asymmetrical and non-symmetrical; Transitive, Intransitive and Non-Transitive and into Reflexive, Irreflexive and non-reflexive. All these properties of relations using appropriate relational Predicate and individual variables can be symbolized.

4.5 Summary

This unit introduced the learner to the analyses of the properties of relations. A relation is, said to be *symmetrical* if when one thing bears a particular relation to a second, the second bear it also to the first. If, however, when one thing bears a relation to a second then the second *cannot* bear it to the first, such a relation is said to be *asymmetrical*, and a relation, which is such that we cannot definitely say whether, it holds between two or more individuals or not is said to non-symmetrical. Secondly, a relation is defined as *reflexive* if a thing has that relation to itself. If, on the other hand, no individual bears that relation to itself, we say it is an irreflexive relation and when a relation is such that some individuals bear that relation to themselves while others do not we say the relation is non-reflexive. Finally, a relation is said to be *transitive* if it is such that if one thing bears it to a second, and the second bears it to a third, then the first must bear it to third. On the other hand, a relation is called *intransitive* if when one thing bears that relation to a second, and the second to a third, then the first cannot bear it to the third. And a relation, which for any three individuals x,y and z, the relation that holds between x and y on one hand, and y and z on the other hand, does hold between x and z at times is called non-transitive.

4.6 Glossary

Asymmetrical Relation – A property of relations, which is such that when one thing bears a relation to a second then the second, cannot bear it to the first. For example, the relation “taller than”; if “X” is taller than “Y”, then “Y” cannot be taller than “X”.

Intransitive Relation – A relation which is such that if one thing bears that relation to a second, and the second to a third, then the first cannot bear it to the third. For example, if x is the father of y and y is the father of z, x cannot be the father of z. This is expressed in $(\forall x)\ (\forall y)\ (\forall z)\ [(Fxy \bullet Fxz) \supset \sim Fxz]$.

Irreflexive Relation – A relation which a thing or an individual cannot bear to itself. For example, if x is taller than y, x cannot be taller than (itself) x. Thus $(\forall x) \sim Txx$ expresses an irreflexive relation.

Non-reflexive Relation. A relation, which some things or individuals bear to themselves while others do not. For example, “x loves y”; it is possible for x to love y and at the same time x will not love (itself) “x”. Just as x can love y and also love itself (x).

Non-transitive A type of relation which is such that for any three individuals x, y and z, the type of relation that holds between x and y on one hand, and y and z on the other hand, does hold between x and z at times. In this sense, if x loves y on one hand, and y loves z on the other hand, x at times may love z also and at other times may not love z.

Non-symmetrical Relation – A relation, which we cannot definitely say whether, it holds between two or more individuals or not. For example loves, if "x loves y, it does not follow that y loves x nor does it follow that y does not love x.

Reflexive Relation – A relation, like identity, which a thing bears to itself. For any relation between x and y, if x shares that relation to itself, we have a reflexive relation. If x is, for example as strong as y, it means that x is as strong as itself.

Symmetrical relation – A relation which is such that if one thing bears it to a second, the second must bear it also to the first. If x is married to y, then y must be married to y.

Transitive Relation – A relation which is such that if one thing bears it to a second, and the second bears it to a third, then the first must bear it to the third. If x is stronger than y and y is stronger than z, then x is stronger than z.

4.7 Check your Progress

Exercise 4.7

Using the appropriate relational Predicate and individual variables, symbolize each of the following expressions; and state whether the relation is symmetrical, transitive or reflexive.

1. ____________is stronger than _________
2. ____________is as strong as__________
3. ____________is the father of __________
4. ____________is west of ______________
5. ____________ loves ___________
6. ____________ can out smart __________
7. ____________ is equal to _____________
8. ____________ is better than __________
9. ____________ and________are as strong as __
10. ____________ is the uncle of_____and ________

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MODULE 2

Unit 1 Formal Proof of Validity in Quantificational Logic

Contents

1.1 Introduction

This study unit introduces the learner to how to construct formal proofs of validity for arguments whose validity depends upon the internal structures of the simple propositions that make up the argument. It introduces some new rules that are specific to Quantificational Logic. These rules are primarily concerned with dropping and adding of quantifiers hence will enable the learner to handle statements that have quantifiers.

1.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. understand some new rules specific to quantificational logic used in formal proof of validity
2. construct formal proof of validity for arguments in quantificational logic

1.3. Formal Proof of Validity in Quantificational Logic

One principal purpose of symbolization in logic is to put arguments into a form convenient for testing their validity. This accordingly invites the need for us to now focus on how to construct formal proofs of validity for arguments whose validity depends upon the internal structures of the simple propositions that make up the argument. In doing this, we have to add to the twenty rules of inference and Replacement some new rules that are specific to Quantificational Logic; that is, rules that deal with the inner structures of simple propositions.

1.3.1 Rules of Quantification and Recapitulation of Rules of Inference

The rules of Quantification are four in number and operate to strengthens our capacity to construct formal proofs of validity of arguments forms that have quantifiers; they primarily are concerned with dropping and adding of quantifiers.

1. Universal Instantiation (U.I)

$$\frac{(\forall x)\ \psi x}{\therefore\ \psi v \text{ (where v is any individual symbol)}}$$

A universal propositional function is true if all its substitution instances are true, the rule of Universal Instantiation thus permits the substitution instance of a Universal Generalization. In essence, this rule says that any substitution instance of a propositional function can be validly inferred from its Universal quantification. This means that the rule of Universal Instantiation permits us to substitute for instance “Ha” for (∀x) Hx, that is, drop the Universal quantifier and replace or substitute the variables attached to the predicate with any individual constant. It is important to underscore the point that the variable x in Hx could be substituted by any constant, say Hb, Hc, Hd etc.

2. Universal Generalization (U.G)

$$\frac{\psi y}{\therefore (\forall x)\ \psi x}$$

∴ (∀x) ψx (where y denotes any arbitrarily selected individual and ψy is not within the scope of any assumption containing the special symbol “y”)

Universal Generalization is a rule which says that from the substitution instance of a propositional function with respect to the name of any arbitrarily selected individual, one may validly infer the Universal quantification of that propositional function. In essence, this rule permits us to make conclusion from a particular instance or individual within the Universe of discourse to all the individuals, infact, the rule permits us to introduce the universal quantifier to an unquantified wff of predicate logic. Thus if “Ha” is an individual or element from a universe of discourse we can validly introduce the universal quantifier and replace the individual constant with a variable, that is, infer (∀x) Hx. It is in this connection that it is said that Universal Generalization permits us generalize, that is, go from a special substitution instance to a generalized or universally quantified expression.

3. Existential Instantiation (E.I)

(∃x) ψx

:. ψv (where v is any individual constant other than y having no pervious occurrence in the context)

This rule says that from the existential quantification of a propositional function, we can infer the truth of its substitution instance with respect to any individual constant, other than y, that occurs nowhere earlier in the context. In other words, the rule permits us to infer a proposition from an existential propositional function by replacing the individual variable with an individual constant outside the scope of the individual variable. That is, it enables us to drop the Existential quantifier and substitute the variable attached to the predicate letter with an individual constant, which has not occurred earlier in the proof in the same context. Thus, we can infer validly “Ha” from (∃x) Hx, but in so doing we should make sure that “a” has not occurred earlier in the proof in the same context.

4. Existential Generalization (EG)

ψv

∴ (∃x) ψx (where v is any individual symbol)

The rule of EG says that from any true substitution instance of a propositional function, we can validly infer the existential quantification of that propositional function. The point here is that if we take “v” to denote any individual constant, we can infer the existential generalization of a propositional function by replacing the individual name or constant with an individual variable. Simply understood the rule permits us to introduce an Existential quantifier to a predicate wff of the form “Ha” and to replace the individual constant “a” with an individual variable, say x.

These four additional rules of inference, that is UI, UG, EI and EG permit the transformation of non-compound, generalized propositions into equivalent compound propositions to which our twenty rules of inference and replacement may be readily applied, and also permit the transformation of compound propositions into equivalent non-compound propositions.

It would be expedient here to refresh our familiarity with the rules of inference and replacement.

Although, it has been stated that most textbooks on logic give nine elementary valid argument forms, in this text, we will adopt a union of the rules as given by living M. Copi in two books, namely *Introduction to Logic* and *Symbolic Logic.* A union of the rules as given in these books give us ten elementary valid arguments that can be used in constructing formal proofs of validity. An outline of these rules is as follows:

1. **Modus Ponens (M.P)**
   p ⊃ q
   p /∴ q

2. **Modus Tollens (M.T)**
   p ⊃ q
   ~q /∴~p

3. **Hypothetical Syllogism (H.S)**
   p ⊃ q
   q ⊃ r /∴ p ⊃ r

4. **Disjunctive Syllogism (D.S)**
   p v q
   ~ p /∴ q

5. **Constructive Dilema (C.D)**
   (p ⊃ q) • (r ⊃ s)
   p v r /∴ q v s

6. **Destructive Dilema (D.D)**
   (p ⊃ q) • (r ⊃ s)
   ~q v ~s/∴ ~p v ~r

7. **Absorption (Abs.)**
   p ⊃ q /∴ p ⊃ (p • q)

8. **Simplification (Simp.)**
   p • q /∴ p

9. **Conjunction (Conj.)**
   p
   q /∴ p • q

10. **Addition (Add.)**
    p /∴ p v q

The validity of the above argument forms can be established using the truth tables method. Their significance, however, find relief in the construction of formal proofs of validity for a wide range of more complicated arguments. There is, at this point, the need to explain these rules and also give substitution instances of each of them. Accordingly, beginning with the first;

$$\begin{array}{l} \boldsymbol{p \supset q} \\ \boldsymbol{p \;/\therefore\; q} \end{array}$$

that is, Modus Ponens (abbreviated (M.P) says that in any implicational argument form once we can establish the antecedent, we can conclude the consequent. In other words, if we affirm the antecedent of an implication, the consequent follows, that is, we also affirm the consequent. The inference is said to be in the mood, or *modus ponendo ponens* ; the expression signifies that by affirming (in the premiss(es) we affirm (in the conclusion). This rule is derived from the Latin *ponene*, which means to take a stand or to affirm. Thus, if an implicational proposition says "if it is raining, then it is wet outside", and we affirm that "it is raining" (the antecedent) then it follows with logical necessity that "it is wet outside" (that is, we can validly affirm the consequent).

It is important to note that this rule applies to any implicational expression whatsoever. It follows, therefore, that in an argument form, no matter how complex, if and where the major connective is an implication, that is, the horse show" ⊃" symbol, we are permitted to affirm the consequent if we find any occurrence of the antecedent

The second

$$\begin{array}{l} \boldsymbol{p \supset q} \\ \boldsymbol{\sim q \;/\therefore\; \sim p} \end{array}$$

called Modus Tollens (abbreviated M.T) is to the effect that given any implicational proposition, once the consequent is denied it follows that the antecedent, must be denied, or can be validly denied. Here, the Latin *"tollere*", meaning *'to lift up or deny'*, is understood in the sense that by denying (in the premiss(es) we deny (in the conclusion). Like in Modus Ponens the major connective of propositional or argument forms where Modus Tollens apply must be an implication, that is, be the horseshoe "⊃" symbol; it could be simple or complex.

The third rule:

$$\begin{array}{l} \boldsymbol{p \supset q} \\ \boldsymbol{q \supset r \;/\therefore\; p \supset r} \end{array}$$

called Hypothetical syllogism (abbreviate H. S) asserts that if we have two conditional propositions such that the consequent of the first is the antecedent of the second, it follows that the antecedent of the first conditional implies the consequent of the second conditional, that is, the consequent of the second conditional follows from the antecedent of the first conditional. It is important to note here that ordinarily a syllogism is a deductive argument in which a conclusion is inferred from two premisses; in this sense, the term aptly characterized means a syllogism that contains hypothetical (conditional) propositions that are such that the consequent of the second premiss of the conditional follows from the antecedent of the first premiss conditional.

The fourth rule, called Disjunctive Syllogism (abbreviated D.S) is of the form:

$$\begin{array}{l} \boldsymbol{p \vee q} \\ \boldsymbol{\sim p \;/\therefore\; q} \end{array}$$

and it says that given any disjunctive proposition, to deny the first disjunct implies asserting the second disjunct; that is, the rule is such that in an argument form if one premiss is a disjunction and another premiss is the denial of one (normally the first) of the two disjuncts, we are permitted to conclude the other disjunct. In other words, if the major connective in an argument form is the vel: "v" symbol, we can validly assert the second disjunct if we deny, or find an occurrence of the denial of, the first disjunct

The fifth rule, which has the form:

$$(p \supset q) \bullet (r \supset s)$$
$$p \vee r \; / \therefore q \vee s$$

is called Constructive Dilemma (abbreviated C.D) is to the effect that given a conjunction of two conditional or implication propositions, if we assert the disjunction of their antecedents, the assertion of the disjunction of their consequents necessarily follows. That is, the rule permits us to conclude the disjunction of the consequents of two implicational propositions once the disjunction of the antecedents has been asserted or is established.

The sixth rule, which is Destructive Dilemma (abbreviated D.D) is expressed as follows:

$$(p \supset q) \bullet (r \supset s)$$
$$\sim q \vee \sim s \; / \therefore \sim p \vee \sim r$$

It is just like Constructive Dilemma, in the sense that both involve a choice between two alternatives in the conjunction of two implications. But unlike in Constructive Dilemma where the choice is such that the assertion of a disjunction of the antecedents implies the assertion of a disjunction of the consequents; in Destructive Dilemma the choice is between the negations of a disjunction of the consequents and the negations of a disjunction of the antecedents. This rule therefore permits us to conclude the disjunction of the negations of the antecedents of two implicational propositions connected by a conjunction once we find an occurrence of the disjunction of the negations of their consequents.

The seventh rule called Absorption (abbreviated Abs.) has the form:

$$p \supset q \; / \therefore p \supset (p \bullet q)$$

and it asserts that given an implicational proposition the inference of the antecedent implies both the antecedent and the consequent; the rule permits us to, given an implication, conclude that the antecedent implies both itself and the consequent.

The eight rule, called Simplification (abbreviated Simp.) according to which

$$p \bullet q \; / \therefore p$$

merely says that given a conjunctive proposition, we are permitted to conclude the first conjunct as a separate proposition; that is, the rule allows the separation of conjoined propositions.

The ninth rule, called Conjunction (abbreviated Conj.) says

$$p$$
$$q \; / \therefore p \bullet q$$

which is to the effect that if we know two separate facts or propositions to be true, we can assert that the first and second are true. In this sense, the rule enables us to combine independent propositional forms using the connective and "•"; it permits propositions assumed to be true to be combined in one compound proposition.

Finally, rule ten, called Addition (abbreviated Add.) according to which

$$p / \therefore p \vee q$$

This rule, also called *Logical Addition*, allows us to introduce any propositional form whatsoever to a given propositional form with the disjunction operator, the vel: "v". The idea here is that it a propositional form or a premiss is true, then its disjunction with any propositional form or premiss as a component will be true, no matter what the other disjunct is.

These ten Rules of Inference we had demonstrated enables us to establish the validity of a wide range of arguments, which can also be shown to be valid using the truth tables with the advantage however of elegance and convenience. The procedure generally is that we build bridges between the premisses and conclusion, using these elementary valid argument forms; that the conclusion of the argument is arrived at from the premisses by our bridge building using these valid argument forms exclusively proves that the argument is valid.

It was demonstrated also that there are some obviously valid arguments whose validity cannot be proved using only the ten elementary valid argument forms. For example:

~ (P v Q) /:.~Q

In this connection, it becomes necessary that we introduced some other rules to complement the ten rules. These are the rules of replacement. Continuing the numbering from the ten elementary valid argument forms, they are:

11 De Morgan's Theorems (DeM.)

$\sim (p \bullet q) \equiv (\sim p \vee \sim q)$

$(p \bullet q) \equiv (\sim p \vee \sim q)$

$\sim (p \vee q) \equiv (\sim p \bullet \sim q)$

$(p \vee q) \equiv (\sim p \bullet \sim q)$

12. Commutation (Comm.): $(p \vee q) \equiv (q \vee p)$

$(p \bullet q) \equiv (q \bullet p)$

13. Association (Assoc.):

$[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$

$[p \bullet (q \bullet r)] \equiv [(p \bullet q) \bullet □ r]$

14. Distribution (Dist.):

$[p \bullet (q \vee r)] \equiv [(p \bullet q) \vee (p \bullet r)]$

$[p \vee (q \bullet r)] \equiv [p \vee q) \bullet (p \vee r)]$

15. Double Negation (D.N.): $p \equiv \sim\sim p$

16. Transposition (Trans.): $(p \supset q) \equiv (\sim q \supset \sim p)$

17. Material Implication (Impl.): $(p \supset q) \equiv (\sim p \vee q)$

18. Material Equivalence (Equiv.):

$(p \equiv q) \equiv [(p \supset q) \bullet (q \supset p)]$

$(p \equiv q) \equiv [(p \bullet q) \vee (\sim p \bullet \sim q)]$

19. Exportation (Exp.): $[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$

20. Tautology: $p \equiv (p \vee p)$

$p \equiv (p \bullet p)$

These rules of replacement operate to demonstrate that the logical constants are inter definable, that is the operators on proposition are mutually interdependent. It is therefore possible to transform a wff containing any of the constants into another wff in which that particular constant does not appear at all but in place of it another constant(s). When we transform such wff and the truth-table definition of the formulae yield the same truth function, we say they are equivalent. It was shown, for example that we can transform any wff containing any number of occurrence of $\bullet$ into an equivalent wff in which $\bullet$ does not appear at all instead certain complexes of "~" and "v" arise. Similarly, any wff containing $\supset$ can be transformed into an equivalent wff containing ~ and v, but not $\supset$. Further any wff containing $\equiv$ can be transformed into any equivalent containing $\supset$ and $\bullet$, but not $\equiv$ - and thus in turn by the previous steps it can be further transformed into one containing ~ and v but neither $\equiv$ nor $\supset$ nor $\bullet$. Thus, for every wff of propositional logic there is an equivalent wff, expressing precisely the same truth function.

