**Course Code**: PHL 301

**Course Title**: Symbolic Logic

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# COURSE GUIDE

# PHL301: SYMBOLIC LOGIC

**NATIONAL OPEN UNIVERSITY OF NIGERIA**
**(NOUN)**

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## COURSE INTRODUCTION

Welcome to the Course: Symbolic Logic, PHL301, dear students. This course is designed to facilitate an understanding of the meaning and nature of logic in order to have deeper understanding of the correct rules of inference that makes distinction of right reasoning from wrong reasoning. Some of the primary questions to be considered in this course are: What is the meaning of propositional calculus? What is the significance of mathematical logic to valid inferences? How can we identify logic in the natural sense as against in the artificial sense?

This course builds on your previous studies in PHL105 Introduction to Logic I; PHL152 Introduction to Logic II; as well as some studies in GST203, Introduction to Logic and Critical Thinking In PHL 301, the learners are encouraged to be prepared for a more rigorous and engaged period of scholarly discipline with artificial logic which will not only sharpen their criticality and capacity for sound judgment but also be able to establish the non-negotiable place of logic in human endeavours.

In symbolic logic, as here conceived, we are concerned with investigating the validity or invalidity of deductive arguments. The concern is two-fold. First, we attempt to discover basic valid deductive argument-forms. Second, we attempt to determine the validity or invalidity of particular deductive arguments. The two concerns are obviously intimately connected, as we shall see in due course. In pursuing these concerns, we shall employ special symbols for the words, connectives or propositions that constitute argument.

We are all familiar with attempts to achieve precision, brevity and clarity by using special symbols. In mathematics, for example, ‘I’ stands for ‘one’, ‘10’ for ten, ‘100’ for ‘one hundred’, and so on. Similarly, the symbol ‘+’ represents ‘plus’ or ‘sign of addition’, ‘-’ ‘minus’ or ‘sign of subtraction’, ‘=’ represents ‘equals’ or ‘sign of equality’, and so on. These signs are compounded to get complexes like ’20 + 5 = 25’, representing ‘twenty plus five equals twenty-five’ and ’30 – 6’ = 24’ meaning ‘thirty minus six equals twenty-four’. Similarly, ‘2x2x2x2x2x2x2x2x2x2’ can be rewritten as ‘$2^{10}$’.

The use of symbols to represent propositions, terms and connectives is one of the hallmarks of modern logic. Symbols used in logic help us to exhibit, with greater precision, clarity and brevity, the logical structures or forms of propositions sentences or statements and arguments whose forms may tend to be obscured in natural language. It also frees us from psychological attachment to, or influence of, what a sentence, statement or proposition says in natural language. The words ‘proposition’ ‘sentence’ and ‘statement’ shall be taken as synonyms and interchangeably.

## COURSE OBJECTIVES

The objectives of this course include the capacity to be able to:

- Identify and comprehend the idea of propositional calculus;
- Explain the idea of predicate calculus;
- Discuss the rules of inference and rules of replacement function;
- Enumerate and be able to apply the laws and principles of logic to the right proposition; and
- Transfer natural logic into artificial language through artificial/symbolic notations.

## WORKING THROUGH THIS COURSE

For an adequate understanding of the contents of this course, students are encouraged to possess a copy of the course guide which outlines what is expected of them. It will guide students to read through the study text in a coherent and logical manner and thereby enhance their understanding of the fundamental ideas expressed in each of the thematic considerations included in the modules of this course.

In addition to the above, students are required to be actively involved in forum discussion and facilitation. Hence, attendance plus class participation are very important. There are also interesting readings that are necessary to enhance understanding of the course. Lecture notes are mere guidelines.

Furthermore, students are encouraged to develop very important periods of solitude studies in order to be able to grasp the salient rules of logic and how they can be applied. The exercises placed at the end of each module are starters that can assist learners with these endeavors.

## STUDY UNITS

This course, Symbolic Logic, is divided into two main parts, namely:

(1) Propositional Calculus, also called Logic of Propositions, Propositional Logic, Sentential Logic or Truth-Functional Logic, and

(2) Predicate Calculus, also called Predicate Logic, Logic of Predicates, Quantification Theory or Quantificational Logic.

The foregoing two chief divisions concerning the important aspects of PHL301, culminates into a total of 13 units spread across 4 modules. They are outlined below:

**Module 1: Propositional Calculus I**

Unit 1: Meaning of propositional variables, Propositional constants and Logical Connectives

Unit 2: Symbolising Propositions

Unit 3: Truth-Conditions and Truth-Tables

Unit 4: Tautology, Contradiction and Contingent Truths

**Module 2: Propositional Calculus II**
Unit 1: Validity/Invalidity of Arguments
Unit 2: Basic Valid Argument Forms
Unit 3: Logically Equivalent Formulas
Unit 4: Method of Natural Deduction

**Module 3: Predicate Calculus I**
Unit 1: Introducing Predicate logic
Unit 2: Symbolising Propositions in Predicate Logic
Unit 3: Truth and Falsity in Predicate Logic

**Module 4: Predicate Calculus II**
Unit 1: Validity in Predicate Logic
Unit 2: Invalidity in Predicate Logic

## PRESENTATION SCHEDULE

This course will involve intensive interaction between learners and facilitators. Learners are encouraged to practice the module exercises provided for them and then confront their facilitator with difficulties. The participation of learners go on to contribute to class participation which may attract some points in the final evaluation of learners registered for the course.

## ASSESSMENT

Students will be assessed with the regular TMAs which are 30% of total marks and the pen-on-paper assessment which comprises of a total of 70%. This brings the total marks to 100%

## FOR OPTIMAL PERFORMANCE IN THIS COURSE

For students to perform optimally in this course, s/he must:

- Have 75% of attendance through active participation in both forum discussions and facilitation;
- Read each topic and solve exercises in the course materials before it is treated in class;
- Submit every assignment as and when due; failure to do so will attract penalties;
- Know that regular discussion and sharing of ideas among peers will enhance understanding the contents of the course;
- Download videos, podcasts and summary of group discussions for personal use;

- Attempt each self-assessment exercise in the main course material;
- Take the final exam; and
- Approach the course facilitator when there is a challenge with any aspect of the course.

## FACILITATION

This course operates a learner-centred online facilitation. To support the student's learning process, the course facilitator will introduce each topic for discussion before, opening the floor for discussion. Each student is expected to read the course materials, as well as other related texts, and raise critical issues which s/he shall bring forth in the forum discussion for clarification. The facilitator will summarize forum discussion, assist students with exercises that they find difficult, provide relevant materials, videos and podcasts to the class; and disseminate all relevant information via email and SMS as might be required.

## REFERENCES/FURTHER READINGS/WEB SOURCES

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic*. Ibadan: Book Wright Nigeria Publishers

In addition to the afore-stated works, the following online sites can also assist students to acquire additional publications:

- www.pdfdrive.net
- www.bookboon.com
- www.sparknotes.com
- http://ebookee.org
- https://scholar.google.com
- https://books.google.com

**Module 1: Propositional Calculus I**

Unit 1: Meaning of propositional variables, Propositional constants and Logical Connectives

Unit 2: Symbolising more Complex Propositions

Unit 3: Truth-Conditions and Truth-Tables

Unit 4: Tautology, Contradiction and Contingent Truths

## Unit 1: Meaning of Propositional Variables, Propositional Constants and Logical Connectives



### 1.1 Introduction

Dear students, welcome to the first part of the course PHL301. We shall begin our journey into symbolic logic with the ideas of propositional variable and propositional constants. These are important to aid our understanding of what is to be encountered ahead. In this part of Symbolic Logic, we shall investigate the validity and invalidity of arguments or argument-forms which depend on combining atomic or simple propositions into compound propositions to form arguments. Such compounds are called truth-functional compounds because their truth or falsity depends on (or is a function of) the truth or falsity of their constituent propositions. There are broadly three types of symbols needed to explore the principles and techniques of testing the validity or invalidity of arguments in propositional calculus.

### 1.2 Learning Outcomes

By the end of this unit, learners should be able to:

1. Understand and explain the meaning of propositional variables
2. Discuss the idea of propositional constants
3. Identify and be able to apply the logical connectives

### 1.3 Propositional Variables, Propositional Constants and Logical Connectives

**Propositional Variables:** In Symbolic Logic, we want to be able to talk, not only about specific arguments, but also about argument-forms. We use propositional variables to talk about any proposition whatsoever, and about the arguments of which they are constituent parts. We shall employ lower-case letters ‘p’ to ‘z’ as variables. Each variable may be used to stand for any proposition whatsoever, provided that no letter shall be taken to stand for more than one proposition in the same context. Thus, we may talk about any two propositions, ‘p’ and ‘q’, or about any three propositions, ‘p’, ‘q’, and ‘r’, and so on.

**Propositional Constants:** Our concern with the validity or invalidity of specific arguments makes it necessary for us to have a way of representing specific propositions which make up arguments. We shall represent every distinct simple or atomic proposition with any upper-case letter from ‘A’ to ‘Z’. We are free to represent any proposition with any letter, provided that, in the same context,

(i) the same proposition is not represented by two letters, and

(ii) two distinct propositions are not represented by the same letter.

Thus, the distinct simple propositions in the compound proposition,
It is raining and the sun is shining
may be represented as follows:
R: It is raining
S: The sun is shining
The compound thus becomes, in part,

R and S

Propositional variables and propositional constants are together called propositional letters.