The upshot of this is that because we can transform propositions containing a particular operator into an equivalent proposition in which that operator is replaced we may always replace a proposition with an equivalent proposition, since this replacement will always result in the same truth value, no matter the truth value of the component propositions.

1.3.2 Application of Rules of Quantification and Inference

With the four Rules of Quantification, namely, UI, UG, EI and EG and the twenty rules of inference we are now in a position to construct formal proofs of validity of arguments whose validity depends on the inner structure of some non-compound propositions within them, as in the following:

(1)	1.	$(\forall x)$ [~Ax v ~Bx]	
	2.	$(\exists x)$ [Zx $\bullet$ Ax] /$\therefore$ $(\exists x)$ [~Bx $\bullet$ Zx]	
	3.	Za $\bullet$ Aa	2 EI
	4.	~Aa v ~ Ba	1 UI
	5.	Aa $\bullet$ Za	3 Comm
	6.	Aa	5 Simp
	7.	~ ~ Aa	6 DN
	8.	~ Ba	4,7 DS
	9.	Za	3 Simp
	10.	~Ba $\bullet$ Za	8,9 Conj
	11.	$(\exists x)$ [~Bx $\bullet$ Zx]	10 UG

In the above argument we first eliminated the quantifiers and replaced the individual variables in the argument with individual names by applying EI and UI respectively to lines 2 and 1 and generated lines 3 and 4 before applying the relevant rules to derive our conclusion, which was itself derived by applying EG to line 10. Thus while we used EI and UI to eliminate the quantifiers so that we can apply the rules of inference, when we arrived at the conjunction of ~Ba and Za we used EG to introduce the quantifier and to replace the individual constants with individual variables. Special note need to be taken in that we first applied EI before UI notwithstanding that the first quantified premiss is a Universally quantified one. The point is that because an existential quantification of propositional logic is true if it has at least one true substitution instance, it is customary to treat existential quantifiers first before the Universal quantifiers.

(2)	1.	(∃x)) [Ax • Bx]	
	2.	(∀x) [(Bx v Cx) ⊃ Dx] / ∴ ~ (~Da v ~ Aa)	
	3.	Aa • Ba	1EI
	4.	(Ba v Ca) ⊃ Da	2 UI
	5.	Ba (Aa	3 Comm
	6.	Ba	5 Simp.
	7.	Ba v Ca	6 Add
	8.	Da	4,7 MP
	9.	Aa	3 Simp
	10.	Da (Aa	8,9 Conj.
	11.	~(~Da v ~ Aa)	10 DeM

In the above argument, the conclusion is not quantified and the individual constants or names appearing in the conclusion is “a”. In taking our first step which is eliminating the quantifiers and replacing the individual variables in the premisses with individual constants we were guided by the occurrence of “a” in the conclusion, hence we replaced the individual variables with “a” in applying EI and UI to lines 1 and 2. After eliminating the quantifiers and replacing the variables, we then were in a position to and did apply the rules of inference until we derived our conclusion.

(3)	1.	(∃x) [Dx •Gx]	
	2.	(∀x) [Gx ⊃ Px]	
	3.	(∀x) [~Mx ⊃ ~(Dx • Px)]	
	4.	(∀x) [Mx ⊃ Tx] ∴/((∃x) [Dx • Tx]	
	5.	Da • Ga	1 EI
	6.	Ga ⊃ Pa	2 UI
	7.	~Ma ⊃ ~ (Da • Pa)	3 UI
	8.	Ma ⊃ Ta	4 UI
	9.	Ga • Da	5 Comm
	10.	Ga	9 Simp
	11.	Pa	6,9 MP
	12.	Da	5 Simp
	13.	Da •Pa	12,11 Conj
	14.	~~(Da • Pa)	13 DN
	15.	~~Ma	7,14 MT
	16.	Ma	15 DN
	17.	Ta	8,16 MP
	18.	Da • Ta	12,17 Conj
	19.	(∃x) [Dx •Tx]	18 EG

The validity of the same argument could also be proved by applying some other rules as below:

(3b)	1.	(∃x) [Dx • Gx]
	2.	(∀x) [Gx ⊃ Px]
	3.	(∀x) [~Mx ⊃ ~(Dx • Px)]
	4.	(∀x) [Mx ⊃ Tx] ∴/((∃x) [Dx • Tx]

5.	Db • Gb	1 EI
6.	Gb ⊃ Pb	2 UI
7.	~Mb ⊃ ~(Db • Pb)	3 UI
8.	Mb ⊃ Tb	4 UI
9.	Gb • Db)	5 Comm
10.	Gb	9 Simp
11.	P	6,9 MP
12.	Db	5 Simp
13.	Db • Pb	12,11 Conj
14.	(Db • Pb) ⊃ Mb	7 Trans.
15.	Mb	14,13 MP
16.	Tb	8,15 MP
17.	Db • Tb	12, 16 Conj
18.	(∃x) [Dx • Tx]	17 EG

The actual difference in the proof of validity of arguments 3(a) and (b) lie not in the individual constant "a" and "b", but in the application of Transposition rule in line 14 of 13(b) instead of Double Negation in lines 14 and 15 of 13(a).

(4.)

1.	(∃x) [Bx (Rx] • [Jx (Hx]	
	∴/((∀x) [(Bx v Jx) ⊃ (Hx v Rx)]	
2.	(By ⊃ Ry) • (Jy ⊃ Hy)	1 EI
3.	By ⊃ Ry	2 Simp
4.	~By v Ry	3 Impl.
5.	(~By v Ry) v Hy	4 Add
6.	~By v (Ry v Hy)	5 Assoc
7.	(Ry v Hy) v ~ By	6 Comm
8.	(Jy ⊃ Hy) • (By ⊃ Ry)	2 Comm
9.	Jy ⊃ Hy	8 Simp
10.	~Jy v Hy	9 Impl.
11.	(~Jy v Hy) v Ry	10 Add
12.	~Jy v (Hy v Ry)	11 Assoc
13.	~Jy v (Ry v Hy)	12 Comm
14.	(Ry v Hy) v ~Jy	13 Comm
15.	[(Ry v Hy) v ~ By] • [(Ry v Hy) v ~Jy)	7,14 Conj
16.	[(Ry v Hy) v (~By • ~Jy)	15 Dist
17.	(~By • ~Jy) v (Ry v Hy)	16 Comm
18.	~(By v Jy) v (Ry v Hy)	17 DeM
19.	~(By v Jy) v (Hy v Ry)	18 Comm
20.	(By v Jy) ⊃ (Hy v Ry)	19 Impl.
21.	(∀x) [(Bx v Jx) ⊃ (Hy v Rx)]	20 UG

The point to remark in proving the formal validity of argument 4 above is that in applying the rule of Universal instantiation to line 1 we substituted an individual variable "y" for another individual variable "x". This is done because our conclusion is universally quantified, if we, therefore, substitute any individual constant for "x" before applying Universal Generalization then we are bound to make a wrong inference. The rule of Universal Generalization says that y denotes any arbitrarily selected individual, and since the truth of any substitution instance of a propositional function follows from its Universal quantification, we can infer the substitution instance that results from replacing "x" by "y". The gist here is that where the conclusion of an argument is universally quantified, in applying Universal instantiation we should substitute the individual variable "y" for the individual variable "x"

(5.)	1.	$(\forall x) \supset Rx\ (\ Sx)] \bullet (\exists x)\ (Gx \bullet Hx)]$	
	2.	$(\forall x)\ \ [Hx \supset Rx]\ \therefore/((\exists x)\sim[\sim Sx \bullet \sim Fx]$	
	3.	$[(\exists x)\ (Gx \bullet Hx)] \bullet [(\forall x)\ (Rx \supset Sx)]$	1 Comm
	4.	$(\exists x)\ [Gx \bullet Hx]$	3 Simp
	5.	$(\forall x)\ \ [Rx \supset Sx]$	1 Simp
	6.	$Ga \bullet Ha$	4 EI
	7.	$Ha \supset Ra$	2 UI
	8.	$Ra \supset Sa$	5 UI
	9.	$Ha \bullet Ga$	6 Comm
	10.	Ha	9 Simp
	11.	Ra	7,10 MP
	12.	Sa	8,11 MP
	13.	Sa v Fa	12 Add
	14.	$\sim(\sim Sa \bullet \sim Fa)$	13 DeM
	15.	$(\exists x)\sim[\sim Sx \bullet \sim Fx]$	14 EG

In the above argument, the requirement to the effect that we have to handle EI before UI, is the reason why we did not begin our proof by applying the rule of instantiations. We rather had to first apply commutation to line 1 so that we would be enabled to simplify line 1 such that the conjunction of the Universally and Existentially quantified expressions are separated. This is deemed necessary so that we will follow procedure and apply EI first as is exemplified in line 6. After applying EI, we went ahead to apply UI before applying the twenty rules of inference.

It is, however significant to highlight here that like the first ten rules of inference, the four rules of quantification, that is, UI, UG, EI and EG can be applied only to whole lines in a proof.

1.4 Conclusion

Four additional rules of inference, that specifically deal with quantification, that is UI, UG, EI and EG permit the transformation of non-compound, generalized propositions into equivalent compound propositions to which our twenty rules of inference and replacement may be readily applied, and also permit the transformation of compound propositions into equivalent non-compound propositions. Thus with the rules of UI, UG, EI and EG we are equipped to construct formal proofs of validity for arguments whose validity depends upon the internal structures of the simple propositions that make up the argument.

1.5 Summary

This unit to strengthen our capacity to construct formal proofs of validity of arguments whose validity depends upon the internal structures of the simple propositions that make up the argument introduced the learner to four additional rules, namely UI, UG, EI and EG. The rule of Universal Instantiation (UI) permits us to substitute for instance "Ha" for $(\forall x)$ Hx, that is, drop the Universal quantifier and replace or substitute the variables attached to the predicate with any individual constant. The rule of Universal Generalization (UG) permits us generalize from a special substitution instance to a generalized or universally quantified expression, that is, from “Ha”, we can validly infer (∀x) Hx. The rule of Existential Instantiation says that from the existential quantification of a propositional function, we can infer the truth of its substitution instance with respect to any individual constant, other than y, that occurs nowhere earlier in the context. Thus, we can infer validly “Ha” from (∃x) Hx, but in so doing we should make sure that “a” has not occurred earlier in the proof in the same context. Finally, Existential Generalization (EG) says that from any true substitution instance of a propositional function, we can validly infer the existential quantification of that propositional function. The rule thus permits us to introduce an Existential quantifier to a predicate wff of the form “Ha” and to replace the individual constant “a” with an individual variable, say x.

1.6 Glossary

Absorption (abbreviated Abs.) One of the ten elementary valid argument forms which is to the effect that if ***p*** implies **q** we can validly infer that ***p*** implies both ***p*** and ***q***. This symbolized as p ⊃ (p • q).

Addition (abbreviated as Add.) One of the ten elementary valid argument forms which is to the effect that given any propositional form we can validly infer another propositional form with the disjunction connective. Expressed as ***p*,** then ***p v q*** the rule states that if a propositional form or a premiss is true, a disjunction of it with another propositional form or premiss as a component will be true no matter what the other disjunct is.

Association (Abbreviated as Assoc.) One of the rules of replacement applied primarily to a valid regrouping of disjunctive and conjunctive propositions. The rule permits us to replace validly ***[p v (q v r)]*** with ***[(p v q) v r]*** and vice versa, and also ***[p•(q• r)]*** with ***[(p• q) • r]*** and vice versa.

Commutation (abbreviated Comm.) This is a type of logical equivalence; one of the rules of replacement that permits the valid reordering of the components of conjunctive or disjunctive propositions. The rule permits us to replace ***p v q*** with ***q v p*** and vice-versa, and ***p • q*** with ***q • p*** and vice versa.

Conjunction (abbreviated “Conj ”) is the name of a rule of inference, one of the ten elementary valid argument forms; it permits the combination of independent propositions assumed to be true in one compound proposition. Thus if we have ***p, q*** we can validly infer ***p • q***.

Constructive Dilemma (abbreviated “C.D.”) One of the ten elementary valid argument forms which permits the inference of the disjunction of the consequents of two implicational propositions once the disjunction of their antecedents is established. Thus we can from ***(p⊃ q) • (r ⊃ s)*** validly infer ***q v s*** if ***p v r*** is established, that is, if ***(p ⊃ q) • (r ⊃ s)*** is true, and ***p v r*** is also true, then ***q v s*** must be true.

De Morgan’s theorem *(DeM .)* One of the rules of replacement which is such that it permits the mutual replacement of a conjunction with a disjunction and vice versa, in such a manner that negated variables eliminate their negations both inside and outside the scope of a parentheses and unnegated variables acquired negations also outside and inside the scope of a parentheses. Thus ***~(p • q)*** transforms into ***~ p v ~ q; p • q*** into ***~(~ p v ~ q)*** and vice versa; similarly ***~(p v q)*** transforms into ***~ p • ~ q***, ***~ (p v q)*** into **~ p • ~ q** and vice versa.

Destructive Dilemma One of the ten elementary valid argument follows; it permits the conclusion of a negated disjunction of the antecedent of a conjoined implications once the disjunction of the negations of their consequents is established. Thus, we can from $(p \supset q) \bullet (q \supset s)$ validly infer $\sim p \vee \sim r$ if $\sim q \vee \sim s$ is established, this means that if $(p \supset q) \bullet (r \supset s)$ and $\sim q \vee \sim s$ is true then $\sim p \vee \sim r$ must be true. 138, 144

Disjunctive Syllogism ***(D. S.)*** One of the ten elementary valid argument forms which is such that given any disjunctive proposition, to deny the first disjunct validly implies asserting the second disjunct.

Distribution One the rules of replacement which is applied to conjunctive and disjunctive wff's in such a manner that whenever we have conjunction as a major connective with disjunction as minor or vice versa in an expression, we distribute the variables such that the conjunction turns out to be distributed into itself and a disjunction, and the disjunction into itself and a conjunction. Thus $[p \bullet (q \vee r)]$ replaces $[(p \bullet q) \vee (p \bullet r)]$ and vice versa ; also $[p \vee (q \bullet r)]$ replaces $[(p \vee q) \bullet (p \vee r)]$ and vice versa.

Double Negation – One of the rules of replacement expressing logical equivalence by permitting the valid mutual replacement of any symbol by the negation of the negation of that symbol; that is, permits the inference of a symbol from a canceling out is its consecutive denials or negations. This is symbolized as $p \equiv \sim\sim p$ and vice versa. 161-166

Existential Generalization (E.G.) A rule of inference in the theory of quantification which permits the valid substitution of a propositional function with the existential quantification of that propositional function, that is from □***v***, we can validly infer $(\exists x)$ ***x***□.

Existential Instantiation (E.I.) A rule of inference admissible in quantification theory which permits one to validly infer from the existential quantification of a propositional function the truth of its substitution instance with respect to any individual constant that does not occur earlier in that context; that is, from $(\exists x)$ □***x*****,** we can infer ***v***□.

Exportation (Exp.) The name of a rule of inference expressing the logical equivalence that permits the mutual replacement of a conditional with a conjunctive antecedent, by an association of conditionals, that is, $(p \bullet q) \supset r$ is logically equivalent and therefore validly replaces $p \supset (q \supset r)$, and vice versa.

Formal Proof of Validity – The deduction of the conclusion of an argument from its premisses by a sequence of statement forms each of which is either a premiss of the given argument, or follows from the preceding statement form of the sequence by one of the rules of inference.

Instantiation A process in quantification theory of substituting an individual constant for individual variables, thereby converting a propositional function into a proposition.

Material Equivalence (Equiv) a rule of logical inference that permits the transformation of bi-conditionals into a conjunction of conditionals, and into a disjunction of conjunctions and vice versa. Accordingly $p \equiv q$ and $[(p \supset q) \bullet (q \supset p)]$ are materially equivalent and therefore mutually replaceable. Similarly, $p \equiv q$ and $[(p \bullet q) \vee (\sim p \bullet \sim q)$ are materially equivalent and therefore mutually replaceable.

Material Implication (abbreviated ***Impl***.) the name of a replacement rule that permits, subject to the introduction or elimination of a negation sign, the transformation of a conditional into a disjunction and vice-versa. Accordingly $p \supset q$ and $\sim p \vee q$ mutually replaces one another and are therefore logically equivalent.

Modus Ponens (M.P.) One of the ten elementary valid argument forms according to which it the antecedent of a conditional premiss is affirmed, the consequent of the premiss follows, that is, would be validly affirmed. This is expressed by $p \supset q$, p, therefore q.

Modus Tollens (M.T.) One of the ten elementary valid argument forms according to, which if the consequent of a conditional premiss if denied, the antecedent of that premiss would be validly denied. This is expressed by $p \supset q$, $\sim q$ therefore $\sim p$.

Replacement, Rule of A rule which permits logically equivalent expressions to replaces each other wherever they occur. Ten such rules that evince logical equivalence forms part of the rules of inference.

Rules of Inference – The rules that permit valid inferences from statements assumed as premisses. In this Self Study Material, we have thirty rules of inferences: ten elementary valid argument forms, ten rules of replacement, that is, logically equivalent expressions that are mutually replaceable, the rule of conditional proof, the rule of indirect proof, four rules governing instantiation and generalization in quantified logic, and four rules of Quantifier negation.