**Logical Connectives** (also called logical constants): There are five more or less basic logical connectives or constants, namely:

(i) ‘~’ (called ‘curl’ or ‘tilde’ or ‘wave’), for the negation word, ‘not’ or its equivalents;

(ii) ‘.’ (called ‘dot’), for the conjunction word ‘and’ or its equivalents

(iii) ‘v’ (lower-case letter, ‘v’, called ‘vee’ ‘wedge’ or ‘vel’), for the disjunction word ‘or’ or its equivalents;

(iv) ‘⊃’ (called ‘horse shoe’), for the conditional phrase, ‘if… then’ or its equivalents, and

(v) ‘≡’ (called ‘three bars’ or ‘triple bar’), for the biconditional phrase, “…if and only if...’, or its equivalents.

These connectives or constants, as the names indicate, are used in connecting propositional letters, whether variables or constants.

Self-Assessment Exercise

1. The sign for disjunction is which of these? (a) – (b) v (c) € (d) N

2. “The Ground is Wet *and* Rain did not fall” is best captured as: (a) G v F (b) F v G (c) G. F (d) G. -F

**1.4**

the relevant connectives that are central to them. We have considered the idea of propositional constant and variables as well.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic*. Ibadan: Book Wright Nigeria Publishers

## 1.6 Possible Answers to SAE

1. (b); 2. (d)

## Unit 2: Symbolising Propositions



### 2.1 Introduction

In this unit, we are going to learn how to symbolize propositions following what we have learned from the previous unit. What then do we mean by symbolize? How can this be attained?

### 2.2 Learning Outcomes

After our study in this unit, learners will be able to:

2.2.1 Symbolise propositions
2.2.2 Understand rules involved in the connectives during symbolising

### 2.3 Symbolising Propositions

#### 2.3.1 Negation

To obtain the negation of a proposition, we place the negation sign, (that is '~') to the left side of the symbol of the propositional constant or variable. Thus, if we represent the sentence 'The sun is shining' with the upper-case letter 'S' each of the following sentences, which are all different ways of negating the sentence 'The sun is shining', will be represented as '~S':

1. *The sun* is *not* shining
2. *It is not the case that* the sun is shining
3. *It is false that* the sun is shining
4. *It is not true that* the sun is shining
5. The sentence, 'The sun is shining', *is false.*
6. The sentence, 'The sun is shining', *is not true*

### 2.3.2 Conjunction

To obtain the conjunction of two propositions, we place the conjunction sign, '.', in between the signs for the two propositions, called conjuncts. Thus, if we use the upper-case letter 'W' for the proposition 'Today is Wednesday' and the letter 'U' for the proposition 'We are at the National Open University, Abuja, the compound sentence (called a conjunction) "Today is Wednesday and we are at the National Open University, Abuja' will be symbolised as follows:

W. U

Similarly, the proposition:

Alhaji Shehu Shagari and General Ibrahim Babangida are former Heads of State of Nigeria

can be rewritten explicitly into a conjunction as follows:

Alhaji Shehu Shagari is a former Head of State of Nigeria and General Ibrahim Babangida is a former Head of State of Nigeria

If we use the letter 'S' to represent 'Alhaji Shehu Shagari is a former Head of State of Nigeria', and 'B' to represent 'General Ibrahim Babangida is a former Head of State of Nigeria'. The conjunction will be symbolised as follows:

S. B

However, though all conjunctions must contain the conjunction-word 'and' or any of its equivalents some propositions containing such conjunction-words are not conjunctions. For example, the proposition:

*Obafemi Awolowo and Nnamdi Azikiwe were contemporaries*

is not a conjunction, for the reason that the proposition cannot be rewritten as an explicit conjunction. This is because the sentence expresses a relation of being contemporaries between Awolowo and Azikiwe; it is thus a relational proposition.

Similarly, the sentence:

*The Green Eagles lost and the spectators went on the rampage*

is not a genuine logical conjunction, in spite of the fact that it contains the conjunction-word 'and'. The reason is that the sentence contains a time-order which is essential to the meaning of the whole sentence, thus making it impossible to commute the conjuncts and retain the original meaning of the sentence. For, to say

*The spectators went on the rampages and the Green Eagles lost*

suggests a different time-order from the original sentence. Finally, the following sentence may not be straightforwardly interpreted as a conjunction.

*The population of Lagos is larger than those of Oyo and Ogun States.*

This is because the sentence may mean,

*The population of Lagos is larger than that of Oyo State, and the population of Lagos is larger than that of Ogun State.*

in which case it is a conjunction. However, it may also be interpreted as

*The population of Lagos is larger than those of Oyo and Ogun States put together*

which is not a conjunction. This means that the original sentence is ambiguous.

To sum up, for a sentence to be a genuine conjunction,

1. it must be capable of being rewritten as an explicit conjunction;
2. its conjuncts must be commutable without change in meaning, and
3. if the sentence is ambiguous, then one of its interpretations may be a genuine conjunction.

'But', 'although', 'both' 'and', however', 'not only…but also…, 'despite', 'yet' 'while', 'albeit', are all possible synonyms for 'and'.

### 2.3.3 Disjunction

To symbolise a disjunction of two propositions, we place the disjunction sign, 'v', in between the signs for the two propositions, called disjuncts. Thus, if we use the upper-case letter 'I' to represent the sentence, 'Ibadan is the capital of Oyo State' and use the letter 'O' for 'Ogbomoso will be the capital of new Oyo State', the disjunction:

*Ibadan is the capital of Oyo State or Ogbomoso will be the capital of new Oyo State*

will be represented as follows:

I v O

Similarly, the proposition: 'Oyo State University of Technology will be located in Ibadan or Ogbomoso' can be written out explicitly into a disjunction as

*Oyo State University of Technology will be located in Ibadan or Oyo State University of Technology will be located in Ogbomoso*

If we use 'I' represent*: Oyo State University of Technology will be located in Ibadan* and 'O' to represent: *Oyo State University of Technology will be located in Ogbomoso* the disjunction will be symbolised as follows:

I v O

Again, the proposition,

*You will not be issued a passport unless you submit your application on time.*

can be taken to express the following explicit disjunction:

*Either you submit your application on time or you will not be issued a passport.*

If we use 'S' to represent 'You submit your application on time' and use 'I' to represent 'You will be issued a passport', the disjunction will be symbolised as follows:

S v ~ I

(if we add the negation sign '~', to 'I').
You must note that 'either or', 'except', 'unless' and 'or else' are synonyms for 'or'.

### 2.3.4 Conditional

A conditional is a compound proposition consisting of two simple or atomic propositions joined together by the phrase 'if…then…' or its equivalent. The part of the conditional before 'then' (or immediately after 'if') is called the antecedent, while the part after 'then' is called the consequent. To symbolise a conditional, we place the conditional sign '⊃' in between the letters for the antecedent and the consequent of the conditional. Thus, if we use the upper-case letter 'A' for 'If there is a United States of Africa,' and use the letter 'M' for 'It will become a permanent member of the Security Council of the United Nations', the conditional:

> *If there is a United States of Africa, then it will become a permanent member of the Security Council of the United Nations.*

This will be represented as: $A \supset M$

The same expression will be used for each of the following propositions, which mean roughly the same as the original:

1. A only if M
2. M if A
3. A implies M
4. A is a sufficient condition for M
5. M is a necessary condition for A
6. Only if M, A
7. M provided that A
8. M in case A
9. M when A

### 2.3.5 Biconditional

A biconditional is a compound proposition consisting of two simple or atomic propositions joined together by the biconditional phrase '…if and only if…' or its equivalent. The parts of the biconditional are called components. To symbolise a biconditional, we place the biconditional sign '≡' in between the letters for the components of the biconditional. Thus, if we use the letter 'S' for 'Southern Sudan will know peace' and the letter 'E' for the proposition, 'UN peace-keeping operation is effective', the biconditional proposition,

*Southern Sudan will know peace if and only if UN peace keeping is effective,* will be represented as follows:

$$S \equiv E$$

The same expression will be used in representing each of the following propositions.

1. UN peace-keeping operation being effective is both a necessary and sufficient condition for Southern Sudan to know peace.
2. If UN peace-keeping is effective, then Southern Sudan will know peace, and if Southern Sudan is to know peace, then UN peace-keeping operation must be effective.
3. Southern Sudan knowing peace implies and is implied by UN peace-keeping operation being effective.
4. Southern Sudan knowing peace entails and is entailed by UN peace-keeping operation being effective.

**2.3.6 Symbolising more complex propositions**

It is obvious that the above examples of symbolised expressions represent 'basic' forms of negation, conjunction, disjunction, conditional and biconditional. Propositions can be more complex than these, and our symbolic apparatus is able to cope with expressions with varying degrees of complexity. It must be stressed that converting propositions in natural language into symbols is not a mechanical process, and it takes some ingenuity to capture the sense of the original in symbols. For example, let us symbolise the following proposition.

(1) If either the government relents or <u>News watch</u> wins her case, then the people will be happy.

Let us begin by determining how many component propositions are in the compound proposition. For each component proposition, let us have a separate letter of the alphabet. Note that no letter may be used for more than one atomic or simple proposition in the same context, and the same proposition may not be assigned two different letters. It is useful to first build up a 'dictionary' indicating which letter represents which atomic proposition. Thus, we will have:

G: Government relents
N: Newswatch wins her case
P: The people will be happy

(Note that the choice of letters is quite arbitrary, provided only that once a letter is chosen to represent a simple proposition, the letter is used throughout the context for that proposition. However, as in the above example, the choice of letters may be based on a key word in the proposition or on some other criterion).