Simplification (Simp.) - One of the ten elementary valid argument forms which as a rule of inference permits the separation of conjoined statement forms. If the conjunction of p and q is given, the rule permits the inference that p. This is expressed as $p \bullet q$, therefore p.

Tautology (abbreviated **Taut**) is the name of an expression of logical equivalence, a rule of inference that permits the mutual replacement of p by (p v p), and the mutual replacement of p by (p • p).

Universal Instantiation (U.I.) A rule of inference in quantificational logic that permits the valid inference of any substitution instance of a propositional function from the universal quantification of that propositional function.

Universal Generalization (U.G.) A rule inference in quantificational logic that permits the valid inference of a generalized, or universally quantified expression from an expression that is given as true of any arbitrarily selected individual.

1.7 Check your Progress

Exercise 1.7

(A) Construct a formal proof of validity for each of the following arguments.

(i) $(x)(Ax \supset \sim Bx)$
$(\exists x)(Cx \bullet Ax)$
$\therefore (\exists x)(Cx \bullet \sim Bx)$

(ii) $(x)(Dx \supset \sim Ex)$
$(x)(Fx \supset Ex)$
$\therefore (x)(Fx \supset \sim Dx)$

(iii) $(x)(Gx \supset Hx)$
$(x)(Ix \supset \sim Hx)$
$\therefore (x)(Ix \supset \sim Gx)$

(iv) $(\exists x)(Jx \bullet Kx)$
$(x)(Jx \supset Lx)$
$\therefore (\exists x)(Lx \bullet Kx)$

(v) $(x)(Mx \supset Nx)$
$(\exists x)(Mx \bullet Ox)$
$\therefore (\exists x)(Ox \bullet Nx)$

(vi) $(\exists x)(Px \bullet \sim Qx)$
$(x)(Px \supset Rx)$
$\therefore (\exists x)(Rx \bullet \sim Qx)$

(vii) $(x)(Sx \supset \sim Tx)$
$(\exists x)(Sx \bullet Ux)$
$\therefore (\exists x)(Ux \bullet \sim Tx)$

(viii) $(x)(Vx \supset Wx)$
$(x)(Wx \supset \sim Xx)$
$\therefore (x)(Xx \supset \sim Vx)$

(ix) $(\exists x)(Yx \bullet Zx)$
$(x)(Zx \supset Ax)$
$\therefore (\exists x)(Ax \bullet Yx)$

(x) $(x)(Bx \supset \sim Cx)$
$(\exists x)(Cx \bullet Dx)$
$\therefore (\exists x)(Dx \bullet \sim Bx)$

(xi) $(x)(Fx \supset Gx)$
$(\exists x)(Fx \bullet \sim Gx)$
$\therefore (\exists x)(Gx \bullet \sim Fx)$

(B) Construct a formal proof of validity for each of the following arguments, in each case using the suggested notations.

(i) No athletes are bookworms. Carol is a bookworm. Therefore Carol is not an athlete. *(Ax, Bx c)*
(ii) All dancers are exuberant. Some fencers are not exuberant. Therefore some fencers are not dancers. *(Dx, Ex, Fx)*
(iii) No gamblers are happy. Some idealists are happy. Therefore some idealists are not gamblers. *(Gx, Hx, Ix)*
(iv) All jesters are knaves. No knaves are lucky. Therefore no jesters are lucky. *(Jx, Kx Lx)*
(v) All mountaineers are neighborly. Some outlaws are mountaineers. Therefore some outlaws are neighborly. *(Mx, Nx, Ox)*
(vi) Only pacifists are Quakers. There are religious Quakers. Therefore pacifists are sometimes religious. *(Px, Qx, Rx)*
(vii) To be a swindler is to be a thief. None but the underprivileged are thieves. Therefore swindlers are always underprivileged. *(Sx, Tx, Ux)*
(viii) No violinists are not wealthy. There are no wealthy xylophonists. Therefore violinists are never xylophonists. *(Vx, Wx, Xx)*
(ix) None but the brave deserve the fair. Only soldiers are brave. Therefore the fair are deserved only by soldiers. (Dx: x deserves the fair; Bx: x is brave; Sx: x is a soldier)
(x) Everyone that asketh receiveth. Simon receiveth not. Therefore Simon asketh not. *(Ax Rx, Sx)*

1.8 References/Further Reading

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Unit 2 Conditional Proof

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2.1 Introduction

This study unit introduces the learner to a method of formal proof of validity or technique called Conditional Proof. Generally, the conditional proof is a device for shortening and simplifying proof. However, in this Unit, we treat conditional proof as an additional rule, having, as it were, the same status with the Elementary Valid Argument forms and the Rules of Replacement. This latter position adopted in this unit does not, however mean that this method and the rules are akin, it differs, indeed from the rules in several important respects. It allows us to draw conclusions based on the assumption that a certain condition is true.

An important advantage of the conditional proof method is that it allows us to draw conclusions based on limited information. By assuming a certain situation to be true, we can draw conclusions that may not be immediately obvious based on the available information. This can be especially useful in situations where we do not have complete information or where we are trying to prove a statement that seems counterintuitive.

2.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. understand a new technique used to test the validity of an argument, particularly those that the conclusion is conditional.
2. construct formal proof of validity for arguments using the Conditional Proof method
3. with much easy show that a proposition's truth follow from another proposition than to prove it independently.

2.3 The Method of Conditional Proof

The method of Conditional Proof (C.P.) is different in kind from the rules of inference or replacements. There are a certain types of arguments, which cannot be tested with any of the rules discussed in the previous unit without further support. Further, C.P is not a system of proof, which does away with the twenty rules. It rather increases the rules to twenty-one. As a rule, the method of Conditional Proof is compulsorily used in testing the validity when the conclusion is conditional. This rule is characteristic of C.P in the sense that nowhere else it is used. Hence, this rule can be designated as the rule of C.P.

2.3.1 Conditional Proof in Proof of Validity

The rule of conditional proof allows us to deduce a conditional proposition from a conjunction of premisses by a sequence of elementary valid arguments, which satisfy the relevant rules of inference. That is, all premisses in C.P. are be supported by rules of inference. The additional premiss, which is a characteristic mark of C.P., is always the antecedent of the conclusion and the construction of proof always begins with antecedent of the conclusion as the premiss. This premiss itself is called C.P.

Simply presented, the Rule of conditional proof could be stated as follows: at any point in a proof, any proposition ***P*** may be put down as a line of the proof with the justification *"Assumed Conditional Proof" (A.C.P.)*, provided that a mark is made against the number of that line.

Following Irving Copi in *Symbolic Logic,* we, in this unit, shall adopt the use of a bent arrow with its head pointing at the assumption from the left, its shaft bent down to run along all lines within the scope of the assumption, and then bent inward to mark the end of the scope of that assumption. (For the starring method, see Purtill, Richard 1971:61 – 63). An assumption would be discharged after our arrow bends inward, and, in this respect, if an assumption is a proposition p and the last proposition where the arrow bends inwards is q we would write immediately under the arrow $p \supset q$. We then cite all lines lying from the assumption to the line that discharges it as being justified by the Rule of Conditional Proof (RCP).

This is illustrated below as follows:

	$p \supset q$	
1	→ P	
2		
3		
4		
5	q	
6	$p \supset q$	1-5 RCP

In this method of proof, no arrowed line would be the conclusion of an argument and no arrowed line would be cited after its assumption is discharged. The justification for this rule is as follows: if by assuming a proposition p we can prove together with certain premisses, another proposition q, then we have shown that, if those premisses are true, then if p is true, then q is true. The upshot of this is that the chief use of the Rule of Conditional Proof (R.C.P.) is with respect to conditional statements; and here our proof is conducted by assuming as an extra premiss the antecedent of our conclusion and then work towards deriving the consequent of the conclusion. That is, if we have in our conclusion $(A \bullet B) \supset C$, after setting out the premisses and the conclusion we proceed as follows:

A • B

C

(A • B) ⊃C

Nevertheless, we also use the rule to establish conclusions of other forms of arguments, that is, conclusions that are not expressed in conditional form. However, when we have such argument forms it is required that first we convert such expressions to conditional propositions and then proceed with the proof; and after working out the conclusion, we convert them back to their original presentation. For example, if we have an argument with its conclusion as ~p v q we begin by first transforming it into an implication, then prove the validity and transform it once more back into its original disjunctive expression, as illustrated below

~p v q
I I I
p ⊃ q by Implication

p

q

p ⊃ q

~ p v q Implication

Similarly if we have p v q, it first transforms to ~ ~p v q by Double Negation, then to ~ p ⊃ q by Implication. A conclusion of the form p ≡ q is proved by using two separate Conditional Proofs to establish p ⊃ q and q ⊃ p and thus conjoin them applying Material Equivalence that is,

p ≡ q
III
(p ⊃ q) • (q ⊃ p)

p

q

p ⊃ q

q

p

q ⊃ p

(p ⊃ q) • (q ⊃ P) Conj.

p ≡ q Equiv.

A conclusion of the form $\sim(\sim p \bullet \sim q)$ can be established by proving $\sim p \supset q$. Applying De Morgan $\sim(\sim p \bullet \sim q)$ transforms first into $p \vee q$ and applying Material Implication into $\sim p \supset q$, that is,

$\sim(\sim p \bullet \sim q)$

III

$p \vee q$

III

$\sim p \supset q$

$\sim p$

q

$\sim p \supset q$

III

$p \vee q$ Impl.

III

$\sim(\sim p \bullet \sim q)$ De Morgan

Further, a conclusion either of the form p or ~p can be established first by addition. In any of the cases, if we have $p \vee p$ or $\sim p \vee \sim p$, applying Implication this transforms to $\sim p \supset p$ or $p \supset \sim p$. Then $\sim p \supset p$ transforms into $p \vee p$ which applying Tautology yields p. In the same vein, $p \supset \sim p$ transforms into $\sim p \vee \sim p$ which also applying Tautology yields ~p, that is,

p

III

$p \vee p$ Add

III

$\sim\sim p \vee p$ DN

III

$\sim p \supset p$ Impl.

$\sim p$

p

$\sim p \supset p$

III

$\sim\sim p \vee p$ Impl.

III

$p \vee p$ DN

III

p Taut.

With these set down procedures we shall take samples of argument forms and applying the *Rule of Conditional Proof* (RCP) prove their validity.

(1)

1.	C ⊃ (N v A)		
2.	N ⊃ S		
3.	A ⊃ S		
4.	S ⊃ ~D /∴ D ⊃ ~C		
5.	D	A C P	
6.	~~ D	5 DN	
7.	~ S	4,6 MT	
8.	~ A	3,7 MT	
9.	~ N	2,7 MT	
10.	~N • ~A	9,8 Conj	
11.	~(N v A)	10 DeM	
12	~ C	1,11	
13.	D ⊃ ~ C	5 – 12	R C P

In applying the Rule of Conditional Proof above, we set out the argument in lines 1 – 4 and used the stroke "/" to indicate and separate the conclusion from the premisses. In line 5 we assumed the antecedent of the conclusion as an added and new premiss justifying this by writing *Assumed Conditional Premiss* (abbreviated as A C P). Applying the 20 rules of inference in lines 6-12 we worked towards and deduced the consequent of the conditional conclusion. On deducing or arriving at the consequent the assumption was discharged; the bent arrow with its head pointing at the assumption from the left bent inward to mark the end of the scope of the assumption. Line 13 cites all the lines lying from the assumption, that is line 5 to the line that discharges it, that is, the line indicating, the consequent of the condition, which is line 12, that is lines 5 – 12 as being justified by the Rule of Conditional Proof.

One significance of the introduction of R C P comes out in high relief here; it makes proofs simpler, which would otherwise be complicated if conducted with the 20 rules of inference. The above argument was proved using the 20 rules but the deduction procedure was more exerting and complicated.

(2)	1.	~A v ~B	
	2.	B • D	
	3.	~A ⊃ [~(C • D) v F] / ∴ C ⊃ F	
	4.	C	A C P
	5.	B	2 Simp
	6.	~B v ~A	1 Comm
	7.	~~ B	5 DN
	8.	~ A	6,7 DS
	9.	~(C • D) v F	3,8 MP
	10.	D • B	2 Comm
	11.	D	10 Simp
	12.	C • D	4,11 Conj.
	13.	~ ~ (C • D)	12 DN
	14.	F	9, 13 DS
	15.	C ⊃ F	4 – 14 R C P

Conditional proof method could also be applied to arguments whose part(s) of the conclusion is/are not in conditional form provided the major operator is the "⊃", that is, it is fundamentally a conditional. For example

(3) 1.~ (~A v B) v ~(C v D) /∴ D ⊃ ~ (~A • ~E)

2.	D	A C P
3.	D v C	2 Add
4.	C v D	3 Comm
5.	(~A v B) ⊃ ~(C v D)	1 Impl.
6.	~ ~(C v D)	4 DN
7.	~ (~A v B)	5, 6 MT
8.	A • ~ B	7 DeM
9.	A	8 Simp
10.	A v E	9 Add
11.	~ (~A • ~E)	10 DeM
12.	D ⊃ ~ (~A • ~E)	2-11 RCP

It is to be noted further that argument 3 has in its conclusion a variable or constant that has no occurrence in the premisses; this is E. But using the rule of addition we were enabled to introduce it in line 10.

It is also pertinent to remark that the assumed new premiss, that is, the antecedent of the conclusion need not appear, even be utilized, in the proof unless it is necessary and germane in proving our argument. For example, in the proof below, the assumed premiss does not appear in the premisses and is not utilized in proving the consequent:

(4) 1. X ⊃ ~(X v A)

2. (X • R) v S /∴ Z ⊃ S

3.	Z	A C P
4.	S v (X • R)	2 Comm
5.	(S v X) • (S v R)	4 Dist.
6.	S v X	5 Simp
7.	~X v ~(X v A)	1 Impl
8.	~X v (~X • ~A)	7 DeM
9.	(~X v ~X) • (~X v ~A)	8 Dist
10.	~X v ~X	9 Simp
11.	~ X	10 Taut
12.	X v S	6 Comm
13.	S	12,11 DS
14.	Z ⊃ S	3-13RCP

R C P as already highlighted is also used to establish the validity of arguments that are not expressed in conditional form. This is illustrated by the following examples:

(5) 1. (A • B) v ~(C ⊃ D)

2. (C ⊃ D) /∴ (A • B) v (Q ≡ V)

III

~ ~(A • B) v (Q ≡ V)

III

~(A • B) ⊃ (Q ≡ V)

3.	~ (A • B)	A C P
4.	~(C ⊃ D) v (A • B)	1 Comm
5.	~ ~ (C ⊃ D)	2 DN
6.	A • B	4,5 DS
7.	~ (A • B) v (Q ≡ V)	3 Add
8.	~ ~ (A • B)	6 DN
9.	Q ≡ V	7,8 DS
10.	~ (A • B) ⊃ (Q ≡ V)	3-9 RCP
11.	~ ~ (A • B) v (Q ≡ V)	10 Impl.
12.	(A • B) v (Q ≡ V)	11 DN

In the above proof our conclusion is a disjunction, we thus had to transform into a conditional by applying the rule of Double Negation and Implication. Thereafter we assumed the antecedent of the transformed conclusion and worked towards the consequent; at getting at the consequent in line 9 we applied RCP to 3-10 and then proceeded to apply Implication and Double Negation to arrive at the original conclusion.

6) 1. [W ⊃ (~X • ~Y)] • [Z ⊃ ~(X v Y)

2. (~A ⊃ W) • (~B ⊃ Z)

3. (A ⊃ X) • (B ⊃ Y) / ∴ X ≡ Y

III

(X ⊃ Y) • (Y ⊃ X)

III

X ⊃ Y

Y ⊃ X

4.	X	A C P
5.	X v Y	4 Add
6.	[Z ⊃ ~(X v Y] • [W ⊃ (~X • ~Y)]	1 Comm
7.	Z ⊃ ~ (X v Y)	6 Simp
8.	~ Z	7,5 MT
9.	(~B ⊃ Z) • (~A ⊃ W)	2 Comm
10.	~B ⊃ Z	9 Simp.
11.	~ ~ B	10,8 MT
12.	B	11 DN
13.	(B ⊃ Y) • (A ⊃ X)	3 Comm
14.	B ⊃ Y	13 Simp
15.	Y	14,12 MP
16.	X ⊃ Y	4-15 RCP
	17. Y	ACP
18.	Y v X	17 Add
19.	X v Y	18 Comm
20.	~(~X • ~Y)	19 DeM
21.	W ⊃ (~X • ~Y)	1 Simp.
22.	~ W	21,20MT
23.	~A ⊃ W	2 Simp.
24.	~ ~ A	23,22MT
25.	A	24 DN
26.	A ⊃ X	3 Simp.
27.	X	26,25MP
28.	Y ⊃ X	17-27RCP
29.	(X⊃Y)•□(Y⊃X)	16,28Conj
30.	X ≡ Y	29Equiv

In proving the validity of the above argument, applying R C P, we began by applying the rule of Equivalence to $X \equiv Y$ which is the conclusion of the argument, this yielded $(X \supset Y) \bullet (Y \supset X)$; then we applied Simplification to separate $(X \supset Y) \bullet (Y \supset X)$. We then worked out $X \supset Y$ in lines 4-15 and $Y \supset X$ in lines 17 – 27, we then applied the rule of Conjunction (Conj) to lines 16 and 28 to derive $(X \supset Y) \bullet (Y \supset X)$ in line 29 and applied Material Equivalence to line 29 to derive $X \equiv Y$ in line 30.