Next, let us proceed step-by-step, as it were, putting the propositional letters in place, and then the connectives, in the order of their scope; thus

1. If either G or N, then P
2. If G v N, then P
3. G v N ⊃ P

However, the resulting formula in (3), as it stands, is ambiguous. For one thing, it is not clear if the whole expression is a disjunction or a conditional. So, we need to introduce 'punctuation marks' to eliminate the ambiguity. The punctuation marks consist of parentheses '()', brackets '[]' and braces '{}'. Once such punctuation marks are introduced, the above formula (or schema) becomes

(G v N) ⊃ P

which shows clearly that the original proposition is a conditional. Talking of the scope of connectives, we can see that in the above schema, the conditional sign '⊃' has a larger scope than the disjunction sign 'v', The scope of a connective depends on the part(s) of the expression it governs. In the above expression, 'v' governs only the antecedent, whereas '⊃' governs the whole expression.

Let us attempt to symbolise the following more complex proposition:

> On the one hand, either old age is valued or we fail to respect our elders and should suffer the same fate; or, on the other hand, it is not the case that old age is valued and experience respected.

We begin by formulating a dictionary as follows:

O: Old age is valued
F: We fail to respect our elders
S: We should suffer the same fate
E: Experience is respected

To proceed, let us introduce the letters, then the connectives in the order of their scope, thus:

1. On the one hand, either O or F and S; or, on the other hand, it is not the case that O and E.
2. On the one hand, O v F. S; or, on the other hand, it is not the case that O. E.
3. On the one hand, O v F. S; or, on the other hand ~ O. E
4. O v F. S v ~O. E

Quite clearly, the formula, as it stands, is ambiguous. But from the way we have proceeded, it is obvious that the whole expression is a disjunction and the connective with the largest scope is the second 'v' from the left. Once we introduce the necessary punctuation, the expression becomes:

[O v (F. S)] v ~(O. E)

### 2.3.7 Well-formed formulas

All the above symbolised expressions represent well-formed formulas or schemata. It is important to stress that:

1. A negation is well-formed if and only if the negation sign '~', is placed to the left of the negated expression. Otherwise, it is not well formed. Using propositional

variables to illustrate, the negation of ‘p’ is ‘~p’, the negation of ‘p. q’ is ‘~(p. q)’, the negation of ‘p v q’ is ‘~(p v q)’, the negation of ‘p ⊃ q’ is ‘~(p ⊃ q)’ and the negation ‘p ≡ q’ is ‘~(p ≡ q)’.

2. A conjunction, disjunction, conditional or biconditional is well-formed if and only if the connective is placed between the components of the conjunction, disjunction, conditional or biconditional. Otherwise, it is not well-formed. Using propositional variables, the conjunction of ‘p’ and ‘q’ is ‘p. q’. Similarly, the disjunction of ‘p’ and ‘q’ is ‘p v q’, the conditional of ‘p’ and ‘q’ is ‘p ⊃ q’, and the biconditional of ‘p’ and ‘q’ is ‘p ≡ q’. Similarly the conjunction of ‘p . q’ and ‘r . s’ is ‘(p . q). (r . s)’ and the conditional of ‘(p . q)’ and ‘(r . s)’ is ‘(p . q) ⊃ (r . s)’, and so on.
3. An expression involving punctuation is well-formed if and only if both parts of the punctuation are introduced. Otherwise, it is not well-formed. Thus, for example, all the following expressions are well-formed:

   (p ⊃ q) . [(r ⊃ s) ⊃ (p v r)]
   [(p . q) . (r v s)] ⊃ p
   {[(p ⊃ q) . (r ⊃ s)]. (p v r)} ⊃ (q v s)

However, all the following schemata are not well-formed:

(p ⊃ q ⊃ r)
p ⊃ (q ⊃ r)] ⊃ (p v q)
[p ⊃ (q . r)] ⊃ {[q ⊃ (s . t)] ⊃ (p ⊃ s)

for the simple reason that the parentheses, brackets and braces are incomplete.

## 2.4 Summary

In this unit, we have been able to engage in using connectives to symbolise. We can already see that connectives play a crucial role in the ways that propositions are symbolised and understood.

## 2.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 2.6 Unit Exercises

**I.** Symbolise the following statements, using the dictionary provided:

N – NITEL's equipment problem worsens.

P – NEPA raises charges.

C – ACB raises charges.

S – NIPOST requests more Federal Government aid.

W – Nigeria Airways buys five more planes.

1. NEPA raises charges but ACB does not raise charges.
2. Either NEPA or ACB both raise charges.
3. NEPA and ACB both raise charges.
4. NEPA and ACB do not both raise charges.
5. NEPA and ACB both do not raise charges.
6. NEPA or ACB raises charges, but they do not both do so.
7. Nigeria Airways buys five more planes and either NEPA raises charges or NIPOST requests more Federal Government aid.
8. It is not the case that either NITEL's equipment problem worsens or NIPOST requests more Federal Government aid.

**II.** Symbolise the following statements, using the dictionary provided below:

L – Leventis team manager is angry.

F – The fans are bewildered.

R – Rangers team is declared the winner.

1. Rangers team is not declared the winner, and either Leventis team manager is angry and the fans are bewildered.
2. Rangers team is declared the winner if and only if Leventis team manager is not angry and the fans are not bewildered.
3. Either Leventis team manager is angry and the fans are not bewildered, or he is not angry and Rangers team is not declared the winner.
4. If Leventis team manager is angry, then either the fans are bewildered or Rangers team is not declared the winner.
5. If Leventis team manager is angry, then the fans are bewildered and if he is not angry, Rangers team is declared the winner.

**III.** Symbolise the following compound statements using letters as abbreviations for simple proposition:

1. If all blacks have everything in common, then one of you can be no richer than the other, if you say truly that you are all blacks.
2. If your wife and children love you and wish that you should be happy, no one can doubt that they are very ready to promote your happiness.
3. It is not the case that the President will introduce a decree concerning S. A. P. protests; moreover, it is also not the case that the A. F. R. C. will pass the decree.

4. If Nduka Odizor enters the tennis match, he is sure to win; although he will enter if and only if the prize money exceeds 5 million naira.
5. Either Segun Odegbami is the captain or if Kadiri Ikana is the captain then Ohenhen Ogboe is not the captain.
6. If either Shagari or Awolowo is the President and Awolowo is not, then if Shagari is the President, then Awolowo is not and Shagari is not.

## Unit 3: Truth-Conditions and Truth-Tables



### 3.1 Introduction

Compound (or molecular) propositions are also called truth-functional molecules or compounds. An expression is truth-functional if its truth-value (that is, its truth or falsity) depends on, or is determined by, the truth-values of its constituent elements. In other words, the truth-value of a truth-functional compound is a *function* of the truth-values of its constituent elements. For example, the truth-value of the propositional function 'p . q', which is a conjunction, is a function of the truth-values of the variables 'p' and 'q'. In this sense, negation, conjunction, disjunction conditional and biconditional are truth-functional. Let us now describe their *truth-conditions*, that is, the conditions under which they are true or false.

Be it noted that in this course, it is assumed that every simple proposition is either true or false, so the course can also be called two-valued logic. In real life, however, we do know that some sentences are true and some are false, while some sentences are neither. However, the principles and techniques developed in the course are designed to deal with only sentences which are assumed to be either true or false.

### 3.2 Learning Outcomes

By the end of this unit, learners should be able to:

1. Use the truth-table
2. Apply the rules of getting valid arguments from the truth-tables

### 3.3.1 Truth-Conditions and Truth-Tables

The truth-conditions will be exhibited in an array of T's ('T' for 'true') and F's ('F' for 'false') called the truth-table. Where there is only one component or atom, say, 'p', the array will have only two (vertical) rows, since 'p' has either the truth-value 'T' or the truth-value 'F', thus:

| p |
|---|
| T |
| F |

Where there are two components, say, ‘p’ and ‘q’, the truth-table will have four (vertical) rows. This is because there are four possible combinations of truth-values, namely, one where both components are true, another where both components are false, and two where one component is true and the other false, as follows:

| p | q |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

(1) **Negation**

Consider the proposition:

Chief Olusegun Obasanjo is the President of the Federal Republic of Nigeria

The statement can be negated in a number of ways, but let us do it in the simplest way, thus:

Chief Olusegun Obasanjo is *not* the President of the Federal Republic of Nigeria.

If our original proposition is true, then obviously its negation is false, and if the original sentence is false then its negation is true. This observation can be generalised by saying that if a proposition is true, then its negation is false, and if a proposition is false, then its negation is true. This position can be represented schematically as follows:

| p | ~p |
|---|---|
| T | F |
| F | T |

(2) **Conjunction**

Consider the conjunction

Olusegun Obasanjo is a retired army general and Theophilus Danjuma is a retired army general

Now, if we were to ask you to accept that the above compound proposition is true, you would do so only if both were indeed retired army generals, that is, if it is true both that Obasanjo is a retired army general and that Danjuma is a retired army general. From this, we can generalise that a conjunction is true if and only if both conjuncts are true; it is false if any or both conjuncts are false. Thus, the conjunction is false if at least one of the conjuncts is false. This result can be schematically represented as follows, taking any two propositions, say, ‘p’ and ‘q’:

| p | q | p . q |
|---|---|---|

| | | |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

(3) **Disjunction**

Consider the compound propositions:

1. Either the President is in Lagos or he is in Abuja
2. Either the First Lady will watch the movie or go to sleep.

Though both propositions are disjunctions, there seems to be a subtle difference between the two. Thus in (1), since the president cannot be in Lagos and Abuja at the same time, it is obvious that both propositions cannot be true at any point in time. Ordinarily, proposition (1) will be considered true if the President is in Lagos or if the President is in Abuja; it will be false only if the president is neither in Lagos nor in Abuja. Similarly proposition (2) will be considered true if the First Lady watches the movie or if she goes to sleep. It will be considered false only if she does neither. In (2), however, it is indeed possible for the First Lady to watch the movie *and* go to sleep, or not to watch movie *but* go to sleep. This means that it is possible for both disjuncts to be true. This latter use of 'or' is said to exhibit the *weak*, *inclusive* or *non-exclusive* sense of it. The former is said to be the *strong*, *exclusive* or *non-inclusive* sense of it. For our present purposes, the truth-functional 'or' will be understood in the *non-exclusive* sense, since it sufficiently captures the other sense. Moreover, an *exclusive* disjunction can be expressed by such phrases as 'but not both'. In the *non-exclusive* sense, a disjunction is true if at least one of the disjuncts is true; it is false only if both disjuncts are false. This situation can be schematically represented as follows:

| p | q | p v q |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

(4) **Conditional**

Consider the following propositions which contain the expression 'if …then…'

1. If all human beings are mortal beings, and all Nigerians are human beings, then all Nigerians are mortal beings.
2. If Garubada is a bachelor, then he is unmarried.
3. If blue litmus paper is placed in acid, then the paper will turn red.
4. If the Green Eagles win the World Cup, the government will give each of the players a house in Abuja.