2.3.2 The Strengthened rule of Conditional Proof

In this method of Conditional Proof, the construction of proof does not necessarily assume the antecedent of the conclusion. The structure of this method is that an assumption is made initially. There is no need to know the truth-status of the assumption because an assumption may be false, but the conclusion can still be true. Further, the assumption can be any component of any premiss or conclusion. The method is called the strengthened rule because we enjoy more freedom in making assumption or assumptions, which means that plurality of assumptions is allowed. It strengthens our repertoire of testing equipment. Another feature of this method is the limit of assumption. The last step is always outside the limits of assumption. If there are two or more than two assumptions in an argument, then there will be a distinct last step with respect to each assumption. This last step can be regarded as the conclusion relative to that particular assumption. It shows that the last step is deduced with the help of assumption in conjunction with the previous steps in such a way that the rules of inference permit such conjunction. Before the conclusion is reached, the function of assumption also ceases. Then it will have no role to play. Then, automatically, the assumption is said to have been discharged.

In applying this strengthened rule, we would have two or more arrowed lines, that is, assumptions and discharge lines. The procedure in this connection is that after one assumption of limited scope has been discharged, another such assumption may be made and then discharged. Alternatively, a second assumption of limited scope may be written within the scope of the first. What this comes to is that scopes of different assumptions may follow each other, or one scope may be contained entirely within another. If the scope of an assumption does not extend all the way to the end of a proof, then the final line of the proof does not depend on that assumption, but has been proved to follow from the original premisses alone. The upshot is that we need not restrict ourselves to using as assumptions only the antecedents of conditional conclusions. Any proposition can thus be taken as an assumption of limited scope, for the final line, that is, the conclusion will always be beyond its scope and independent of it. This could be illustrated as follows:

$p \supset q$
p A. C P.
r A. C. P.
s
$r \supset s$ R. C. P.
q
$p \supset q$ R. C. P.

$p \supset [(q \bullet r) \supset s]$

p

$(q \bullet r)$

s

$(q \bullet r) \supset s$

$p \supset [(q \bullet r) \supset s]$

To demonstrate the application of conditional proofs within conditional proofs, we shall work out some examples as in below:

(1)	1.	$A \vee (B \supset D)$	
	2.	$[B \supset (B \bullet D)] \supset (F \vee G)$	
	3.	$(F \supset A) \bullet (G \supset H)$ /∴ $A \vee H$	
			III
			$\sim\sim A \vee H$
			$\sim A \supset H$
	4.	$\sim A$	A.C.P.
	5.	$B \supset D$	1,4 DS
	6.	B	A.C.P.
	7.	D	5,6 M.P
	8.	$B \bullet D$	6,7 Conj
	9.	$B \supset (B \bullet D)$	6 – 8 R.C.P.
	10.	$F \vee G$	2, 9 MP
	11.	$A \vee H$	3, 10 CD
	12.	$\sim \sim A \vee H$	11 DN
	13.	$\sim A \supset H$	12 Impl.
	14.	H	13,4 MP
	15.	$\sim A \supset H$	4 – 14 RCP
	16.	$\sim \sim A \vee H$	15 Impl
	17.	$A \vee H$	DN

(2)	1.	$A \supset (B \supset C)$
	2.	$(D \bullet C) \supset E$
	3.	$F \supset (D \bullet \sim E)$ /∴ $\sim A \vee (\sim B \vee \sim F)$

4.	E	A C P
5.	B ⊃ W	1,4 MP
6.	B	A C P
7.	W	5,6 MP
8.	W ⊃ (G ⊃ S)	2 Exp.
9.	G ⊃ S	7,8 MP
10.	~G v S	9 Impl
11.	~ (G • ~S)	10 DeM
12.	~ U	3,11 MT
13.	B ⊃ ~ U	6 – 12 RCP
14.	E ⊃ (B ⊃ ~ U)	4 – 13 RCP

In proving argument No. 1 it is clear that the second assumption is contained entirely within the first and that the final line, that is the conclusion is beyond its scope and somewhat independent of it. Infact the second assumption is not in any way part of the conclusion. This underscores the point that RCP allows us to introduce any proposition whatsoever, that is, assume any premiss in proving the validity of an argument.

In proving argument No.2, lines 4 – 13 lie within the scope of the first assumption, while lines 6-12 lie within the scope of the second assumption. The point on relief here is that the scope of an assumption p in a proof consists of all lines p through q, where the line following q is of the form p ⊃ q and is inferred by R.C.P. from that sequence of lines. Accordingly, the second assumption in argument 9 above lies within the scope of the first because it lies between the first assumption and line 14, which is derived by RCP from the sequence of lines 4 through 13.

2. 3.3. Conditional Proof of Validity in Quantificational Logic

The Rule of Conditional Proof is also applicable in Quantificational Logic. Like in the above, the procedure is to begin by assuming the antecedent of the conclusion of the argument that we want to prove as an extra premiss and then work to derive the consequent. Once we deduce the consequent of the conclusion, then we deduce the entire conclusion by conditional proof. The distinguish feature of applying CP in quantificational logic is that the antecedent of the conclusion is a statement function, not a complete statement. Thus, only the statement function is assumed as the first line in the conditional sequence**.** The quantifier is added after the sequence is discharged. The following are examples of the application of conditional proof to Quantificational theory.

(1)	1.	(∀x) [Ax ⊃ Bx]	
	2.	(∀x) [Cx ⊃ AX] / ((∀x) [Cx ⊃ Bx]	
	3.	Cy	A CP
	4.	Ay ⊃ By	1 UI
	5.	Cy ⊃ Ay	2 UI
	6.	Ay	5,3 MP
	7.	By	4,6 MP
	8.	Cy ⊃ By	3 – 7 RCP
	9.	(∀x) Cx ⊃ Bx	8 UG

We started our proof above by assuming Cy which is the instantiation mode of the antecedent of our conclusion; thereafter we applied UI to lines 1 and 2 and then worked towards deriving the consequent. After deriving By we discharged our arrow and concluded $Cy \supset By$; and finally applied UG to arrive at our conclusion in its quantified form.

(2) 1. $(\forall x)Mx \supset Rx]$
2. $(\forall x)\ [Rx \supset Hx]$
3. $[(\exists x)\ (Mx \bullet Bx)] \bullet \{(\forall x)\ [Hx \supset (Dx \supset Px)]\}$

$\therefore /(\ (\exists x) \sim(Dx \bullet \sim Px)$

III

$\sim Dx \vee Px$

III

$Dx \supset Px$

4.	Da	ACP
5.	$(\exists x)\ (Mx \bullet Bx)$	3 Simp
6.	$(\forall x)\ [Hx \supset (Dx \supset Px)] \bullet [(\exists x)\ (Mx \bullet Bx)\ ]$	3 Comm
7.	$(\forall x)\ [Hx \supset \{Dx \supset Px)]$	6 Simp
8.	$Ma \bullet Ba$	5 EI
9.	$Ma \supset Ra$	I UI
10.	$Ra \supset Ha$	2 UI
11.	$Ha \supset (Da \supset Pa)$	7 UI
12.	$Ma \supset Ha$	9,10 HS
13.	Ma	8 Simp
14.	Ha	12,13 MP
15.	$(Ha \bullet Da) \supset Pa$	11 Exp.
16.	$Ha \bullet Da$	14,7 Conj
17.	Pa	15,16 MP
18.	$Da \supset Pa$	4-17 RCP
19.	$\sim Da \vee Pa$	18 Impl.
20.	$\sim(Da \bullet \sim Pa)$	19 DeM
21.	$(\exists x)\ \sim(Dx \bullet \sim Px)$	20 UG

We can also apply the strengthened Conditional Proof in arguments involving quantifiers, as indeed we had in propositional logic. This is exemplified as in

(3) 1. $(\forall x)\ [\sim(Kx \bullet Lx) \supset \sim(Mx \vee Nx)]$
2. $(\forall x)\ \{\sim[Px \vee Qx) \supset Rx]\ (\ \sim(Kx \vee Sx)\}$
$\therefore /\ (\forall x)\ [Mx \supset (Px \supset Rx)]$

3.	My	ACP
4.	~(Ky • Ly) ⊃ ~(My v Ny)	1 UI
5.	~[(Py v Qy) ⊃ Ry] ⊃ ~(Ky v Sy)	2 UI
6.	My v Ny	3 Add
7.	~~(My v Ny) ⊃ ~~(Ky • Ly)	4 Trans
8.	(My v Ny) ⊃ (Ky • Ly)	7 DN
9.	Ky • Ly	8, 6 MP
10.	Ky	9 Simp
11.	Ky v Sy	10 Add
12.	~~(Ky v Sy)	11 DN
13.	~~[(Py v Qy) ⊃ Ry]	5,12 MT
14.	(Py v Qy) ⊃ Ry	13 DN
15.	Py	ACP
16.	Py v Qy	15 Add
17.	Ry	14,16 MP
18.	Py ⊃ Ry	15–17 RCP
19.	My ⊃ (Py ⊃ Ry)	3-18 RCP
20.	(∀x) [Mx ⊃ (Px ⊃ Rx)]	19 UG

2.4 Conclusion

The Conditional Proof is an additional rule that allows us to draw conclusions based on the assumption that a certain condition is true. In procedure, we begin using the rule by introducing an assumption, which is the antecedent of the conditional we wish to derive. We then proceed "normally", that is, using our rules of inference and equivalence rules, until we derive the consequent of the desired conditional. Finally, we invoke CP to infer the conditional. The goal is to demonstrate that if the ACP [Assumed Conditional Proof] were true, then the desired conclusion necessarily follows. The validity of a conditional proof does not require that the ACP [Assumed Conditional Proof] to be true, only that if it were true it would lead to the consequent."

The conditional proof will often simplify a proof, especially one that has a conditional in the conclusion, making the proof shorter or easier to solve. It is also notable that we use the rule to establish conclusions of other forms of arguments, that is, conclusions not expressed in conditional form. However, when we have such argument forms it is required that first we convert such expressions to conditional propositions and then proceed with the proof; and after working out the conclusion, we convert them back to their original presentation. These procedures are made possible because the operators on propositions are inter-definable.

2.5 Summary

Conditional proofs exist to connect several conjectures that are otherwise unproven, so that the proof of one conjecture may immediately imply the validity of several others. It is much easier to prove the truth of a proposition by deriving it from another proposition than by independently proving it.

This unit strengthens our repertoire of testing the validity of arguments by allowing us to deduce a conditional proposition from a conjunction of premisses by a sequence of elementary valid arguments, which satisfy the relevant rules of inference.

An important advantage of the conditional proof method is that it allows us to draw conclusions based on limited information. By assuming a certain situation to be true, we can draw conclusions that may not be immediately obvious based on the available information. The method also makes proofs simpler, which would otherwise be complicated if conducted with the 20 rules of inference.

2.6 Glossary

Conditional Proof – A method of proving the validity of an argument by assuming at any point in a proof a proposition ***"p"*** and proving together with certain premisses another proposition ***q***; the point is that if those premisses are true then if ***p*** is true, then ***q*** is true.

2.7 Check your Progress

Exercises

(1) Using the Conditional Proof Method, prove the validity of the following arguments.

(1)
1. $(J \bullet K) \supset L$
2. $M \supset J$
3. $\sim K \supset \sim N$ / $\therefore$ $(M \bullet N) \supset L$

(2)
1. $U \vee \sim V$
2. $W \equiv V$
3. $\sim W \supset \sim X$ / $\therefore$ $\sim X \vee V$

(3)
1. $(K \vee L) \supset [(M \vee N) \supset B]$ / $\therefore$ $K \supset (M \bullet N)$

(4)
1. $P \supset (D \bullet \sim H)$
2. $(F \bullet S) \supset B$
3. $(B \bullet D) \supset \sim H$ / $\therefore$ $F \supset (S \supset \sim P)$

(5)
1. $M \vee \sim R$
2. $(K \supset G) \bullet (M \supset S)$
3. $S \supset K$ / $\therefore$ $\sim (R \bullet \sim G)$

(6)
1. $(K \supset L) \bullet (M \supset N)$
2. $(L \vee N) \supset \{[O \supset (O \vee P)] \supset (K \bullet M)\}$

/ $\therefore K \equiv M$

(7)
1. $A \supset (B \vee C)$
2. $E \supset (C \vee P)$
3. $\sim C$ / $\therefore$ $\sim (B \vee P) \supset \sim (A \vee E)$

(8)
1. $(T \supset E) \bullet (A \supset L)$ / $\therefore$ $(T \vee A) \supset (E \vee L)$

(9)
1. $A \supset B$
2. $B \supset [(C \supset \sim \sim C) \supset D]$ / $\therefore$ $A \supset D$

(10)
1. $A \equiv B$
2. $(Z \bullet X) \vee (Q \supset R)$ / $\therefore$ $(A \equiv B) \bullet [(Z \bullet X) \vee N]$

(ii) Construct a formal proof of validity for each of the following arguments, using the Rule of Conditional wherever deemed applicable.

(1.) 1. $(\forall x)\ [\sim Bx \supset \sim Ax]$
2. $(\forall x)\ [(Ax \bullet Bx) \supset Dx]$ / $\therefore$ $(\forall x)\ [Ax \supset Dx]$

(2) 1. $(\forall x)\ [Rx \supset Sx]$
2. $\sim Tb \bullet \sim Sb$ / $\therefore$ $\sim Rb$

(3) 1. $(\forall x)\ [Zx \supset Xx]$
2. $(\exists x)\ [Zx \bullet Xx]$ / $\therefore$ $Xx \vee Ux$

(4) 1. $(\forall x)\ [Dx \supset \sim Ex]$
2. $(\exists x)\ [Kx \bullet Dx]$ / $\therefore$ $(\exists x)\ [Da \bullet Fa]$

(5) 1. $(\forall x)\ [(Bx \vee Gx) \supset Fx]$
2. $(\forall x)\ [(Fx \vee Vx) \supset Nx]$ / $\therefore$ $(\exists x) \sim [Bx \bullet \sim Nx]$

(6) 1. $(\exists x) \sim[Cx \bullet \sim(Fx \vee Gx)]$
2. $(\forall x)\ [Fx \supset Rx]$
3. $\sim(\forall x)\ [Cx \supset Rx]$ / $\therefore$ $(\exists x)\ [Cx \bullet Kx]$

(7) 1. $(\forall x)\ [Hx \supset \sim Jx]$
2. $(\forall x)\ [\sim Hx \supset Ax]$ / $\therefore$ $(\forall x)\ [\sim Ax \supset \sim Jx]$

(8) 1. $[(\forall x)\ (\sim Ex \supset \sim Dx)] \bullet [(\exists x)\ (Bx \bullet Rx)]$
2. $(\forall x)\ (\sim Dx \supset \sim Rx]$ / $\therefore$ $(\exists x)\ [\sim Ex \bullet \sim Hx]$

(9) 1. $(\forall x)\ [(Gx \bullet Hx) \supset (Ix \bullet Jx)]$
2. $(\forall x)\ [(Gx \bullet Kx) \supset Hx]$ / $\therefore$ $(\forall x)\ [(Gx \bullet Kx) \supset Ix]$

(10) 1. $(\forall x)\ [Ax \supset (Sx \equiv Wx)]$
2. $(\exists x)\ [(Ax \bullet (Sx \bullet \sim Wx)]$
3. $(\forall x)\ [Ax \supset Ix]$ / $\therefore$ $(\forall x)\ [Ix \supset Ax]$

2.8 References/Further Reading

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Unit 3 Indirect Proof

Contents

3.1 Introduction

This study unit introduces the learner to a method of formal proof of validity or technique called Indirect Proof. It is a device for shortening and simplifying proofs, whereby we prove a conclusion by showing its negation to be self-contradictory. Like we did with Conditional Proof, we treat Indirect Proof, in this unit, as an additional rule, having, as it were, the same status with the rules of inference. Similarly also this not mean that Indirect Proof and the rules are akin, it differs, indeed from the rules of inference in several important respects. It allows us to draw conclusions based on the assumption that a certain condition is true.

3.2 Intended Learning Outcomes

It is expected that at the end of this unit, you will be able to:

1. understand a new technique used to test the validity of an argument by reasoning in reverse direction.
2. construct formal proof of validity for arguments using the Indirect Proof method
3. with much easy show that a proposition's truth follow from another proposition than to prove it independently.

3.3 The Method of Indirect Proof

Indirect Proof method is also known as *Reductio Proof* because of its similarity with traditional *Reductio ad absurdum (Raa)* technique, a method very common in the construction of proof of geometrical theorems. This method is characterized, as it were, by a special feature, it involves proving an argument valid by showing that a *counterexample* leads to either an absurdity or to a contradiction. Thus, in order to prove a certain statement, its contradiction is assumed to be true from which the conclusion, which contradicts our assumption, is logically deduced. If A contradicts ~ B, then either A must be false or ~ B must be false. A cannot be false because it is logically deduced from what is purported to be true. Therefore ~B must be false, which means that B must be true. This is how a theorem in geometry or an argument in logic is, sometimes, proved. . In geometry, Euclid in deriving his theorems usually begins by assuming the opposite of what he wants to prove. Rene Descartes in his celebrated *methodic doubt* that culminated in the *Cogito ergo sum* began by denying his existence since the senses are deceptive or at times misleading. He ran into contradiction or absurdity because he realized that for him to think he must exist, hence the famous but, nevertheless, controversial dictum: *Cogito ergo sum "I think, therefore I am".*

3.3.1 Indirect Proof in Formal Proof of Validity

Indirect proof involves the assumption of the contrary (that is denial) of what we want to prove; if our assumption leads to a contradiction or *"reduces to an absurdity"* then that assumption must be false, and so its denial – the conclusion to be proved must be true. The point here is that if an argument is valid and we assume the contrary, that is assume that the argument is invalid by making the premiss(es) true and the conclusion false, then we are bound to deduce a contradiction from our assumption.

The method of conducting Indirect Proof (I. P.) thus involves beginning by assuming as an extra premiss the negation of the conclusion of our argument, and then work towards deriving a contradiction from the premisses of the argument by any sequence of the rules of inference. An outline of this procedure is as follows:

1. Assume as an extra premiss the negation of the conclusion of the argument (this is indicated by writing A.I.P. meaning *Assumed Indirect Premiss* besides the extra premiss).
2. From the Assumed premiss and the other premiss(es) derive a contradiction.
3. The negation of the Assumed Premiss (which is normally the original conclusion of the argument) is written as the final line of the proof. This is justified by the point that if a contradiction is derived from the negated Assumed Premiss, it means that the original conclusion is valid.
4. To indicate the method of Proof applied write I P besides the last line.
5. Use (as in conditional proof) a bent arrow with its head pointing at the A IP from the right, its shaft bent down to run along all lines within the scope of the assumption, and then bent inward to mark the end of the scope of the assumption and proof of I.P. This procedure is illustrated as follows:

1.	~ (A v F)	
2.	(~A • ~N) ⊃ (O v N)	
3.	~N v (E • ~N)] /∴ O v N	
4.	~ (O v N)	AIP
5.	~ O • ~N	4 DeM
6.	~ O	5 Simp
7.	~A • ~F	1 DeM
8.	~ A	7 Simp
9.	~N • ~O	5 Comm
10.	~ N	9 Simp
11.	~A • ~N	8,10 Conj.
12.	O v N	2,11 MP
13.	N	12,6 DS
14.	N • ~ N	13,10 Conj.
15.	O v N	4-14 IP

In argument 1 above we began by assuming as an extra premiss a negation of the conclusion; the conclusion is O v N and our Assumed premiss is ~(O v N) as shown in line 4. We then worked towards deriving a contradiction and this is achieved in line 14 where we have N • ~N, we then concluded our proof using I.P to arrive at our original conclusion O v N.

The point is significant that what is important in the Indirect Proof method is that a contradiction be derived; it is immaterial which of the variables is used to derive this. In fact, it is possible to derive different contradictions from the same argument. This could be exemplified using argument 1 above as follows:

1.	~ (A v F)	
2.	(~A • ~N) ⊃ (O v N)	
3.	~ N v (E • ~N) /∴ O v N	
4.	~ (O v N)	AIP
5.	~ O • ~N	1 DeM
6.	~ N • ~ O	5 Comm
7.	~ A • ~F	1 DeM
8.	~ N	6 Simp
9.	~ A	7 Simp
10.	~A • ~ N	9,8 Conj
11.	O v N	2,10 MP
12.	N v O	11 Comm
13.	O	12,8 DS
14.	~ O	5 Simp
15.	O •~O	13, 14 Conj
16.	O v N	4 – 15 IP

Let us look at the following other arguments using I.P

(2)	1.	Z ⊃ (~A v G)	
	2.	(Z ⊃ A) • [~ E ⊃ (G v Z)] /∴ E v G	
	3.	~ (E v G)	AIP
	4.	~ E • ~G	3 DeM
	5.	~ E	4 Simp
	6.	~ E ⊃ (G v Z) • (Z ⊃ A)	2 Comm
	7.	~E ⊃ (G v Z)	6 Simp
	8.	G v Z	7,5 MP
	9.	~G • ~E	4 Comm
	10.	~ G	9 Simp
	11.	Z	8,10 DS
	12.	~A v G	1,11 MP
	13.	Z ⊃ A	2 Simp
	14.	A	13,11 MP
	15.	~ ~ A	14 DN
	16.	G	12,15 DS
	17.	G • ~ G	16,10 Conj
	18.	E v G	3-17 IP

(3)	1.	(R ⊃ W) ⊃ (X • Y)	
	2.	~(~X • ~Z) ⊃ (~Y v R) /∴ R	
	3.	~R	AIP
	4.	~R v W	3 Add
	5.	R ⊃ W	4 Impl

6.	X • Y	1,5 MP
7.X		6 Simp
8.	X v Z	7 Add
9.	(X v Z) ⊃ (~Y v R)	2 DeM
10.	~Y v R	9,8 MP
11.	Y • X	6 Comm
12.	Y	11 Simp
13.	~ ~ Y	12 DN
14.	R	10,13 DS
15.	R • ~R	14,3 Conj.

In argument 3, the Indirect Proof could be conducted without showing clearly a contradiction; it is obvious that line 14 and 3 are contradictory R•~R. The proof is therefore successful and we may well end our proof at line 14. However, to maintain consistency in our pattern we instead of stopping at line 14 conjoined lines 14 and 3 in line 15 to indicate clearly the contradiction. Nevertheless, the validity of our proof above, if terminated at line 14 would not be vitiated.

More exerting proof is illustrated in argument no. 4 below.

(4)	1.	(E v N) ⊃ [(W v S) ⊃ (~M • P)]	
	2.	(M ⊃ ~R) ⊃ G /∴N ⊃ (~G ⊃ ~S)	
	3.	~ [N ⊃ (~G ⊃ ~S)]	AIP
	4.	~ [~N v (~G ⊃ ~S)]	3 Impl.
	5.	~ [~N v (~ ~G v ~S)]	4 Impl.
	6.	~ [~N v (G v ~S)]	5 DN
	7.	N • ~ (G v ~S)	6 DeM
	8.	N	7 Simp
	9.	~ (G v ~S) • N	7 Comm.
	10.	~ (G v ~S)	9 Simp
	11.	~G • S	10 DeM
	12.	~ G	11 Simp
	13.	S • ~G	11 Comm
	14.	S	13 Simp.
	15.	N v E	8 Add
	16.	E v N	15 Comm
	17.	(W v S) ⊃ (~M • P)	1,16 MP
	18.	S v W	14 Add
	19.	W v S	18 Comm
	20.	~M • P	17,19 MP
	21.	~ M	20 Simp
	22.	~ M v ~R	21 Add
	23.	M ⊃ ~R	22 Impl.
	24.	G	2,22 MP
	25.	G • ~ G	24,12 Conj.
	26.	N ⊃ (~G ⊃ ~S)	3-25 IP

It is important to note in the above proof, the stepwise application of the rule of implication in lines 4 and 5.

(5)	1.	(~Z ⊃ ~A) ⊃ (~F • ~ U) /∴ ~U v (Q v ~Z)	
	2.	~ [~U v (Q v ~Z)]	AIP
	3.	U • ~ (Q v ~Z)	2 DeM
	4.	U	3 Simp
	5.	~ (Q v ~Z) • U	3 Comm
	6.	~ (Q v ~ Z)	5 Simp
	7.	~Q • Z	6 DeM
	8.	Z • ~Q	7 Comm
	9.	Z •	8 Simp
	10.	Z v ~ A	9 Add
	11.	~A v Z	10 Comm
	12.	A ⊃ Z	11 Impl
	13.	(A ⊃ Z) ⊃ (~F • ~ U)	1 Trans
	14.	~F • ~ U	13, 12 MP
	15.	~U • ~ F	14 Comm
	16.	~ U	15 Simp
	17.	U • ~U	4,16 Conj
	18.	~U v (Q v ~Z)	2 – 17 I P

(6)	1.	(~Z v ~ O) • (~Z ⊃ ~E)	
	2.	~O ⊃ ~U /∴ E ⊃ ~ U	
	3.	~ (E ⊃ ~ U)	AIP
	4.	~ (~E v ~ U)	3 Impl
	5.	E • U	4 DeM
	6.	E	5 Simp
	7.	~ ~ E	7 DN
	8.	(~Z ⊃ ~ E) • (~Z v ~ O)	1 Comm
	9.	~Z ⊃ ~ E	8 Simp
	10.	~ ~ Z	9,7 MT
	11.	~Z v ~ O	1 Simp
	12.	~ O	11,10 DS
	13.	Z ⊃ ~ O	11 Impl.
	14.	Z ⊃ ~ U	13,2 HS
	15.	Z	10 DN
	16.	~ U	14,15 MP
	17.	U • E	5 Comm
	18.	U	17 Simp
	19.	U • ~ U	18,16 Conj
	20.	E ⊃ ~ U	3 – 19 IP

3.3.2 Indirect Proof of Validity in Quantificational Logic

We also apply the Rule of Indirect Proof in Quantificational Logic. This, as already elaborated above, involves the assumption of the negation of the conclusion as an extra premiss and then work towards deducing a contradiction from the premisses and the assumed extra premiss, which is the negated conclusion. In this respect if have the argument:

(1) 1. $(\forall x)\ [Px \supset (Qx \supset Rx)[$
2. $(\forall x)\ (Px \supset Qx)$
3. $(\forall x)\ [\sim Sx \supset (Rx \vee Px)]\ \therefore /\ (\ (\exists x)\ \sim[\sim Sx \supset \sim Rx]$

our first step would be to negate the conclusion which would yield $\sim (\exists x)\ \sim[S\ (\sim Rx]$ as the assumed indirect premiss.

It is nevertheless possible to conduct a formal proof of validity for this argument, but the introduction of a rule called the Rule of Quantifier Negation (QN) also called Exchanging Quantifying Expression (EQ) would be very useful hence facilitate our capacity to demonstrate the validity of arguments when applying the Indirect Proof Method

3.3.2.1 The Rule of Quantifier Negation (QN)

The rule of quantifier Negation has to do with the relationship between Universal and Existential Quantifiers. It involves the introduction of the negation symbol ~ to the Quantifiers of Quantificational Logic. We had in a previous unit developed four Equivalence rules by introducing the negation symbol ~ to the quantifiers, and it is these four equivalence rules of relationship between the Existential and Universal quantifiers as set again below that is called the rule of Quantifier Negation:

1. $[(\forall x)\ \psi x] \equiv [\sim(\exists x)\ \sim\psi x]$
2. $[(\exists x)\ \psi x] \equiv [\sim(\forall x)\ \sim\psi x]$ where ψ stands for any
3. $[(\forall x)\ \sim\psi x] \equiv [\sim(\exists x)\ \psi x]$ predicate whatsoever
4. $[(\exists x)\ \sim\psi x] \equiv [\sim(\forall x)\ \psi x]$

Being equivalence rules, the implication is that any of the above logically equivalent expressions may replace each other where they occur. In this sense $(\forall x)\ \psi x$ may replace $\sim(\exists x)\ \sim\psi x$ whenever it occurs and vice versa; this applies to rules 2 – 4 above also.

Let us now illustrate the application of QN in Indirect Proof Method. Our argument would thus be set out as follows:

1.	$(\forall x)\ [Px \supset (Qx \supset Rx)]$	
2.	$(\forall x)\ [Px \supset Qx]$	
3.	$(\forall x)\ [\sim Sx \supset (Rx \vee Px)]\ \therefore\ (\ \sim(\exists x)\ \sim[\sim Sx \bullet \sim Rx]$	
4.	$\sim(\exists x)\ \sim[\sim Sx \bullet \sim Rx$	AIP
5.	$(\forall x)\ [\sim Sx \bullet \sim Rx]$	4 QN
6.	$\sim Sa \bullet \sim Ra$	5 UI
7.	$Pa \supset (Qa \supset Ra)$	1 UI
8.	$Pa \supset Qa$	2 UI
9.	$\sim Sa \supset (Ra \vee Pa)$	3 UI
10.	$\sim Sa$	6 Simp
11.	$Ra \vee Pa$	9,10 MP
12.	$\sim Ra \bullet Sa$	6 Comm
13.	$\sim Ra$	12 Simp
14.	Pa	11,13 DS
15.	Qa	8,14 MP
16.	$Qa \supset Ra$	7,14 MP
17.	Ra	16,15 MP
18.	$Ra \bullet \sim Ra$	17,13 Conj
19.	$\sim(\exists x)\ \sim(\sim Sx \bullet \sim Rx)$	5 – 18 IP

(2) 1.	(∀x) (Ex ⊃ Fx)	
2.	(∀x) (Gx ⊃ Hx)	
3.	[(∀x) (Fx ⊃ Ix)] • [(∃x) (Ex • Gx)]	
4.	(∀x) (Hx ⊃ Jx) /∴ (∃x) (Ix • J)	
5.	~ (∃x) (Ix • Jx)	AIP
6.	(∀x) ~ (Ix • Jx)	5 QN
7.	(∀x) (Fx ⊃ Ix)	3 Simp
8.	[(∃x) (Ex • Gx)] • [(∀x (Fx ⊃ Ix)]	3 Comm
9.	((∃x) (Ex • Gx)	8 Simp
10.	Ea • Ga	9 EI
11.	~(Ia • Ja)	6 UI
12.	Ea ⊃ Fa	1 UI
13.	Ga ⊃ Ha	2 UI
14.	Ha ⊃ Ja	4 UI
15.	Fa ⊃ Ia	7 UI
16.	~Ia v ~Ja	11 DeM
17.	Ea	10 Simp
18.	Fa	12,17 MP
19.	Ia	15,18 MP
20.	Ga • Ea	10 Comm
21.	Ga	20 Simp
22.	Ha	13,21 MP
23.	Ja	14,22 MP
24.	~ ~Ia	19 DN
25.	~Ja	16,24 DS
26.	Ja • ~Ja	23,25 Conj
27.	(∃x) (Ix • Jx)	5-26 IP

(3) 1.	(∀x) [Ex ⊃ Fx]	
2.	(∀x) [Ex ⊃ (Fx ⊃ Gx)]	
3.	(∀x) [Fx ⊃ (Gx ⊃ Rx)] /∴ (∀x) (Ex ⊃ Rx)	
4.	~(∀x) (Ex ⊃ Rx)	AIP
5.	(∃x)~ [Ex ⊃ Rx)	4 QN
6.	~(Ez ⊃ Rz)	6 EI
7.	Ez ⊃ Fz	1 UI
8.	Ez ⊃ (Fz ⊃ Gz)	2 UI
9.	Fz ⊃ (Gz ⊃ Rz)	3 UI
10.	~(~Ez v Rz)	7 Impl.
11.	Ez • ~Rz	11 DeM
12.	Ez	12 Simp
13.	Fz	8,13 MP
14.	~Rz • Ez	12 Comm
15.	~Rz	16 Simp

16.	Fz ⊃ Gz	9,13 MP
17.	Gz	17,14 MP
18.	Gz ⊃ Rz	10,14 MP
19.	Rz	19,18 MP
20.	Rz • ~Rz	20,16 Conj
21.	(∀x) [Ex ⊃ Rx]	5 – 20 IP

3.4 Conclusion

Indirect Proof is an additional rule that proves a proposition by showing that its denial conjoined with other propositions previously proved or accepted leads to a contradiction. In indirect proof, also known as proof by contradiction, one assumes the opposite of what is to be proved, and then derives a contradiction from that assumption. The contradiction shows that the original assumption is false, which means that the statement to be proved must be true. In procedure, we begin by assuming as an extra premiss the negation of the conclusion of our argument, and then work towards deriving a contradiction from the premisses of the argument by any sequence of the rules of inference. The point is that if an argument is valid and we assume the contrary, that is assume that the argument is invalid by making the premiss(es) true and the conclusion false, then we are bound to deduce a contradiction from our assumption.

Indirect proof will often simplify a proof, thereby making proofs shorter or easier to solve. The method of proof is called "indirect" because from taking what seems to be the opposite stance from the proof's declaration, then trying to prove that. If you "fail" to prove the falsity of the initial conclusion, then the argument must be valid.

3.5 Summary

The indirect proof method shows the truth of a statement by assuming its negation and then showing that this leads to a contradiction.

This unit strengthens our repertoire of testing the validity of arguments by allowing us to assume as an extra premiss the negation of the conclusion of the argument, and from the assumed premiss and the other premiss(es) derive a contradiction. The derivation of a contradiction from the negated assumed premiss, means that the original conclusion is valid.

It is pertinent to, however, note that as in Conditional Proof we begin proof in Indirect Proof with assumption; we assume as an extra premiss, the contradiction of what has to be proved, that is we begin by denying the conclusion. Remarkably, Indirect Proof differs from Conditional Proof in that in the former what is assumed is a part of the argument whereas in the case of the later it is not.

3.6 Glossary

Indirect Proof A method of proving the validity of an argument by showing that a counterexample leads either to an absurdity or to a contradiction. That is the rules of inference that permit (i) inferring not ~p having derived a contradiction from p and (ii) inferring p having derived a contradiction from ~p

Quantifier Negation (QN) – Also called ***Exchanging Quantifying Expression (EQ)*** it has to do with the introduction of the negation symbol ~ to the quantifiers of Quantification theory

Reductio ad absurdum: Reductio ad absurdum (Latin: reduction to the absurd) is a form of argument in which a proposition is disproved by assuming the opposite of what is to be proved and deducing its implications to absurd , i.e., self-contradictory consequence.

3.7 Check your Progress

Exercises

For each of the following argument forms, construct an Indirect Proof.