5. If it is raining then there is a rainbow.

Though all the above sentences are conditional in form there are subtle differences between them: (1) describes a logical implication, where the antecedent contains the premises, and the consequent represents the conclusion; and (4) describes a decision or resolution; (5) is more difficult to characterise. However, a close perusal of the various uses of the connective ‘if…then…’ in the above examples will show that the minimum claim being made in each is that if the antecedent is true then the consequent is also true. In other words, it is not the case that the antecedent is true and the consequent false. Thus, for example, the government will be liable to being called dishonourable only if the Green Eagles win the World Cup and it did not give the players the promised house, that is, if the antecedent of its promise is true and the consequent is false. That is why the conditional is sometimes said to be the conditional affirmation of the consequent. It is this minimum meaning that is given to ‘if…then…’ This has the advantage of enabling us to deal uniformly and unambiguously will all expressions containing ‘if…then…’; this meaning is called *material implication*. Thus, to say that ‘p materially implies q’ simply means that it is not the case that ‘p’ is true and ‘q’ is false, which means that ‘p ⊃ q’ is false only if ‘p’ is true and ‘q’ is false. This position can be schematically represented as follows:

| p | q | p ⊃ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

(5) **Biconditional**

The biconditional, as the name indicates, consists of two conditional propositions. Thus, for example, the biconditional.

There will be peace in South Sudan if and only if the rebels are defeated militarily

can be broken up into two conditional propositions, as follows:

1. There will be peace in South Sudan if the rebels are defeated militarily.
2. There will be peace in South Sudan only if the rebels are defeated militarily.

Rewritten in ‘standard form’, they become,

1. If the rebels are defeated militarily then there will be peace in South Sudan.
2. If there will be peace in South Sudan then the rebels are defeated militarily.

‘There will be peace in South Sudan if and only if the rebels are defeated militarily’ is therefore a shorthand way of writing the conditional propositions in (1) and (2) above. If we use ‘S’ to represent: ‘There will be peace in South Sudan’ and we use ‘R’ to represent

‘The rebels are defeated militarily’, we have the result that ‘S ≡ R’ which is a shorthand way of expressing ‘(S ⊃ R) . (R ⊃ S)’.

We can now generalise this situation by saying that the biconditional of any two propositions, ‘p’ and ‘q’, namely, ‘p ≡ q’, is a shorthand way of expressing two conditional propositions involving ‘p’ and ‘q’, namely, ‘p ⊃ q’ and ‘q ⊃ p’. Similarly, the truth-conditions of ‘p ≡ q’ are derived from the truth-conditions of ‘(p ⊃ q) . (q ⊃ p)’, thus:

| p | q | (p ⊃ q) | . | (q ⊃ p) |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | F |
| F | F | T | T | T |

from which we know that ‘p ≡ q’ is true either if both ‘p’ and ‘q’ are true or if both ‘p’ and ‘q’ are false; it is otherwise false. That is to say, the biconditional is true if ‘p’ and ‘q’ have the same truth value and false if ‘p’ and ‘q’ have different truth-values. This result can be schematically represented as follows:

| p | q | p ≡ q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

In other words, if one truthfully asserts a biconditional proposition, then one is claiming that the propositions biconditionally asserted are either both true or both false.

### 3.3.2 Uses of the Truth-Table

The truth-table, an array of T’s and F’s, has many uses in truth-functional logic. We have already seen one of its uses, namely, to exhibit the truth-conditions of our truth-functional connectives. Let us note the patterns that emerge from that exercise. We saw that when there is only one component in the formula or expression, as in a negation, the truth-table has only two vertical rows. That is because the single component has only two possible truth-values, ‘true’ and ‘false’, thus:

| P |
|---|
| T |
| F |

However, where we have two distinct components, as in a *conjunction, disjunction,* etc., the vertical rows become four, indicating all the possible combinations of truth-values that the two components can have. Thus, if the components are ‘p’ and ‘q’, it is possible for both components to be true, for both to be false, or for one to be false and the other true, thus:

| p | q |
|---|---|
| T | T |
| T | F |
| F | T |
| F | F |

This gives the formula that to determine the correct number of vertical rows, we raise the number 2 to the power of the number of distinct components. Thus, if we have one distinct component, we get $2^1$, i.e., 2 x 1 = 2 rows. If we have two distinct components, we get $2^2$ = 2 x 2 = 4 rows. Similarly, if we have three distinct components, say, 'p', 'q' and 'r', we get $2^3$ = 2 x 2 x 2 = 8 rows, thus:

| p | q | r |
|---|---|---|
| T | T | T |
| T | T | F |
| T | F | T |
| T | F | F |
| F | T | T |
| F | T | F |
| F | F | T |
| F | F | F |

In the eight rows, we have one (horizontal) where all the components are true and another where all the components are false; the other rows have various combinations of T's and F's. But here, too, a pattern emerges: the first (vertical) row from the right takes one 'T' and one 'F' alternately until we have the eight rows; the second row takes double that, that is, four T's and four F's alternately. The same principles apply if we have four or more components. for four distinct components, say 'p', 'q', 'r' and 's', we have $2^4$ = 2 x 2 x 2 x 2 = 16rows; the first (vertical) row from the right takes one 'T' and one 'F' alternately, the second two T's two F's alternately, the third, four T's and four F's alternately and the last eight T's and F's alternately. Lastly, if we have $2^5$ = 2 x 2 x 2 x 2 x 2 = 32rows arranged in a similar fashion, it is obvious that beyond five components the truth-table becomes very tedious. This fact points to the limit of the usefulness of the truth-table. Now, once values have been thus assigned to every component, it would be time to assign values to the connectives in accordance with the truth-conditions discussed earlier. Further, in assigning values to connectives, we begin with the connective with the smallest scope, and end with the connective with the largest scope. We shall see what this amounts to in practice, in the next unit.

## 1.5 Summary

In this unit, we have increased our knowledge on logic. We have looked at the rules of each of the connectives and how they can be used to determine truth value of propositions. The next unit will provide more information over how this can be done.

## 1.6 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 1.7 Unit Exercises

I. Which of the following statements are true and which are false?

1. Chinua Achebe is the author of *Things Fall Apart* v Chinua Achebe is the author of *The Man Died.*
2. ~Chinua Achebe is the author of *The Man Died* v ~Abuja is the new federal capital of Nigeria.

*3. Accra is the capital of Benin Republic v ~(Shakespeare is the author of *Julius Caesar.* Accra is the capital of Benin Republic.

4. ~[~(Ibadan is the largest city in Nigeria v Lagos is an industrial area) v (Enugu is a coal city. ~Accra is the capital of Benin Republic)]
5. (Ibadan is the largest city in Nigeria v ~Kano State is an oil-producing state) v ~(~Ibadan is the largest city in Nigeria. ~Kano State is an oil-producing state)

II. If P . Q and R are true statements, and S and T are false statements which of the following are true?

1. ~P v Q
2. (P . S) v (Q . T )
3. ~(S v T) . (~S v T)
4. ~(S . ~T) v (Q . ~R)
5. ~[(P . Q) v ~(Q . P)]
6. [(P v Q) . (~R v S)] . (P v ~S)
7. ~[(P . Q) v ~(Q . P)]
8. [(P . Q) v R] ⊃ T.
9. ~[(R . ~T) v T] ⊃ (Q . ~T)
10. [(P . ~R) ⊃ S] ⊃ T
11. P ≡ [(P . R) v S]
12. R ⊃ [(S ≡ T) . (R v Q)]

## Unit 4: Tautology, Contradiction and Contingent Truths



### 4.1 Introduction

The focus of this unit is to examine the three possible ways of determining truth values of propositions. What are these three ways? These three ways are the title of this unit. We shall then consider the conditions that will make truth-values to be situated into any of these three.

### 4.2 Learning Outcomes

After our study in this unit learners will be able to:

1. Explain truth-table technique
2. Identify and situate truth-values into any of tautology, contradiction and contingent truths

### 4.3 Tautology, Contradiction and Contingent Truths

The truth-table technique may be used to determine whether a statement or statement-form is a tautology, a contradiction or a contingent truth. For example, take the statement.

> Either Ibadan is the capital of Oyo State or Ibadan is not the capital of Oyo State.