(1) 1. ~(~R v ~ Q)
2. [~(D • V) ⊃ ~ R] • [v ⊃ (U • A)]
/∴ E v A

(2) 1. (A ⊃ B) • (~B • ~ J)
2. ~ A ⊃ (J v F) /∴ F v G

(3) 1. ~W ⊃ ~X
2. Y ⊃ (W ⊃ ~W)
3. A • Z /∴ ~X • A

(4) 1. ~K v ~ M
2. K • R
3. ~M ⊃ [(X • R) ⊃ A] /∴ X ⊃ A

(5) 1. E ⊃ (B ⊃ W)
2. (W • G) ⊃ S
3. U ⊃ (G • ~S) /∴ E ⊃ (B ⊃ ~ U)

(6) 1. ~C v (A v N)
2. (A ⊃ S) • (N ⊃ S)
3. S ⊃ ~ D /∴ C ⊃ ~ D

(7) 1. (A • B) v [(C • D) • E]
2. (A • B) ⊃ E /∴ E

(8) 1. (~Z v X) • (~U v D)
2. ~X v ~ D /∴ Z ⊃ ~ Y

(9) 1. (A • B) ⊃ ~ C
2. C v ~ D /∴ D ⊃ ~ (~A v ~ B)

(10) 1. (~K v M) • (A ⊃ B)
2. (M v B) ⊃ {[Z ⊃ (X v Z)] ⊃ (K • A)}
/∴ K ≡ A

III Construct Indirect proofs for each of the following arguments.

(1) 1. (∀x) [Ax ⊃ (Kx ⊃ Mx]
2. (∃x) (Ax • Bx) /∴ (∃x) (~Mb • ~Kb)

(2) 1. (∀x) (Ax ⊃ Dx)
2. (∀x) (Fx ⊃ ~Dx) /∴ (∀x) [Fx ⊃ ~Ax]

(3) 1. (∃x) (Ax • Bx)

2. $(\exists x)\ (\sim Bx \bullet \sim Ax)\ /\therefore\ (\exists x)\ (\sim Ax \bullet Bx)$

(4) 1. $(\forall x)\ [(Gx \bullet Hx) \supset (Ix \bullet Jx)]$
2. $(\forall x)\ [(Gx \bullet Kx) \supset Hx]\ /\therefore\ (\forall x)\ [(Gx \bullet Kx) \supset Ix]$

(5) 1. $(\exists x) \sim(\sim Rx \vee \sim Sx)$
2. $(\forall x)\ [\sim(\sim Sx \bullet \sim Mx) \supset Bx]\ /\therefore\ (\exists x)\ (Bx \bullet Rx)$

(6) 1. $(\forall x)\ [Ex \supset \sim Dx]$
2. $(\exists x)\ [Kx \bullet Dx]\ /\therefore\ (\exists x)\ [Da \bullet Fa]$

(7) 1. $(\forall x)\ [(Gx \bullet Hx) \supset (Ix \bullet Jx)]$
2. $(\forall x)\ [(Gx \bullet Kx) \supset Hx]\ /\therefore\ (\forall x)\ [(Gx \bullet Kx) \supset Ix]$

(8) 1. $(\forall x)\ [Ax \supset (Sx \equiv Wx)]$
2. $(\exists x)\ [(Ax \bullet (Sx \bullet \sim Wx)]$
3. $(\forall x)\ [Ax \supset Ix]\ /\therefore\ (\forall x)\ [Ix \supset Ax]$

(9) 1. $\forall x)\ [Ax \supset Bx]$
2. $(\forall x)\ [(Ax \bullet Bx) \supset Dx]\ /\therefore\ (\forall x)\ [\sim Dx \supset \sim Ax]$

(10) 1. $(\forall x)\ [Zx \supset Xx]$
2. $(\exists x)\ [Zx \bullet Xx]\ /\therefore\ Xx \vee Ux$

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MODULE 3 Truth-Tree Tests

Unit 1 Truth-Tree Tests of Propositions

Contents

1.1 Introduction

This study unit introduces the learner to a truth tree pattern of lines and symbolized statements that reveals whether it is possible for a given compound statement to be true. If it is possible, the truth tree also shows what truth values may be assigned to the simple components of that statement to obtain a true result.

1.2 Intended Learning Outcomes (ILO's)

It is expected that at the end of this unit, you will be able to

1. construct truth tree involving the reduction of the compound statement into simple statements and/or negated simple statements
2. determine whether individual statements are self-consistent or self-contradictory using the truth tree technique
3. test groups of statements for consistency using the truth tree technique

1.2 Truth Tree Test of Propositions

Truth trees also known as the "*semantic tableau*" test was conceived independently by E. W. Beth and Jaakko Hintikka in the 1950's. Several truth-tree formats have been developed but in this course material we shall follow the format devised by Richard Jeffrey.

A truth tree is a pattern of lines and symbolized statements that reveals whether it is possible for a given compound statement to be true. If it is possible, the truth tree also shows what truth values may be assigned to the simple components of that statement to obtain a true result. Thus, a truth tree accomplishes many of the same things as a truth table, and the use of truth trees can be adapted to determine everything that truth tables can determine. However, truth trees accomplish some things more naturally than anything else, namely, determining whether individual statements are self-consistent or self-contradictory, testing groups of statements for consistency, and testing arguments for validity. Truth trees get their name from the fact that their pattern of lines and symbolized statements resembles an inverted tree, with the trunk at the top and the branches at the bottom.

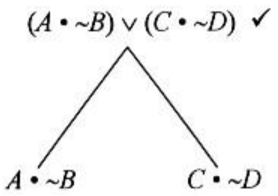
1.3.1 General Procedure for Constructing Truth Tree

1.3.1 (a) Decomposition of Propositions

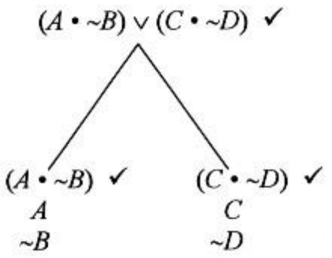
The general procedure for constructing a truth tree involves the reduction of the compound statement into simple statements and/or negated simple statements. This is called *decomposition*. Thus if we have the following compound statement:

(*A* • ~*B*) v (*C* • ~*D*)

to construct a truth tree we begin with the main operator in the statement, which in this case is a vel (v). We note that the truth functional definition of the vel operator stipulates that the given compound statement is true if and only if **(*A* • ~*B*)** is true or **(*C* • ~*D*)** is true (or both). This double alternative for truth is represented in a truth tree by *branching*. Branching is accomplished by splitting the compound statement into its two components and writing those components adjacent to each other beneath the compound statement:



The two branches extending downward from the given statement represent alternate pathways for truth. The given compound statement will be true if (*A* •~*B*) is true or (*C* • ~*D*) is true (or both). Thus, each branch represents a sufficient condition for the truth of the compound statement. In this initial stage of our truth tree, the compound statement is said to have been *decomposed* into its two conjunctive components. We indicate this decomposition by placing a check mark to the right of the compound statement. We now proceed to decompose the conjunctive statements (*A* •~*B*) and (*C* • ~*D*). To accomplish this, we note that the truth functional definition of the dot operator (•) stipulates that a conjunctive statement is true if and only if *both* of its components are true. This requirement that both components be true is represented in a truth tree by *stacking*. Stacking is accomplished by splitting a compound statement into its components and writing those components in a vertical column beneath the statement:

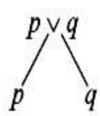


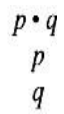
As a result of the stacking arrangement given to *A* and ~*B*, any "movement" (which we will describe shortly) along the left-hand branch (or pathway) must pass through *both* ~*B* and *A*. Similarly, any movement along the right-hand branch (or pathway) must pass through *both* ~*D* and *C*. This arrangement reflects the requirement that both conjuncts must be true for the entire conjunction to be true. Each of the conjunctive statements has now been decomposed into its components, and to indicate this fact, we place a check mark to the right of each.

The truth tree for the given compound statement is now *finished*. A truth tree is said to be finished when the given statement has been completely decomposed into simple statements and/or negated simple statements. In the present example, the given statement has been completely decomposed into *A*, ~*B*, *C*, and ~*D*. When the truth tree is finished, we may proceed to evaluate its paths. Unlike truth tree *construction*, which involves a downward procedure, *evaluation* involves an upward procedure in which we begin at the bottom of the various paths and move upward to the given compound statement. When conducting this evaluation, we ignore all statements that have been checked.

1.3.1 (b) Decomposition Rules

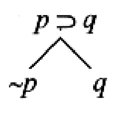
The rationale for decomposing disjunctive statements and conjunctive statements has been explained. Disjunctive statements are decomposed by branching, and conjunctive statements are decomposed by stacking. The general rules are as follows:

(1) 

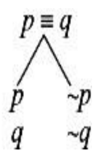
(2) 

We now proceed to explain how statements involving the other three logical operators are decomposed. The rule for negated simple statements is suggested by the example explained above. Negated simple statements are represented in truth trees by single negated letters. No further decomposition is possible.

The rule for conditional statements reflects the fact that *p* ⊃ *q* is logically equivalent to ~*p* v *q*. This equivalence may be established by means of a truth table, but it is also suggested by many instances from ordinary language. For example, the English statement "If you irritate me, then I will slap you" is equivalent in meaning to "Either you do not irritate me, or I will slap you." Thus, the rule for the material conditional is:

(3) 

The decomposition rule for bi-conditional statements is dictated by the truth functional assignments needed to make bi-conditional statements true. The statement form *p* • *q* is true if and only if *p* and *q* have the same truth value; in other words, it is true when either *p* and *q* are both true or *p* and *q* are both false. Therefore, *p* • *q* is logically equivalent to the disjunction [(*p* • q) v (~p • ~*q*)]. Accordingly, the rule for bi-conditionals is as follows:

(4) 

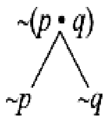
Let us now turn to the rules for negated statements involving the five operators. The easiest of these is the rule for negated negations. The statement form ~ ~*p* is logically equivalent to *p*. Thus, ~ ~*p* is decomposed as follows:

(5) ***~ ~p***
p

This rule is so simple that it may be applied with any of the other rules in a single step.

The decomposition rules for negated conjunctions and negated disjunctions are derived from De Morgan's rule. According to this rule, which may be proved by truth tables, ~ (*p* • *q*) is logically equivalent to ~*p* v ~*q*, and ~ (*p* v *q*) is logically equivalent to ~*p* • ~*q*. Thus, the decomposition rules for these statement forms are as follows:

(6)



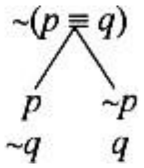
(7) $\sim(p \vee q)$
$\sim p$
$\sim q$

The rule for negated conditionals is dictated by the truth functional assignments under which unnegated conditionals are false. The statement form ~(*p* ⊃ *q*) is true if and only if *p* ⊃ *q* is false; but *p* ⊃ *q* is false if and only if *p* is true and *q* is false. Thus, ~(p ⊃ q) is logically equivalent to *p* • ~*q*. Accordingly, the decomposition rule is:

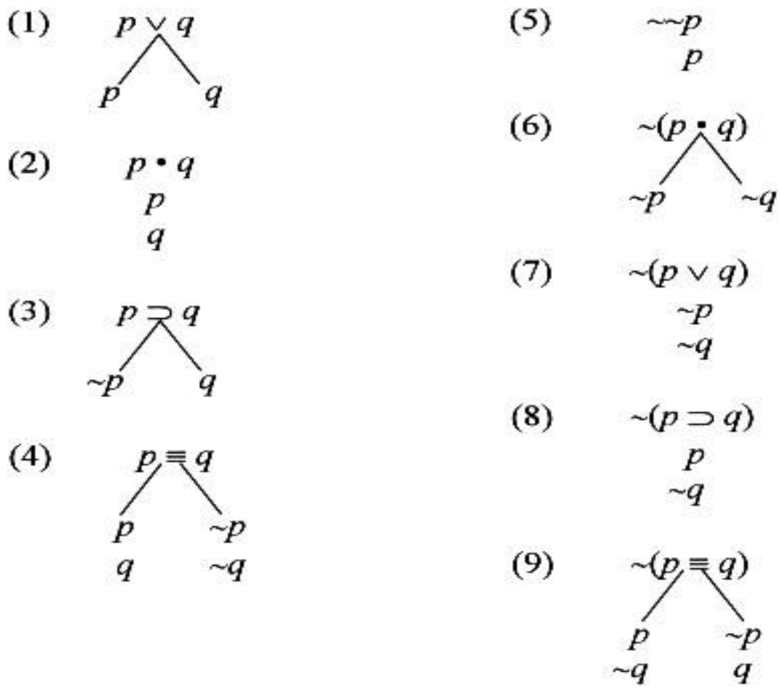
(8) ~(***p*** **⊃** ***q*** **)**
p
~q

Finally, the rule for negated bi-conditionals, like the rule for negated conditionals, is dictated by the truth functional assignments under which un-negated bi-conditionals are false. The statement form *p* • *q* is false if and only if *p* and *q* have opposite truth values; in other words, it is false when either *p* is true and *q* is false or *p* is false and *q* is true. Thus, ~(*p* ≡ *q*) is logically equivalent to [(*p* • ~*q*) v (~*p* • *q*)], and so the decomposition rule for negated bi-conditionals is:

(9)



The decomposition rules are summarized as follows:



With these rules, it will be demonstrated that Truth trees accomplish some things more naturally than anything else, namely, determining whether individual statements are self-consistent or self-contradictory, testing groups of statements for consistency, and testing arguments for validity. We will begin with the application of
truth trees to determining self- consistent and self-contradictory statements.

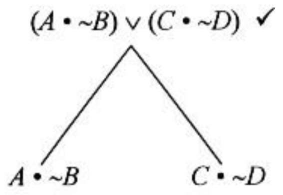
1.3.2 Truth Tree Test of Consistency and Contradiction

Preliminarily, a pair of statements is said to be consistent if and only if there is at least one line on their truth tables in which both statements turn out true. Also *any* group of statements could be tested for consistency--not just pairs of statements. Applying this idea to single statements, we can say that a single statement is *self-consistent* (or internally consistent) if its truth table has at least one line on which the statement turns out true. If there is no such line--that is, if the statement turns out false on every line--the statement is *self-contradictory*. Determining whether a single statement is self-consistent or self-contradictory is the most basic function of a truth tree.

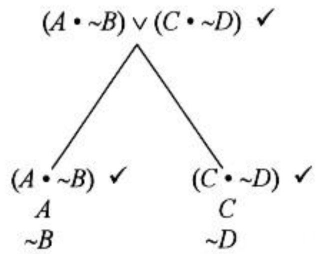
If we take our example of:

$(A \bullet \sim B) \vee (C \bullet \sim D)$

to illustrate how to determine self-consistency and self-contradictory using a truth tree we are faced with the possible ways to assign truth values to A, B, C, and D in such a way that the compound statement turns out true. To use the truth tree to do this, we begin with the main operator in the statement, which in this case is a vel (v). Applying the rule for the decomposition of a disjunctive statement we have:



and then if we applying the stacking rule for the decomposition of a conjunctive statement we have:



The truth tree for the given compound statement is now *finished* that is, the given statement has been completely decomposed into *A*, ~*B*, *C*, and ~*D*. Because the truth tree is finished, we may proceed to evaluate its paths. Evaluating the left-hand path reveals that making ~*B* and *A* true causes the given statement to be true; and evaluating the right-hand path reveals that making ~*D* and *C* true causes the given statement to be true. In other words, the given statement is true if and only if *A* is true and *B* is false, or *C* is true and *D* is false (or both). Because it is possible to assign truth values in such a way as to make the given compound statement true, that statement is self-consistent.

Let us construct truth trees for three additional examples. We now try a conjunctive statement:

$$(K \supset G) \bullet \sim(K \vee G)$$

Because the main operator is the dot, we begin by decomposing the conjunction. As we do so, we place a check to the right of it:

$$\begin{array}{c} (K \supset G) \bullet \sim(K \vee G) \ \checkmark \\ K \supset G \\ \sim(K \vee G) \end{array}$$

Next, we have the option of decomposing the conditional statement first, followed by the negated disjunction, or the negated disjunction first, followed by the conditional statement. When faced with such an option, a handy rule of thumb is: Always decompose the non-branching (stacking) statement(s) first. Following this rule ensures that the resulting truth tree will be simpler. Here, the non-branching statement is the negated disjunction. Thus, we have:

$$\begin{array}{c} (K \supset G) \cdot \sim(K \vee G) \ \checkmark \\ K \supset G \\ \sim(K \vee G) \ \checkmark \\ \sim K \\ \sim G \end{array}$$

Note that the negated disjunction is checked as it is decomposed. Next, we decompose the conditional statement:

$$\begin{array}{cc} \multicolumn{2}{c}{(K \supset G) \cdot \sim(K \vee G) \ \checkmark} \\ \multicolumn{2}{c}{K \supset G \ \checkmark} \\ \multicolumn{2}{c}{\sim(K \vee G) \ \checkmark} \\ \multicolumn{2}{c}{\sim K} \\ \multicolumn{2}{c}{\sim G} \\ \diagup & \diagdown \\ \sim K & G \\ & \times \end{array}$$

The truth tree is now finished, so we may proceed to evaluate its paths. Examining the bottom of the truth tree, we see that it contains two paths that lead upward to the given compound statement. If we take the right-hand path, we encounter first a *G* and then a ~*G*, which is a contradiction. The occurrence of this contradiction means that it is impossible for the path to contain all true statements. In such a case we say that the path is "closed," and we indicate this fact by placing an × at the bottom end. This × may be thought of as "blocking" the path. On the other hand, if we take the left-hand path, we encounter first a ~*K*, then a ~*G*, and finally another ~*K*. Because no contradiction is encountered, it is possible for the path to contain all true statements. In such a case we say that the path is "open."

The fact that at least one path is open indicates that the given compound statement is self-consistent. Also, the appearance of the statements ~*K* and ~*G* in the left-hand path means that the given compound statement turns out true when ~*K* is true and ~*G* is true; that is, when *K* and *G* are both false. This fact can be verified by entering these truth values into the original statement and using them to compute its truth value.