If 'I' is used for 'Ibadan is the capital of Oyo State', this comes to:

$$I \vee \sim I$$

Its truth-table is:

I    I    v    ~ I

This shows that the statement is always true. A statement or statement-form which comes out true under all interpretations is said to be a *tautology* (or to be *tautologous*). Let us take a slightly more complex example. The sentence:

> If Ibadan is the capital of Oyo State and Abuja is the capital of Nigeria, then Ibadan is the capital of Oyo State.

when symbolised, becomes:

$$(I \cdot A) \supset I$$

Its truth-table is:

| I | A | (I . A) ⊃ I |
|---|---|---|
| T | T | T   T |
| T | F | F   T |
| F | T | F   T |
| F | F | F   T |

which shows that the statement is tautologous. Lastly, let us use the tautologous statement-form:

~ (p v q) ≡ (~p . ~q)

Whose truth-table is

| p | q | ~(p v q) ≡ (~p . ~q) |
|---|---|---|
| T | T | F   T   T   F |
| T | F | F   T   T   F |
| F | T | F   T   T   F |
| F | F | T   F   T   T |

The truth-table shows that the statement-form is a tautology.

**Contradiction:** Let us take the following statement:

John is tall and John is not tall

Which, symbolised, becomes:

J . ~J

Its truth-table, which follows:

| J | J . ~J |
|---|---|
| T | F |
| F | F |

Shows that the statement is always false. A statement (or statement-form) which comes out false under all interpretations is called a *contradiction* (or is said to be *contradictory*). Let us take another examples;

> It is false to say that if Ibadan is the capital of Oyo State and Abuja is the capital of Nigeria, then Abuja is the capital of Nigeria.

Symbolised, it becomes:

~ [(I . A) ⊃ A]

and the truth-table is as follows:

| I | A | ~ [(I . A) ⊃ A] |
|---|---|---|
| T | T | F   T   T |
| T | F | F   F   T |

| | | | | |
|---|---|---|---|---|
| F | T | F | F | T |
| F | F | F | F | T |

This shows that the statement is a contradiction. Lastly, let us show that the statement-form '~[~ (p v q) ≡ (~p . ~q)]' is contradictory, thus:

| p . q | | ~ [~ (p v q) ≡ (~p . ~q)] | | | | |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | F |
| T | F | F | F | T | T | F |
| F | T | F | F | T | T | F |
| F | F | F | T | F | T | T |

**Contingent Truth**: Let us take the statement

If Ibadan is the capital of Oyo State, then Abuja is the capital of Nigeria

Which, symoblised, becomes:

$$I \supset A$$

Its truth-table which follows:

| I | A | I ⊃ A |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

shows that the statement is neither a tautology nor a contradiction, since it is neither always true, nor always false. A statement or statement-form which comes out true under some interpretations and false under some interpretations is said to be *contingent*. Let us take a slightly more complex example

> If Ibadan is the capital of Oyo State and Abuja is the capital of Nigeria then Buhari is the President of the Federal Republic of Nigeria

which, symbolised, becomes

$$(I \,.\, A) \supset B$$

Its truth-table which follows,

| I | A | B | (I . A) | ⊃ B |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | T | F |
| T | F | T | F | T |

| | | | | |
|---|---|---|---|---|
| T | F | F | F | T |
| F | T | T | F | T |
| F | T | F | F | T |
| F | F | T | F | T |
| F | F | F | F | T |

shows that the statement is contingent. Lastly, let us show the contingent nature of the statement-form, '~ (p v q) ≡ (~p v ~q)' with the following truth-table:

| p | q | ~ (p v q) | | ≡ | (~p v ~q) | |
|---|---|---|---|---|---|---|
| T | T | F | T | | T | F |
| T | F | F | T | | F | T |
| F | T | F | T | | F | T |

**4.3 2 Logical Equivalence**

The truth-table technique can also be used to determine whether or not two expressions are logically equivalent. The sign for the biconditional is also used to indicate logical equivalence. If two statements or statement-forms are claimed to be logically equivalent, the claim can be tested by joining the expressions together using the biconditional sign and working out the truth-table. For example, let us test the claim that the statement.

If you do not pay on time then you will not get a discount

(symbolised: ~P ⊃ ~D)

is equivalent to the statement:

Either you pay on time or you will not get a discount

(symbolised: P v ~D)

To test this claim, we write out the two statements as a biconditional, thus:

(~P ⊃ ~D) ≡ (P v ~D)

The truth-table is:

| P | D | (~P ⊃ ~D) | ≡ | (P v ~D) |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | T | T |
| F | T | F | T | F |
| F | F | T | T | T |

As the statement of the equivalence using the biconditional sign comes out true under all interpretations, the two statements are shown to be *logically equivalent.* Thus, we can generalise by saying that if two statements are logically equivalent then the statement of their equivalence will be a tautology.

Let us see if p v (q v r)' is logically equivalent to '(p v q) v r':

| p | q | r | [p v (q v r)] ≡ [(p v q) v r] | | | | |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T |
| T | F | T | T | T | T | T | T |
| T | F | F | T | F | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | T | T | T |
| F | F | T | T | T | T | F | T |
| F | F | F | F | F | T | F | F |

Lastly, let us determine the following equivalence

| p | q | ~ (p . q) ≡ (~p . ~q) | | | |
|---|---|---|---|---|---|
| T | T | F | T | T | F |
| T | F | T | F | F | F |
| F | T | T | F | F | F |
| F | F | T | F | T | T |

This last truth-table shows that '~(p . q)' and '~p . ~q' are not logically equivalent, since the statement of their equivalence is not tautologous.

## 4.4 Summary

In this unit, we have learned about the truth-table and how to use it to ascertain truth-values. We have also looked at what condition propositions may be said to be logically equivalent.

## 4.5 References and Further Readings


Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers


## 4.6 Unit Exercises

Use truth-table to determine whether the following truth-functional propositions are tautologies, contradictions or contingents

1. A v (B ⊃ ~C)
2. A ⊃ [B ⊃ C)]
3. [(A . B] . C] v [(A . C) v (A . B)]
4. [A ⊃ (A ⊃ B)] ⊃ B

*5. A ⊃ [A ⊃ (B . ~B)]

6. (A ⊃ B) ≡ (~B ⊃~A)

7. A ≡ [A . (A v B)]
8. {[(A ⊃ B) . (C ⊃ D)] . (B v D)} ⊃ (A v C)
9. [(A ⊃ B) ⊃ C] ≡ [(B ⊃ A) ⊃ C]
10. [A ⊃ (B ⊃ C)] ⊃ [(A ⊃ B) ⊃ (A ⊃ C)]

Determine by truth-tables whether the following pairs of propositions are equivalent:

1. ‘A . B’ and ‘~(~A v ~B)’
2. ‘A . ~B’ and ‘~(A ⊃ B)’
3. ‘~A v (B . C)’ and ‘A . (B ≡ C)’
4. ‘A ⊃ B’ and ‘(~B ⊃ ~A')
*5. ‘(A ⊃ B) . (B ⊃ C)’ and ‘~(C ⊃ A)’
6. ‘~A ⊃ B’ and ‘A ⊃ (A . B)’
7. ‘A ⊃ B’ and ‘(A v B) ≡ B’
8. ‘A . (B v C)’ and ‘(A . B) v (A . C)’
9. ‘(A ⊃ ~B) v A’ and ‘A v ~B’
*10. ‘A’ and ‘A . (A v B)’

## End of Module Tips

In summary, the following are points to note about logical truth, logical falsehood and, contingent truth:

1. A truth functional formula that comes out true under all interpretations of its propositional letters is logically true or a tautology.
2. A formula that comes out false under all interpretations is logically false or contradictory.
3. A formula that is neither logically true nor logically false is contingent.
4. A formula is logically true if and only if its negation is contradictory, and a formula is contradictory if and only if its negation is tautological.
5. A conjunction is tautological if and only if all its conjuncts are tautological.
6. If a conjunction is contingent then all its conjuncts are contingent.
7. There are contradictory conjunctions all of whose conjuncts are contingent.
8. A disjunction is logically true if at least one of its disjuncts is logically true.
9. There are logically true disjunctions none of whose disjuncts is logically true.
10. A disjunction is contingent if and only if at least one of its disjuncts is contingent.
11. If a conditional has a logically true consequent, then it is logically true.
12. If a conditional has a contradictory antecedent, then it is logically true.
13. If a logically true conditional has a logically true antecedent, then it has a logically true consequent.
14. If a logically true conditional has a contradictory consequent, then it has a contradictory antecedent.
15. Substitution of formulas for propositional letters preserves logical truth.
16. Substitution of formulas for propositional letters preserves logical falsehood.
17. A statement is logically true if it can be symbolised by a logically true formula.

All the above points can be proved to be true. Students are encouraged to attempt to prove them as exercises.

## Module 2: Propositional Calculus II

Unit 1: Validity/Invalidity of Arguments
Unit 2: Basic Valid Argument Forms
Unit 3: Rules of Inference
Unit 4: The Rule of Replacement
Unit 5: The Rule of Indirect Proof
Unit 6: Proof of Tautologies
Unit 7: Redundancy of Rules

## Unit 1: Validity/Invalidity of Arguments



### 1.1 Introduction

In this unit, we are going to focus on how to test the validity of propositions via the truth-table. This is in continuance of how the logical connectives and the rules they carry inform how truth-values are assigned. This unit also considers the method of *reductio ad absurdum*.

### 1.2 Learning Outcomes

By the end of this unit, it is assumed that learners should be able to:

1 Formulate means of testing validity via the truth-table
2 Understand the rules involved in testing of validity
3 Identify and apply the method of *reduction ad absurdum*

### 1.3.1 Testing of Validity

The truth-table technique can also be used in testing the validity or invalidity of truth-functional arguments or argument-forms. The procedure is based on the definition that an argument is valid if it is not possible for its premises to be true and its conclusion false. To test a truth-functional argument for validity, using this method, we write out the argument in a horizontal line, starting with the premise or premises and ending with the conclusion. After assigning values to every component and connective, we inspect the array to see if there is any horizontal row where the premise come out true, and the conclusion comes out false. If there is no such row, then the argument is valid, if there is, then the argument is invalid.

Let us look at the following argument:

> If either English Language is required or Mathematics is required, then all students will study Mathematics. Mathematics is required and English Language is required. Therefore, all students will study Mathematics.

Symbolised, we have

$$(E \lor M) \supset S$$

M . E

∴ S

To test the argument for validity, we have:

| E | M | S | (E v M) | ⊃ S | | M . E S |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | T | F |
| T | F | T | T | T | F | T |
| T | F | F | T | F | F | F |
| F | T | T | T | T | F | T |
| F | T | F | T | F | F | F |
| F | F | T | F | T | F | T |
| F | F | F | F | T | F | F |

On inspection, we find that there is no (horizontal) row where the premises are true and the conclusion is false. Therefore, the argument is valid. Again, let us look at the argument:

> If Ayo is elected Head Boy, then either Tosin is elected Head Girl or Bunmi is elected Prefect. Tosin is elected Head Girl. Therefore, if Ayo is elected Head Boy then Bunmi is not elected Prefect.

Symbolised, we have

A ⊃ (T v B)

T

∴ A ⊃ ~B

To test for validity, we have:

| A | T | B | A ⊃ (T | v B) | T | A ⊃ ~B |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | F |
| T | T | F | T | T | T | T |
| T | F | T | T | T | F | F |
| T | F | F | F | F | F | T |
| F | T | T | T | T | T | T |
| F | T | F | T | T | T | T |
| F | F | T | T | T | F | T |
| F | F | F | T | F | F | T |

On inspection, we find that in the first row the premises are true and the conclusion is false, thus showing that the argument is invalid.

**Corresponding Conditional:** Another variety of this method consists in converting the argument into its corresponding conditional. To every argument there is a corresponding conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion of the argument. An argument is valid if its corresponding conditional is a tautology. Let us take the following argument:

> If he uses good bait, then if the fish are biting, then he catches the legal limit. He uses good bait, but he does not catch the legal limit. Therefore, the fish are not biting.

Symbolised, we have

$G \supset (F \supset L)$

$G \,.\, \sim L$

$\therefore \sim F$

Converted to its corresponding conditional the argument becomes:

$\{[G \supset (F \supset L)] \;.\; (G \,.\, \sim L)\} \supset \sim F$

The truth-table is:

| G | F | L | {[G ⊃ (F ⊃ L)] . (G . ~ L)} ⊃ ~ F | | | | | |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | T | F |
| T | T | F | F | F | F | T | T | F |
| T | F | T | T | T | F | F | T | T |
| T | F | F | T | T | T | T | T | T |
| F | T | T | T | T | F | F | T | F |
| F | T | F | T | F | F | F | T | F |
| F | F | T | T | T | F | F | T | T |
| F | F | F | T | T | F | F | T | T |

Since the conditional comes out true under all interpretations, it means that it is a tautology, and that, therefore, the argument is valid. Let us take another argument:

IF Abe wins first prize, then either Betty wins second prize or Cedy is disappointed. Betty does not win second prize. Therefore, if Cedy is disappointed, then Abe does not win first prize.

Symbolised, we have:

$A \supset (B \vee C)$

$\sim B$

$\therefore C \supset \sim A$

Converted into a conditional, the argument becomes

$\{[A \supset (B \vee C)] \,.\, \sim B\} \supset (C \supset \sim A)$

The truth-table is:

| A | B | C | {[A ⊃ (B v C )] | | . | ~ B} ⊃ | | (C ⊃ ~ A) |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | T | F |
| T | T | F | T | T | F | F | T | T |
| T | F | T | T | T | T | T | F | F |
| T | F | F | F | F | F | T | T | T |
| F | T | T | T | T | F | F | T | T |
| F | T | F | T | T | F | F | T | T |
| F | F | T | T | T | T | T | T | T |
| F | F | F | T | F | T | T | T | T |

Since the corresponding conditional is not tautologous, it means that the argument is not valid.

**Self-Assessment Exercise**

1. To every argument there is a corresponding conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion of the argument. **True or False**
2. Is it true that an argument is invalid if its corresponding conditional is a tautology?

### 1.3.2 *Reductio ad Absurdum*

The truth-table technique as presented above is a *decision-procedure*. This means that it can be used to decide, conclusively, whether or not an argument is valid. It can also be used to determine if a proposition is tautologous, contradictory, or contingent, it is also to determine whether or not two propositions are logically equivalent. However, the truth-table has the limitation that if the components in an expression or argument are more than five, the truth-table becomes unwieldy and time-consuming. However, another method has been devised to overcome this problem, and involves using a row or, at most, three rows of the truth table to determine the validity or invalidity of arguments, the logical equivalence of propositions. The method is called the short truth-table technique or *reduction ad absurdum*.

To determine validity or invalidity the method employs two principles that we have already established, namely:

(1) If an argument is valid, then there will be no (horizontal) row of the truth-table where the premises are true and the conclusion is false, and

(2) If an argument is valid, then its corresponding conditional will be a tautology.

To invoke the first principle

(i) we write out the premises and conclusion in a row;

(ii) assuming that the argument is invalid and that, therefore, the premises are true and the conclusion is false, we assign 'T' to each of the premises and 'F' to the conclusion;

(iii) we then try to make good our assumption that the argument is invalid by assigning a value to each of the remaining components or connectives, as appropriate;

(iv) finally, we inspect the resulting array to see if there is any inconsistency in our truth-assignments; if there is an inconsistency, then the argument is valid; if there is no inconsistency, then the argument is, indeed, invalid, as assumed.

For example, the argument:

(~A v B) . (A v C)]
~D ⊃ ~ C
∴ B v D

will be written out as follows:

(1) (~A v B) . (A v C) | ~ D ⊃ ~ C || B v D

Assuming that the argument is invalid and that, therefore, the premises are true and the conclusion is false, we have

(2) (~A v B) . (A v C) | ~ D ⊃ ~ C || B v D
T T T

We then try to make good our assumption by assigning values to the remaining components or connectives, thus:

(3) (~A v B) . (A v C) | ~ D ⊃ ~ C || B v D
T T F T T T F   T T T   F F F

Having assigned a value to every component and connective, we discover on inspection that the first premise, which is a conjunction has been made true only because of the inconsistency of assigning the same truth value, 'T' to both 'A' and '~A'. (This is inconsistent with our truth-conditions for negation where if a formula is true then its negation must be false). The argument is therefore valid since our assumption that it is invalid has been proven wrong by the inconsistency.

On the other hand, the following argument:

P ⊃ (Q v R)
R ⊃ (S . T)
~ S
∴ P ⊃ T

will, on our assumption, be assigned the values

P ⊃ (Q v R) | R ⊃ (S . T) | ~ S || P ⊃ T
T   T   T   F

Making good our assumption, we have

```
P ⊃ (Q v R)   R ⊃ (S . T)   ~ S    P ⊃ T
T T  T  T F   F T   F F T     T    T F  F
```

On inspection, we find that there is no inconsistency in our truth-assignments; therefore, the argument is, indeed, *invalid*, as assumed.

To invoke the second principle,

(i) we convert the argument into its corresponding conditional;

(ii) assuming that the argument is invalid and that, therefore, its corresponding conditional is not a tautology, we place an ‘F’ under the major connective for the conditional;

(iii) we then try to make good our assumption that the conditional is not a tautology, by assigning a value to every other connective or component, as appropriate;

(iv) we finally inspect the resulting array to see if there is any inconsistency in our truth-assignments; if there is an inconsistency then the argument is valid; if there is no inconsistency, then the argument is indeed invalid, as assumed.

Using the following argument:

P ⊃ (Q ⊃ R)
R ⊃ (S . T)
∴ P ⊃ (Q ⊃ S)

we have, as its corresponding conditional,

(1) {[P ⊃ (Q ⊃ R)] . [R ⊃ (S . T)]} ⊃ [P ⊃ (Q ⊃ S)]

Assuming that the argument is invalid and that, therefore, the corresponding conditional is not a tautology, we have

```
(2)  {[P ⊃ (Q ⊃ R)] . [R ⊃ (S . T)]} ⊃ [P ⊃ (Q ⊃ S)]
                                    F
```

We then try to make good our assumption that the expression is not a tautology and therefore has an F under the major connective, thus:

```
(3)  {[P ⊃ (Q ⊃ R)] . [R ⊃ (S . T)]} ⊃ [P ⊃ (Q ⊃ S)]
       T T   T T T    T T T  F T T    F  T F  T  F F
```

Having assigned a value to every connective and component, we discover on inspection that the second premise ‘R ⊃ (S . T)’ cannot be true, as indicated because the consequent ‘S.T’ cannot be true since one of the conjuncts ‘S’ is false. Since we have thus encountered an inconsistency, it means that we have failed to show that the corresponding conditional is not a tautology. The argument is, therefore, valid.

On the other hand, the following argument;

P ⊃ Q
R ⊃ S

Q v R
∴P v S

can be shown to be invalid, thus:

{[(P ⊃ Q) . (R ⊃ S)] . (Q v R)} ⊃ (P v S)
F T T T F T F T T T F F F F F

On inspection, we find no inconsistency in our truth-assignments, meaning that the argument is, indeed, invalid, as assumed.