Here is the third example: (*H* • ~*N*) ≡ (*N* v ~*H*)

First, the bi-conditional is decomposed:

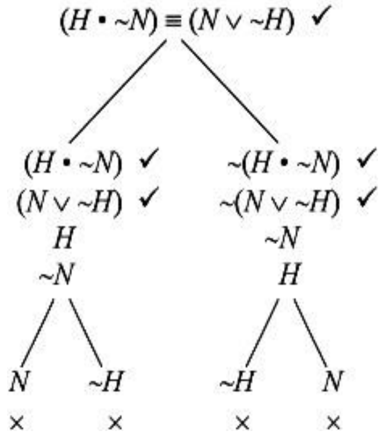
$$\begin{array}{cc} \multicolumn{2}{c}{(H \cdot \sim N) \equiv (N \vee \sim H) \ \checkmark} \\ \diagup & \diagdown \\ (H \cdot \sim N) & \sim(H \cdot \sim N) \\ (N \vee \sim H) & \sim(N \vee \sim H) \end{array}$$

Next, the conjunction and the negated disjunction (which involve no branching) are decomposed:

$$\begin{array}{cc} \multicolumn{2}{c}{(H \cdot \sim N) \equiv (N \vee \sim H) \ \checkmark} \\ \diagup & \diagdown \\ (H \cdot \sim N) \ \checkmark & \sim(H \cdot \sim N) \\ (N \vee \sim H) & \sim(N \vee \sim H) \ \checkmark \\ H & \sim N \\ \sim N & H \end{array}$$

Note that when ~(N v ~H) was decomposed, ~~*H* was replaced with H in a single step.

Lastly, the disjunction and the negated conjunction are decomposed. These steps involve branching:



When we attempt to ascend any of the four paths of this truth tree, we encounter a contradiction. This means that all four paths are closed, and we indicate this fact by placing an "×" at the bottom end of each path. When all of the paths of a truth tree are closed, we say that the truth tree itself is closed, and this tells us that the given proposition is self-contradictory. There is no possibility for the given proposition to be true. At this point it should be apparent that there are alternate ways of constructing truth trees for most statements. For example, we often have the option of decomposing a disjunctive statement first, followed by a conjunctive statement, or a conjunctive statement first, followed by a disjunctive statement. Thus, the question arises whether the method of construction affects the outcome. The answer is no, it does not. Thus, the example we just finished could have been decomposed differently, but no matter which sequence we might have chosen, the resulting truth tree would have turned out closed. A similar remark applies to open truth trees. Mindful of this fact, we can now assert an important rule for truth trees:

Rule 1: *A statement is self-contradictory if and only if it has a closed truth tree.*

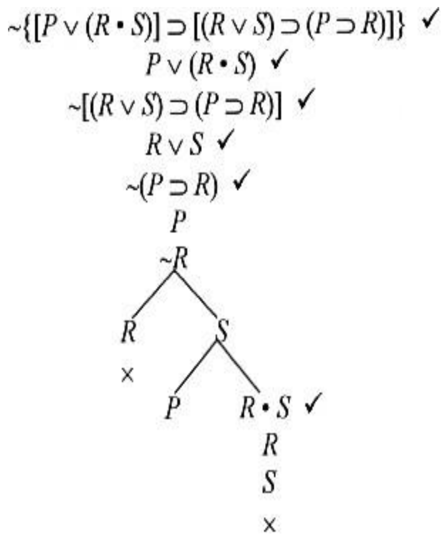
Our fourth example involves a negated conditional statement:

~{[*P* v (*R* • *S*)] ⊃ [(*R* v *S*) ⊃ (*P* ⊃ *R*)]}

First, we decompose the negated conditional statement in braces; next, the resulting negated conditional statement in brackets; and then, the resulting negated conditional statement in parentheses:

$$\begin{array}{c}
{\sim}\{[P \lor (R \bullet S)] \supset [(R \lor S) \supset (P \supset R)]\} \;\checkmark \\
P \lor (R \bullet S) \\
{\sim}[(R \lor S) \supset (P \supset R)] \;\checkmark \\
R \lor S \\
{\sim}(P \supset R) \;\checkmark \\
P \\
{\sim}R
\end{array}$$

Next, we have the option of decomposing either P v (R • S) or R v S. Noting that ~R appears in the existing truth tree and that this letter will contradict the R in R v S, we choose to decompose R v S first, then P v (R • S). This sequence results in a truth tree that is slightly shorter than the truth tree that would have resulted if we had decomposed these statements in reverse order:

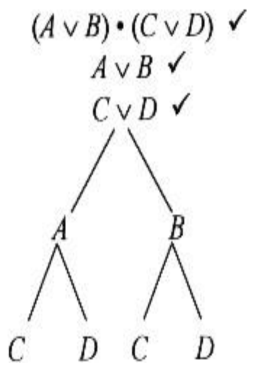


The truth tree is now finished and we can proceed to evaluate its paths. Because a contradiction (R and ~R) appears in the upper left-hand path, the path is closed, and we indicate this fact by placing an × beneath the *R*. Because this path is closed, no further entries need be made beneath the *R*. A contradiction (*R* and ~*R*) also appears in the lower right-hand path, so we place an × at the bottom end of this path as well. One path remains open, however, and this means that the given proposition is self-consistent. Examination of the open path reveals that the proposition turns out true when *P* is true, *S* is true, and *R* is false.

A final point needs to be made about decomposing disjunctions. Consider, for example, the following proposition:

(*A* v *B*) • (*C* v *D*)

The finished truth tree for this proposition is as follows:



It is important to notice that when the second disjunction (*C* v *D*) is decomposed, identical entries must be made under *both A* and *B*. We can express this requirement as a general principle governing the decomposition of any statement. Whenever a statement is decomposed, the resulting components must be attached to the end of every open branch beneath the decomposed statement.

The basic principles of truth tree construction and path evaluation may now be summarized:

(1) Truth tree construction begins with the main operator of a statement.

(2) Truth tree construction proceeds downward, while path evaluation proceeds upward.

(3) Statements that entail stacking should be decomposed before statements that entail branching.

(4) Whenever a statement is decomposed, a check mark is placed to the right of it. Checked statements are ignored in the phase of path evaluation.

(5) A truth tree is finished when the given statement is completely decomposed into simple statements and/or negated simple statements.

(6) A path is closed if and only if it contains a contradiction. Such paths are "blocked" by placing an × at the bottom end.

(7) Any path that is not closed is open.

(8) A truth tree is closed if and only if all its paths are closed.

(9) A statement is self-contradictory if and only if it has a closed truth tree. Otherwise it is self-consistent.

(10) Whenever a statement is decomposed, the resulting components must be attached to the end of every open branch beneath the decomposed statement.

1.3.3 Testing Groups of Statements for Consistency

In the previous section in this unit, we saw how truth trees are used to test individual statements for self-consistency. In this section we use truth trees to test *groups* of statements for consistency. Suppose we are given the following group of statements:

S1, S2, S3, . . . , Sn

These statements are consistent with one another if and only if it is possible to assign truth values to their simple components in such a way that all of the statements turn out true. But if it is possible to assign truth values in this way, then, and only then, will the following conjunction turn out true:

S1 • S2 • S3 • . . . • Sn.

As a result, the statements S1, S2, S3, . . . , Sn are consistent if and only if the conjunction formed from them is self-consistent.

To see this idea from another angle, suppose we construct a truth table for the statements S1, S2, S3, . . . , Sn , and suppose there is at least one line in that truth table in which all of the statements turn out true. The group of statements is then consistent. But each and every one of these statements is true on a line if and only if the conjunction of those statements is true on that line. Thus, the group of statements is consistent if and only if the conjunction of those statements is self-consistent. Stated otherwise, the group of statements is inconsistent if and only if the conjunction of those statements is self-contradictory.

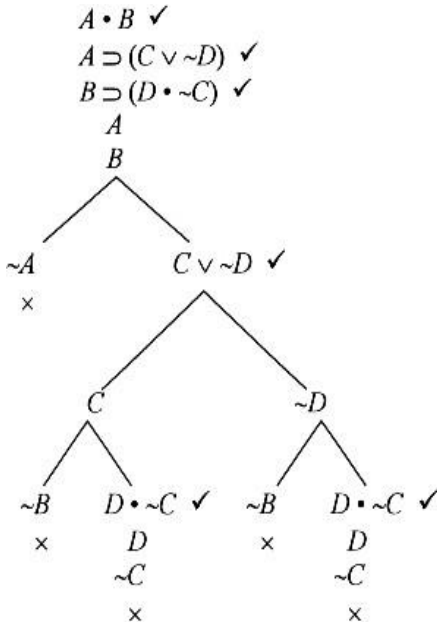
Now, applying truth trees to this idea, the statement S1 • S2 • S3 • . . . • Sn is self-contradictory if and only if it has a closed truth tree. Thus, we have the following rule:

Rule 2: *A group of statements is inconsistent if and only if the conjunction of those statements has a closed truth tree.(Rule 1 is already stated in the previous section)*

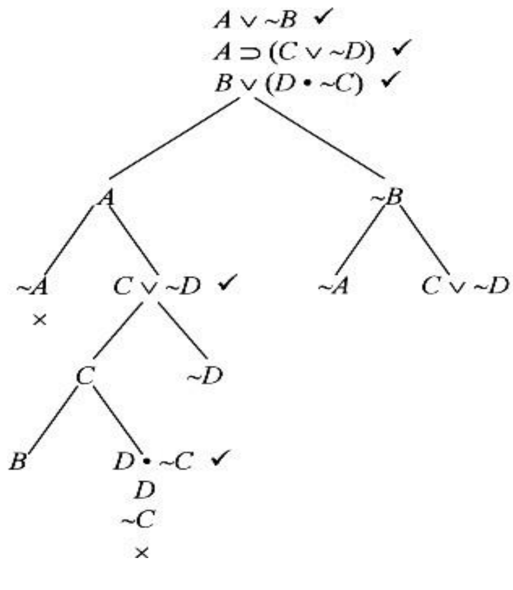
In constructing such a truth tree, we can skip the step of actually writing out the conjunction of the statements. We need only stack the statements in a column and proceed as with any other conjunctive statement. For example, suppose we are given the following group of statements:

A • B, A ⊃ (C v ~D), B ⊃ (D • ~ C)

We begin by stacking these statements in a column and proceed to construct a truth tree in the normal fashion. The finished truth tree is as follows:



Because every path of the truth tree is closed, the truth tree itself is closed. This means that there is no path by which all of the statements can be true. Therefore, the given group of statements is inconsistent. At this point we may observe something that may have been noticed earlier: If any finished *path* of a truth tree is open, the statement we are testing is self-consistent. A path is said to be finished if finishing the entire truth tree would involve making no changes in that path. Thus, if we are not interested in finding *all* of the truth value assignments that make the statement true, it is not necessary to finish the other paths. As an illustration of this fact, consider the following truth tree. Only the left-hand part has been finished:



The path leading upward from B, in the lower left-hand corner, is finished. If the entire truth tree were to be finished, no changes would have to be made to that path. Also, that path reveals no contradictions. Thus, the path is open, and the given group of statements is consistent. The statements turn out true when B is true, C is true, and A is true. If we are only interested in finding one set of truth values that makes the group of statements true, it is not necessary to finish the truth tree.

Note that in this example, the third statement from the top has been checked even though it has not been completely decomposed. It has only been decomposed in regard to the left-hand branch. This amounts to a modification of our convention for checking statements, and it should only be used when we are intentionally leaving part of the truth tree unfinished.

1.4 Conclusion

he truth tree technique has many attractive features: Like formal proofs, it can be employed in other branches of logic; like full truth tables, it is an effective procedure; and like proofs and brief truth tables, it is a practical test for complex arguments. In fact, truth trees provide an alternate decision procedure for assessing validity, logical equivalence, satisfiability and other logical properties of sentences and arguments. The advantage of truth trees is that it is a decision procedure whose complexity is not a function of the number of propositional letters in the formula being analyzed. The method is capable of being adapted to a more expressive logical language.

1.5 Summary

The truth tree method applies immediately to look for counterexamples to a sentence being a contradiction. We make the sentence to be tested the first line of a tree. If there are one or more counterexamples, that is, cases in which the sentence is true, the tree method is guaranteed to find them. If the tree method does not turn up a counterexample, that is, if all paths close, we know there are no cases in which the sentence is true. But if there are no cases in which it is true, the sentence is false in all cases; in other words, it is a contradiction.

In the same vein, truth trees test for consistency; a group of statements is consistent if and only if the conjunction of those statements is self-consistent. Stated otherwise, the group of statements is inconsistent if and only if the conjunction of those statements is self-contradictory.

1.6 Glossary.

Branching: splitting the compound statement into its two components and writing those components adjacent to each other beneath the compound statement: Each branch will represent one way of making true all the sentences which appear along it. The left branch will make line 1 true by making 'A' true. The right branch will make line 1 true by making 'B' true. Since the paths have branched, they represent alternative ways of making line 1 true.

Closed Branch: A branch containing a proposition P and its literal negation ~P. A closed branch is represented by an **X**

Closed Tree: A tree is closed when all of the tree’s branches are closed. A closed tree will have an **X** under every branch.

Counterexample. Counterexample disproves a statement by giving a situation where the statement is false; truth tree as a proof by contradiction, proves a statement by assuming its negation and obtaining a contradiction.

Consistency: A set of one or more propositional logic sentences is consistent if and only if there is at least one assignment of truth values to sentence letters which makes all of the sentences true. The truth tree method applies immediately to test a set of sentences for consistency.
Decomposition: A mechanism that breaks the statement down into its constituent pieces. Decomposition occurs by applying specific decomposition rules. There are 9 decomposition rules, each applying to a specific type of decomposable proposition
Open Branch: An open branch is a branch that is not closed. That is, a branch that does not contain a proposition P and its literal negation ~P.
Stacking: splitting a compound statement into its components and writing those components in a vertical column beneath the statement A stacking rule is a truth-tree rule where the condition under which a proposition 'P' is true is represented by stacking. A stacking rule is applied to propositions that are true under one truth-value assignment.

1.7 Check your Progress

Exercises I

Determine whether the following propositions are self-consistent or self-contradictory by constructing a truth tree for each. For those that are self-consistent, identify at least one set of truth values that makes the proposition true. Test your answer by using these truth values to compute the truth value of the given proposition.

1. (*K* ⊃ ~*M*) • (*M* ≡ ~*K*)
2. (*S* ⊃ ~*R*) ≡ ~(*R* ⊃ ~*S*)
3. ~[(*N* ⊃ ~*B*) ⊃ (*B* ⊃ *N*)]
4. ~[(*E* • *G*) É (E • ~*G*)]
5. [A • ~(*R* v *H*)] • [*R* • ~(A • *H*)]
6. ~{[*P* ⊃ (*B* ⊃ ~*P*)] ⊃ [*B* ⊃ (*P* ⊃ ~*B*)]}
7. [*R* ⊃ (*N* • ~C)] • [*N* ≡ (*C* • *R*)]
8. [(*G* • ~*E*) v (*G* • ~*C*)] • [(*G* v D) ⊃ (*E* • *C*)]
9. [*S* • (*P* v ~*R*)] ≡ [*R* •~(*S* • *P*)]
10. ~{[*C* ⊃ (*K* ⊃ *M*)] ≡ [*K* ⊃ (*C* ⊃ *M*)]}

Exercises II

Use truth trees to determine if each of the following groups of statements is consistent or inconsistent. If consistent, identify one set of truth values that makes the group of statements true. Then, test your answer by entering these truth values into the statements of that group and proceed to prove that each statement turns out true.

1. M ⊃ ~D, K v D, M • K
2. S • R, S ≡ G, ~R v ~G
3. C ⊃ (N • ~H), N ⊃ ~C, C v H
4. (P • ~B) ≡ (Q v D), ~(P v Q), B ⊃ D
5. (S • C) ≡ (E v G), S ⊃ C, E ⊃ G, S • ~G
6. (R ≡ B) ⊃ (R • ~Q), R ⊃ (B • Q), ~B v R
7. T ⊃ (F v ~A), R ⊃ (A v ~F), T • R, ~(A ≡ F)
8. S ⊃ (N • ~C), H ⊃ (C v ~N), H v S
9. (D v ~B) ⊃ (N • T), Q v ~B, D v ~Q, N ⊃ T
10. (B v E) ⊃ ~S, G ⊃ (S • E), A ⊃ (S • B), A v G

1.8 References/Further Reading

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Bonevac, Daniel A. (2002) Deduction: Introductory Symbolic Logic, Wiley-Blackwell

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Unit 2 Truth-Tree Tests of Validity in Propositional Logic

Contents

2.1 Introduction

This study unit introduces the learner to another method of proof of validity that takes the form of *indirect proof* or argument by *reduction ad absurdum* called the *truth tree test*

2.2 Intended Learning Outcomes (ILO's)

It is expected that at the end of this unit, the learner will be to

1. apply a less challenging method of testing validity of arguments
2. explain the rules governing the use of truth tree tests of validity
3. state how counterexamples are identified
4. construct truth tree tests of the validity and invalidity of arguments

2.3 Truth-Tree Tests of Validity in Propositional Logic

The truth tree test of validity takes the form of *indirect proof* or argument by *reduction ad absurdum.* According to Richard Jeffery (1989:31) the truth trees method might aptly be called *refutation trees* or *reduction trees*. It proves the validity of an argument by refuting the hypothesis that the premisses together with the denial of the conclusion form a satifiable set. This means that the method tries to demonstrate that a counterexample to a valid argument will lead to some inconsistency (counterexamples are cases in which the premisses are all true and the conclusion false).