Note that an inconsistency may arise in one of three ways: first, by assigning two truth-values to the same component in the same context; second, by assigning the same truth-value to a formula and its negation in the same context; and third, by assigning a truth-value to the connectives inconsistent with the truth-conditions discussed earlier, for example, to make a disjunction true though both disjuncts are false, or to make a conjunction true though one of its conjuncts is false, or to make a conditional true though the antecedent is true and the consequent false.

Note, further, that in all this, truth-values are not assigned arbitrarily. Thus, if a conditional is false, it is because its antecedent is true and the consequent false. Similarly, a conjunction is true only if both conjuncts are true, and for a disjunction to be true, at least one of its disjuncts must be true. In other words, in assigning truth-values we have to respect the truth-conditions of the connectives already discussed.

However, there is a problem with the short truth-table technique which we need to point out. The problem may arise if we have a conjunction or a biconditional as the conclusion of an invalid argument. The problem arises because in applying this method, the conclusion of the argument has to be assumed to be false, while the premises are true. Though, there is only one possible world where a disjunction or conditional is false, there are two possible worlds where a biconditional is false, and three possible worlds where a conjunction is false. In the following example, the argument though invalid, may be adjudged valid by the occurrence of an inconsistency because of an unlucky assignment of truth-values.

T ≡ U | U ≡ (V . W) | V ≡ (T v X) | T v X ‖ T . X

There will be an inconsistency if all the components are assigned the value ‘false’. However, there will be no inconsistency either if we make T, U, V and W true and X false or if we make T, U and W false and V and X true. Let us look at each possibility:

(1) T ≡ U | U ≡ (V . W) | V ≡ (T v X) | T v X ‖ T . X
F T F | F T F F F | F T F F F | F T F ‖ F F F

The last premises 'T v X' cannot be true, since it is a disjunction both of whose disjuncts are false. This is an inconsistency, giving the false impression that the argument is valid.

(2)

| T ≡ U | U ≡ (V . W) | V ≡ (T v X) | T v X | T . X |
|---|---|---|---|---|
| T T T | T T T T T | T T T T F | T T F | T F F |

However, in (2) there is no inconsistency, thus showing that the argument which appears valid in (1) is really invalid. For completeness let us look at the third possibility:

(3)

| T ≡ U | U ≡ (V . W) | V ≡ (T v X) | T v X | T . X |
|---|---|---|---|---|
| F T F | F T T F F | T T F T T | F T T | F F T |

This, again, contains no inconsistency and is therefore invalid. Incidentally, the argument is invalid, and the appearance of validity in (1) is deceptive. What this example shows is that where the conclusion which has to be false is a conjunction or a biconditional, great care must be taken in using the short truth-table. Trying other possibilities is clearly indicated here, thus requiring more than one row to prove invalidity of such arguments.

**Self-Assessment Exercise**

1. What method has been devised to overcome the limitations of the short truth value table?
2. Identify the ways by which inconsistencies may arise in truth-assignment.

## 1.5 Summary

In this unit, we have deepened our understanding of logic by focusing over how tests of validity on propositions that have been assigned truth-values may be done. The method of *reduction ad absurdum* has also been considered.

## 1.6 References and Further Readings


Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers


**Possible Answers to SAE**

**SAE (1)-** 1. Yes

2. No

**SAE (2) -**1. *Reductio ad Absurdum*

2. i. by assigning two truth-values to the same component in the same context;

ii. by assigning the same truth-value to a formula and its negation in the same context

iii. by assigning a truth-value to the connectives inconsistent with the truth-conditions

## 1.7 Unit Exercises

Use the truth-table technique to determine the validity or invalidity of each of the following arguments:

1. $A \supset B$
   $A \supset C$
   $\therefore B \vee C$

2. $A \vee B$
   $A$
   $\therefore \sim B$

3. $A \supset B$
   $B \supset C$
   $\therefore C \supset T$

*4. $(P \vee Q) \supset (P \,.\, Q)$
   $P \,.\, Q$
   $\therefore P \vee Q$

5. $(A \supset B) \,.\, (C \supset D)$
   $A \vee C$
   $\therefore B \vee D$

6. $(A \vee B) \supset C$
   $C \supset (A \,.\, B)$
   $\therefore (A \vee B) \supset (A \,.\, B)$

7. $A \vee (B \,.\, \sim B)$
   $A$
   $\therefore \sim(B \,.\, \sim B)$

8. $(A \vee B) \supset (A \,.\, B)$
   $\therefore (A \supset B) \supset (B \supset A)$

*9. $A \supset (B \vee C)$
   $(B \,.\, C) \supset \sim A$
   $\therefore \sim A$

10. $(A \vee B) \supset C$
   $C \supset (A \,.\, B)$
   $\therefore (A \,.\, B) \supset (A \vee B)$

II. Use the truth-table technique to determine the validity or invalidity of each of the following arguments:

1. If disparity is to be removed then those educationally disadvantaged states should be given special quotas. If those states that are educationally disadvantaged states should be given special quotas, then some people receive preferential treatment. If some people receive preferential treatment, then disparity is not to be removed. Therefore, disparity is not to be removed.
2. If the hijackers' demands are met then criminals will be rewarded. If the hijackers' demands are not met then the innocent hostages will be killed. So, either criminals will be rewarded or innocent hostages will be killed.
3. *If people are entirely rational then either all of a person's actions can be predicted in advance or the universe is deterministic. Not all of a person's

actions can be predicted in advance. Thus, if the universe is deterministic then people are not entirely rational.

4. If oil production continues to grow then either oil imports will decrease or domestic reserves will be depleted. If oil imports decrease and domestic oil reserves are depleted then the country will soon be bankrupt. Therefore, if oil production continues to grow then the nation will soon be bankrupt.
5. If the South African government stops township violence, then both the African National Congress and the Nkatha Freedom Party will support the government's constitutional proposals. But the Nkatha Freedom Party will not support the government's constitutional proposals. Therefore, the South African government does not stop township violence.

Use short truth-table technique (*reduction ad absurdum*) to determine the validity/invalidity of each of the following arguments:

1. A ⊃ B
   B ⊃ A
   ∴ A v B

2. A ⊃ (B v C)
   A ⊃ ~B
   ∴ A v C

3. A ⊃ B
   A v B
   ∴ B

4. A ⊃ C
   B ⊃ C
   ∴ A ⊃ C

*5. A ⊃ (B . C)
   ∴~(B . C) ⊃ ~A

6. (A ⊃ B) . (C ⊃ D)
   A v C
   ∴ B v C

7. A ⊃ B
   ~B
   ∴ ~A

8. A ⊃ B
   C ⊃ D
   ~B v ~D
   ∴ ~A v ~C

9. B ⊃ ~A
   ~C v B
   ∴ A ⊃ ~C

*10. (A . B) v (~A . ~B
   ∴ A ≡ B

## Unit 2: Basic Valid Argument Forms



### 2.1 Introduction

The aim of this unit is to introduce you to the basic valid arguments in artificial logic. We will be looking at truth-functional argument and the extent to which they may be justified valid.

### 2.2 Learning Outcomes

At the end of this unit, Learners will be able to:

1. Identify and relate with the basic valid arguments
2. Explain the signs related to each basic argument
3. Apply the signs related to each basic argument

### 2.3 Basic Valid Arguments

As we have seen, we can use the truth-table to determine the validity or invalidity of a truth-functional argument. A truth-functional argument is valid if it is not possible for its premises to be true and its conclusion false. Alternatively, a truth-functional argument is valid if its corresponding conditional is a tautology. If an argument is valid, it means that its premises imply its conclusion. The concept of validity and implication are therefore synonymous. Thus, all claims about implication, like validity, can be tested using the truth-table technique. Thus, we know that "~p" implies "~ (p . q)" and "(p v q) ⊃ r" implies "p ⊃ r". Similarly, "p", "q" "p . q" and "~p ⊃ q" each implies "p v q", and each of "p v (q v r)" and "~p ⊃ q" is implied by it. Also, "~p', "q", "q . r", "~p v q", "~q ⊃ ~p", "p ⊃ (q . r)", "(p v r) ⊃ q" each implies "p ⊃ q", and each of " p v q" "~q ⊃~p", "p ⊃ (q v r)" and "(p ⊃ q) v r" is implied by it. Each of these implications can be tested using the truth-table. Students are encouraged to test them as exercises.

In addition, let us consider the following, which may be called laws of implication:

1. Every formula implies itself.
2. If one formula implies a second and the second implies a third, then the first implies the third.
3. A logically true formula is implied by every formula.
4. A logically true formula implies only logically true formulas.
5. A logically false formula implies every formula.

6. A logically false formula is implied only by logically false formulas.
7. Substitution of formulas for propositional letters preserves implication.

8. A formula implies a conjunction if and only if it implies each conjunct of the conjunction.
9. A formula is implied by a disjunction if and only if it is implied by each disjunction.

Each of the above laws can be proved using what we know about disjunction, conjunction, logical truth and so on. Thus, for example we can prove that a logically false formula implies every formula, by arguing that since an implication is false only if the antecedent is true and the consequent false, then an implication in which the antecedent is always false (because it is logically false) can have any formula (whether true or false) as its consequent. Such an implication will come out true under all interpretations because there can be no interpretation where the antecedent is true and the consequent is false.

Finally, there are some basic valid argument-forms that can be constructed using negation, conjunction, disjunction and conditional in various combinations. We shall list the argument-forms under the major propositional types, leaving it to the student to use the truth-table technique to confirm the validity of each argument-form. This will not only reassure the student of the validity of those argument-forms, but will also make her conversant with the valid argument-forms which will be required in the next stage of our study.

(i) **Conjunction**

There are two valid argument-forms that employ conjunction, namely *Simplification* and *Conjunction*.