By definition, a valid argument is one in which it is impossible for the premisses to be true and the conclusion false. In other words, a valid argument is one in which it is impossible for the premisses and the negation of the conclusion all to be true. But this means that an argument is valid if and only if the conjunctive statement formed from the premisses and the negation of the conclusion is self-contradictory. Now, applying truth trees to this idea, we have the following rule:

Rule 3: *An argument is valid if and only if the conjunction of the premisses and the negation of the conclusion has a closed truth tree*

If the truth tree produced from the conjunction of the premisses and the negation of the conclusion is closed, there is no path by which the premisses can be true and the conclusion false, and so the argument is valid. On the other hand, if this truth tree is open, then there is such a path, and so the argument is invalid. The idea behind this rule can be stated symbolically. Suppose we have an argument consisting of the following premisses and conclusion:

P1
P2
P3

M

Pn

C

This argument is valid if and only if the following conjunctive statement has a closed truth tree:

P1 & P2 & P3 & . . . & Pn & ~C

To construct a truth tree for this conjunctive statement, we simply stack the premisses and the negation of the conclusion and proceed as with any other conjunctive statement. Consider, for example, the following argument:

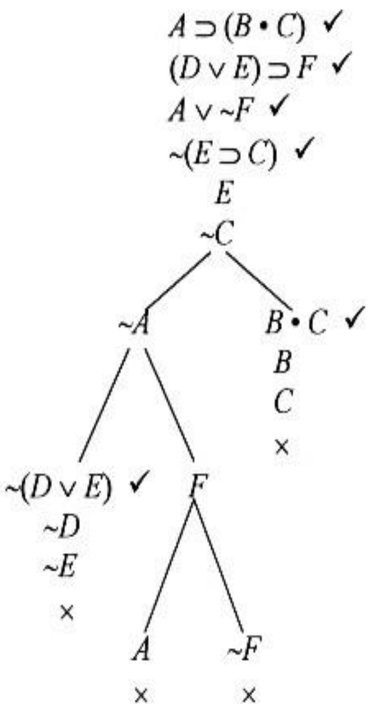
$A \supset (B \bullet C)$

$(D \vee E) \supset F$

$\underline{A \vee \sim F}$

$E \supset C$

Combining these premisses with the negation of the conclusion, we have the following finished truth tree:



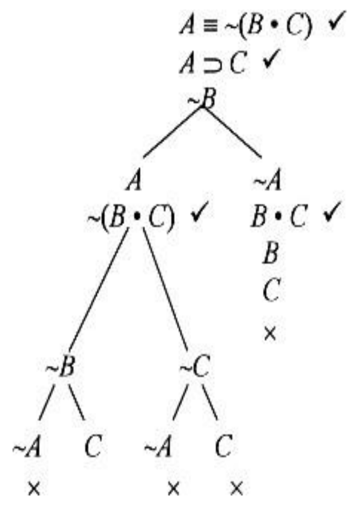
Because all paths are closed, it is impossible for the premisses to be true and the conclusion false. Thus, the argument is valid. Here is another argument:

$A \equiv \sim(B \bullet C)$

$\underline{A \supset C}$

B

The finished truth tree is as follows:



Because one path of this finished truth tree is open, the argument is invalid. The open path reveals that the argument has true premisses and a false conclusion when C is true, B is false, and A is true.

How can we identify Counterexamples?

Counterexamples are used to prove that a statement is invalid.

Identify the hypothesis and the conclusion in the given statement.

The counterexample must be true for the hypothesis but false for the conclusion.

Outline of how to test arguments using the Truth Trees method

1. List the premisses and the *negation* of the conclusion in a vertical column. This is to assume invalidity. This column forms the "trunk" of the tree.

2. Take any compound statement in the trunk, check it off, and draw its truth-conditions at the bottom of the trunk, following the decomposition rules from the chart below. If compound statements remain on any branch, check them off and break them down, listing their truth conditions (by the branching rules) at the bottom of *every* open branch below them. Repeat until each compound statement in each branch has been checked off and broken down (decomposed)

Tip: To save labor and paper, check off and breakdown all non-branching compounds before any branching compounds.

3. If a branch contains contradictory information anywhere along it, including the trunk, then *close* that branch with an x at its bottom. When these steps are complete, then either:

> All branches are closed. This means that the initial assumption of invalidity leads to contradiction; hence the assumption is false. *The argument is valid*.
>
> At least one branch is open. This means that the assumption of invalidity is not contradictory; there is at least one assignment of truth-values, which makes all the premisses true and the conclusion false. *The argument is invalid*.

Each open branch is a counterexample --a vertical, or jagged, representation of an invalidating row of a truth table.

General rules

To list a statement (in trunk or branch) is to assign it the truth-value T. Decomposition rules apply only to whole statements, not to compounds that are components of larger compounds. In deciding which decomposition rule to apply to a statement, look only at the statement's main connective. A compound will branch if and only if it has more than one T in its truth table column.

Branching represents inclusive disjunction: the statement is true under one or the other or both of the conditions that branch below it. Non-branching represents conjunction: the statement is true only under both the conditions listed below it. The decomposed components of a compound must be listed at the bottom of *every* open branch below the compound.

2.4 Conclusion

The truth tree test is designed such that it is guaranteed to turn up at least one counterexample to an argument if there are any counterexamples. If the method finds a counterexample, we know the argument is invalid. If the method runs to completion without turning up a counterexample, we know there are no counterexamples, so we know that the argument is valid.

This method is more nearly mechanical than is natural deduction. This fact makes truth trees less of a challenging, but also less aggravating, because they are easier to do. Truth trees also have the advantage of making the content of propositional logic statements dear, in a way which helps in proving general facts about systems of logic.

2.5 Summary

To use a tree to test for validity, you write down at the root of the tree all premisses and the negation of the conclusion. Then you work through the tree until you find an open and completed branch or all branches are closed. If you found an open and completed branch, then that means that it is possible for all statements in the root of the tree to be true, which in turn means that it is possible for all premisses to be true while the conclusion is false. Hence, the argument is invalid. If all branches closed, the opposite is true, i.e. the argument is valid.

2.6 Glossary

2.7 Check your Progress

1. $F \supset (S \supset R)$
   $\underline{\sim R}$
   $S \supset \sim F$

2. $B \supset (H \bullet T)$
   $\underline{\sim(T \bullet E)}$
   $\sim B$

3. $G \supset A$
   $\underline{G \supset N}$
   $G \supset (A \bullet N$

4. $R \supset (T \vee K)$
   $\underline{\sim(K \bullet P)}$
   $R \supset T$

5. $(\sim S \bullet P)$
   $\underline{S \supset \sim F}$
   $S \equiv F$

6. $J \supset \sim (A \equiv P)$
   $\underline{R \supset (J \bullet \sim P)}$
   $R \supset A$

7. $G \supset D$
   $F \supset T$
   $\underline{\sim(D \vee T)}$
   $\sim (G \vee F)$

8. $E \supset (A \vee T)$
   $\underline{(A \bullet T) \supset M}$
   $E \supset M$

9. $Q \vee P$
$Q \vee T$
$\underline{\sim(T \vee P)}$
$N \supset Q$

10. $\sim [(P \vee B) \equiv (Q \vee D)]$
$P \supset \sim D$
$\underline{Q \supset B}$
$P \supset B$

11. $(N \equiv J) \supset G$
$\sim(G \bullet N)$
$\underline{J \supset \sim D}$
$K \supset G$

12. $\sim[Q \supset (N \bullet C)]$
$C \vee T$
$\underline{T \supset \sim(Q \vee E)}$
$\sim N$

13. $H \supset (Q \supset \sim H)$
$\underline{P \vee E) \supset (Q \bullet L)}$
$\sim (E \bullet H)$

14 . $L \supset (N \supset C)$
$\underline{\sim C \supset (L \vee N)}$
$N \equiv C$

15. $E \vee \sim(G \bullet B)$
$\underline{F \supset (B \bullet K)}$
$E \vee G$

2.8 References/Further Reading

Agler, David W. (2012) Symbolic Logic: Syntax, Semantics, and Proof Barnes and Noble

Bonevac, Daniel A. (2002) Deduction: Introductory Symbolic Logic, Wiley-Blackwell

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Unit 3 Proving Invalidity in Predicate Logic

Contents

3.1 Introduction

This study unit introduces the learner to how to apply the truth tree method to test validity in Predicate (Quantificational) Logic

3.2 Intended Learning Outcomes (ILO's)

It is expected that at the end of this unit, the learner will be able to

1. explain the rules governing proof of invalidity in Predicate Logic
2. construct truth tree tests of invalidity in Predicate Logic

3.3 Proving Invalidity in Predicate Logic

In the previous unit, we underscored the point that to say that an argument is valid is to say that in every possible case in which the premisses are true, the conclusion is true also. We re-express this by saying that an argument is valid if and only if it has no counterexamples, that is, no possible cases in which the premisses are true and the conclusion false. In propositional logic, the trees were really just a labor-saving device. We could always go back and check through all the truth tree lines.

Predicate logic changes everything; although truth trees can be used in predicate logic to prove the validity of many fairly simple arguments, these proofs usually turn out to be longer and more complicated than corresponding proofs by natural deduction. A more practical application for truth trees is to prove the invalidity of invalid arguments. When used for this purpose, one general approach to the subject depends upon the same principles as the finite universe method.

The use of truth trees presented in this section is confined to that approach. The finite universe method depends on the idea that any argument that can have true premisses and a false conclusion in a universe containing only a few members is invalid. First a universe of one is tried, then a universe of two, then a universe of three, and so on. Eventually some universe having a finite size will be found in which it is possible for the premisses to be true and the conclusion false. Identifying an assignment of truth-values in that universe that gives this result proves the argument invalid. In a finite universe, universally quantified statements are equivalent to finite conjunctions of singular statements, and existentially quantified statements are equivalent to finite disjunctions of singular statements. Thus, in a universe consisting of two members, which we will name a and b, the statement

$(\forall x)$ Fx is equivalent to Fa • Fb, and $(\exists x)$Fx is equivalent to Fa v Fb. In a universe consisting of three members, which we will name a, b, c, $(\forall x)$Fx is equivalent to Fa • Fb •

Fc, and $(\exists x)Fx$ is equivalent to Fa v Fb v Fc, and so on. In a universe consisting of only one member, which we will name a, both $(\forall x)Fx$ and $(\exists x)Fx$ are equivalent to Fa.

To use truth trees to prove invalidity in predicate logic follow these three steps:
(1) Select a universe of a certain size.
(2) Translate the argument into conjunctions and/or disjunctions of singular statements as just explained.
(3) Construct a truth tree and evaluate its paths. Step 3 is accomplished in basically the same way as it is for arguments in propositional logic. The difficulty with using truth trees to prove invalidity in predicate logic arises from the fact that universally quantified statements are often expressed as conditionals (which involve branching), and existentially quantified statements are translated in a finite universe as disjunctions (which also involve branching). The result is that truth trees for predicate logic can expand off the page in a horizontal direction. We can often avoid this problem, however, by strategically finishing only one branch (or path) of the truth tree, and leaving the other branches dangling. If we know that the argument in question is invalid, our only interest is in finding one assignment of truth values that makes the premisses true and the conclusion false. If, in constructing our truth tree, we strategically avoid finishing paths that will close, we can usually finish a single open path fairly quickly. This open path will disclose the needed truth value assignment.

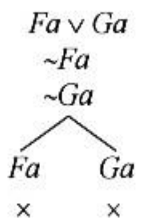
Consider, for example, the following argument:

$$\begin{array}{l} (\exists x)(Fx \vee Gx) \\ \underline{(\exists x){\sim}Fx} \\ (\forall x)Gx \end{array}$$

To prove this argument invalid we first consider a universe containing only one member. If we call that member a, the argument becomes equivalent to:

$$\begin{array}{l} Fa \vee Ga \\ \underline{{\sim}Fa} \\ Ga \end{array}$$

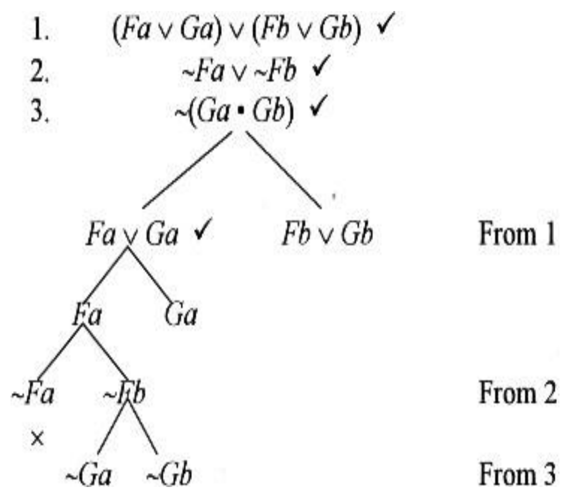
After negating the conclusion, the truth tree for this argument is as follows:



Since the truth tree is closed, it is impossible for the argument to have true premisses and a false conclusion in a universe of only one member, so next we try a universe of two. If we name the second member b, the argument becomes equivalent to:

$$\begin{array}{l} (Fa \vee Ga) \vee (Fb \vee Gb) \\ \underline{{\sim}Fa \vee {\sim}Fb} \\ Ga \bullet Gb \end{array}$$

In performing this translation for a two-member universe, remember that statements with existential quantifiers become disjunctions, and statements with universal quantifiers become conjunctions. After negating the conclusion, we proceed to construct a truth tree; but to keep it simple, we finish only part of it. To help identify the steps, we have numbered the lines and noted the source of the various steps in the right-hand margin. Also, we have adopted our modified convention of placing check marks next to statements that are decomposed in at least one branch:



The two open paths at the bottom of the tree are finished, because if the entire truth tree were finished, nothing would have to be changed in these two paths. The left-hand path reveals that the argument will have true premisses and a false conclusion in a two-member universe when Ga is false, Fb is false, and Fa is true. When these truth-values are assigned to the simple statements in the original argument (as converted to a two-member universe), the premiss turn out true and the conclusion false. Thus, we have proved the argument invalid. If all the paths of this truth tree had turned out to be closed, then we would have to proceed to a universe of three, and continue until at least one open path appeared at the bottom of the tree. When using unfinished truth trees to prove invalidity in predicate logic, the objective is to *avoid* closed branches. Thus, one must use foresight in constructing such a truth tree. If it appears in advance that a certain path will close, then there is no point in working with that path. Instead, another path should be selected that looks more promising. When truth trees are used to prove invalidity, their sole purpose is to disclose an assignment of truth-values that makes the premisses true and the conclusion false. Just one such assignment of truth values is all that we need. There is no point in finding more than one. After an open path has been found, the truth values revealed by that path should be assigned to the simple statements in the argument and then used to compute the truth value of the premisses and conclusion. This will ensure that no mistakes have been made in the construction of the truth tree.

3.4 Conclusion

Everything we learned about truth trees in propositional logic carries over to predicate logic. Someone gives us an argument and asks us whether it is valid. We proceed by searching for a counterexample. We begin by listing the premisses and the denial of the conclusion as the beginning of a tree. Just as before, if we can make these true we will have a case in which the premisses are true and the conclusion false, which is a counterexample and which shows the argument to be invalid. If we can establish that the method does not turn up a counterexample, we conclude that there is none and that the argument is valid.

3.5 Summary

The tree method for predicate logic works in exactly the same way, with just one change: Each branch is no longer a way of developing a line of a truth tree which will make all the sentences along the branch true. Instead, a branch is a way of developing an **interpretation** which will make all the sentences along the branch true. All you have to do is to stop thinking in terms of building a line of a truth table (an assignment of truth values to sentence letters). Instead, start thinking in terms of building an interpretation.

3.6 Glossary

Finite universe method. This method is used to show that an invalid argument is indeed invalid by constructing a small model in which the premises of the argument are all true but the conclusion false. That conjunction, of course, is false, since b is not square. Hence, (x)Sx is false as well.

3.7 Check your Progress

Exercises

Prove the following arguments invalid by constructing (at least) an unfinished truth tree for each. A finished open path will reveal an assignment of truth values that makes the premisses true and the conclusion false for a universe of a certain size. Test your answer by using these truth values to compute the truth value of the premisses and conclusion. All of these arguments will fail in a universe of less than three members.

1. $\underline{(\forall x)\ (Fx \supset Gx)}$
   $(\forall x)\ [Fx \supset (Gx \bullet Hx)]$

2. $(\forall x)\ (Fx \supset Gx)$
   $\underline{\sim Ge}$
   $Fc \bullet \sim Ge$

3. $(\forall x)\ (Fx \supset Gx)$
   $\underline{(\exists x)\ (Gx \bullet Hx}$
   $(\exists x)(Fx \bullet Hx)$

4. $(\forall x)\ (Fx \supset Gx)$
   $\underline{Fc}$
   $(\forall x)\ Gx$

5. $(\forall x)\ (Fx \supset Gx)$
   $\underline{(\exists x) \sim Fx}$
   $(\exists x) \sim Gx$

6. $(\exists x)\ (Fx \vee Gx)$
   $\underline{(\exists x) \sim Fx}$
   $(\exists x) \sim Gx$

7. $(\forall x)(Fx \equiv Gx)$
   $\underline{(\exists x)\ (Fx \vee Gx)}$
   $(\forall x)\ Fx$

8. $(\forall x)\ (Fx \equiv Gx)$
   $\underline{(\exists x)\ Fx}$
   $(\forall x)\ Gx$

9. $(\exists x)\ (Fx \vee Gx)$
   $\underline{(\exists x) \sim Fx}$
   $(\exists x)\ Gx$

10. $(\forall x)\ (Fx \supset Gx)$
    $\underline{(\exists x)\ (Jx \vee \sim Gx)}$
    $(\forall x)\ (Fx \supset Jx)$

3.8 References/Further Reading

Agler, David W. (2012) Symbolic Logic: Syntax, Semantics, and Proof Barnes and Noble

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