(1) **Simplification:** The two forms of Simplification are as follows:

$$p \cdot q \qquad\qquad p \cdot q$$
$$\therefore\ p \quad \text{and} \quad \therefore\ q$$

This means that from the truth of a conjunction we can infer or derive any of its conjuncts. This is because the conjunction of two propositions is true only if both of the two Propositions are true.

**(2) Conjunction:** The form of conjunction is

$$p$$
$$q$$
$$\therefore\ p \cdot q$$

This means that if each of two propositions is true, we can infer or derive their conjunction.

(ii) **Disjunction**

There are two valid argument-forms that employ disjunction, namely Addition and Disjunctive Syllogism. Recall that the disjunction of two propositions is true if at least one of these propositions is true.

(3) **Addition:** The form of Addition is:

$$\begin{array}{l} p \\ \therefore p \vee q \end{array}$$

This means that from the truth of a proposition we can infer a disjunction of which it is a disjunct.

(4) **Disjunctive Syllogism:** The two forms of Disjunctive Syllogism are:

$$\begin{array}{lcl} p \vee q & & p \vee q \\ \sim p & \text{and} & \sim q \\ \therefore q & & \therefore p \end{array}$$

This means that if a disjunction is true and we know that one of the disjuncts is false, then we can infer the other disjunct. In other words, the falsity of one of the disjuncts of a true disjunction implies the truth of the other disjunct.

(iii) **Conditional**

There are five valid argument-forms that utilise the conditional along with others like negation and disjunction, namely, **Modus Ponens, Modus Tollens** , **Hypothetical Syllogism, Constructive Dilemma** and **Destructive Dilemma.**

(5) ***Modus Ponens:*** The form of ***modus ponens*** is:

$$\begin{array}{l} p \supset q \\ p \\ \therefore q \end{array}$$

which means that the conjunction of a conditional with its antecedent implies the consequent. In other words, if a conditional and its antecedent are true we can infer the consequent.

(6) ***Modus Tollens:*** The form of ***Modus Tollens*** *is*:

$$\begin{array}{l} p \supset q \\ \sim q \\ \therefore \sim p \end{array}$$

which means that the conjunction of a conditional with the negation of its consequent implies the negation of its antecedent. In other words, if we have a conditional and the negation of its consequent, we can infer the negation of its antecedent.

(7) **Hypothetical Syllogism:** The form of the **Hypothetical syllogism** is:

$p \supset q$
$p \supset r$
$\therefore p \supset r$

This means that if the consequent of a conditional implies a third proposition, then the antecedent of that conditional implies the third proposition. In other words, if we have a conditional whose consequent implies a third proposition then we can infer that the antecedent of the (first) conditional implies the consequent of the (second) conditional.

(8) **Constructive dilemma** has the following form:

$(p \supset q) . (r \supset s)$
$p \vee r$
$\therefore q \vee s$

This means that the conjunction of two conditionals with the disjunction of the antecedents of the two conditionals implies the disjunction of the consequents of the conditionals. In other words, if we have a conjunction of two conditionals and the disjunction of the antecedents of the two conditionals, then we can infer the disjunction of the consequents of the conditionals.

(9) **Destructive Dilemma** has the following form:

$(p \supset q) . (r \supset s)$
$\sim q \vee \sim s$
$\therefore \sim p \vee \sim r$

This means that the conjunction of two conditionals with the disjunction of the negations of the consequents of the two conditionals implies the disjunction of the negations of the antecedents of the conditionals. In other words, if we have a conjunction of two conditionals and the disjunction of the negations of the consequents of the two conditionals, then we can infer the disjunction of the negations of the antecedents of the conditionals.

## 1.4 Summary

In this unit, we have been able to identify and discourse the conditions under which all the basic arguments can be deemed as valid.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 1.6 Unit Exercises

Test the validity or invalidity of each of the above valid argument-form, using the truth-table technique.

**Unit 3: Logically Equivalent Formulas**



## 1.1 Introduction

In this unit, we shift attention to discuss some logical formulas that are at par. You will notice some logical connectives and equations that are logically equivalent.

## 1.2 Learning Outcomes

At the end of this unit, learners must:

1. Identify some logical connectives that are equivalent
2. Identify some equivalent logical equations

## 1.1 Logically Equivalent Formulas

We have seen that two truth-functional formulas are said to be equivalent if they agree in truth-value under every interpretation of their propositional letters. Thus, "p" is equivalent to "~~p", to "p . p" and to "p v p"; "p . q" is equivalent to "q . p" and to ~(~p v ~q); "p v q" is equivalent to "p v q" and to ~(p . ~q); similarly, "p ⊃ q" is equivalent to "~(p . ~ q)", to "~p v q" and to "~q ⊃ ~p"; and "p ≡ q" is equivalent to "(p ⊃ q) . (q ⊃ p)", to "~ (p . ~q) . ~(q . ~p)", and to "(p . ~q), and so on.

We have also seen that two expressions are logically equivalent if the statement of their equivalence (using the biconditional sign) is tautologous. Logicians have discovered some basic forms of logically equivalent formulas. The proving of their equivalence shall be left as exercises for the student. Two logically equivalent expressions have the same logical force or import; logical equivalence does not mean synonymy or equivalence of meaning. Logical equivalence also indicates various inter-relationships between our logical connectives.

In what follows, we shall list some standard logically equivalent formulas under our major types of compound propositions, namely, negation, conjunction, disjunction, conditional and biconditional:

(i) **Negation:**

(1) **Double Negation:** A proposition is logically equivalent to its double negation, thus;

$p \equiv \sim\sim p$

(2) **DeMorgan's Theorems:** The negation of a conjunction is logically equivalent to the disjunction of the negations of the conjuncts, thus:

$\sim (p \,.\, q) \equiv (\sim p \vee \sim q)$

Thus, the negation of conjunction is not a conjunction but a disjunction. Similarly, the negation of a disjunction is logically equivalent to the conjunction of the negations of the disjuncts, thus:

$\sim (p \vee q) \equiv (\sim p \,.\, \sim q)$

Thus, the negation of a disjunction is not a disjunction but a conjunction.

(ii) **Disjunction and Conjunction**

(3) **Commutativity:** Both disjunction and conjunction are commutative. Therefore, a disjunction is logically equivalent to its inverse, thus:

$(p \vee q) \equiv (q \vee p)$

Similarly, a conjunction is logically equivalent to its inverse, thus:

$(p \,.\, q) \equiv (q \,.\, p)$

(4) **Associativity:** Both disjunction and conjunction are associative. Therefore, regrouping, using our punctuation marks, does not affect the truth-value of a disjunction, thus:

$[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$

Similarly, regrouping does not affect the truth-value of a conjunction, thus:

$[p \,.\, (q \,.\, r)] \equiv [(p \,.\, q) \,.\, r]$

Thus, a regrouped disjunction or conjunction is logically equivalent to the original disjunction or conjunction.

(5) **Tautology or Idempotency:** Both conjunction and disjunction are idempotent. Therefore, repeating a proposition as a disjunction with itself is logically equivalent to the original proposition, thus:

$p \equiv (p \vee p)$

Similarly, repeating a proposition as a conjunction with itself is logically equivalent to the original proposition, thus:

$p \equiv (p \,.\, p)$

(6) **Distributivity:** The conjunction of a proposition with a disjunction is logically equivalent to a disjunction of the conjunctions of the proposition and each of the disjuncts, thus:

$[p \,.\, (q \vee r)] \equiv [(p \,.\, q) \vee (p \,.\, r)]$

Similarly, the disjunction of a proposition with a conjunction is logically equivalent to a conjunction of the disjunctions of the proposition and each of the conjuncts, thus:

$[p \vee (q \,.\, r)] \equiv [(p \vee q) \,.\, (p \vee r)]$

(iii) **Conditional**

(7) **Transposition:** If it is true that:

If the United Nations is to be effective then it must be financially solvent

Then it is true that:
If the United Nations is not financially solvent then it cannot be effective.

This is clear from our characterisation of the conditional as material implication which is false only if the antecedent is true and the consequent is false. So, the falsity of the consequent implies the falsity of the antecedent. This means that a conditional is logically equivalent to the 'contraposition' of the conditional, thus:

$$(p \supset q) \equiv (\sim q \supset \sim p)$$

(8) **Material Implication:** A conditional is logically equivalent to a disjunction; the first disjunct is the negation of the antecedent of the conditional and the other disjunct is the affirmation of its consequent, thus:

$$(p \supset q) \equiv (\sim p \vee q)$$

(9) **Exportation:** If a conjunction implies a proposition, then the first conjunct implies a conditional whose antecedent is the second conjunct and vice versa, thus:

$$[(p \,.\, q) \supset r] \equiv [p \supset (q \supset r)]$$

(iii) **Biconditional**

(10) **Material Equivalence:** As we have seen, the name 'biconditional' connotes logically two conditional propositions with the antecedent and consequent interchanged, thus:

$$(p \equiv q) \equiv [(p \supset q) \,.\, (q \supset p)]$$

Similarly, as we know, a biconditional is true either if both components are true or both components are false. Therefore, a biconditional is logically equivalent to a disjunction of, on the one hand, the conjunction of the components and, on the other, the conjunction of the negations of the components, thus:

$$(p \equiv q) \equiv [(p \,.\, q) \vee (\sim p \,.\, \sim q)]$$

## 1.2 Summary

In this unit, we have been able to identify some of the equations and connectives that are logically equal to one another.

## 1.5 References and Further Readings


Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

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## 1.6 Unit Exercises

Prove the equivalence of each of the pairs of statement-forms, using the truth-table technique.

## Unit 4: Method of Natural Deduction



### 1.1 Introduction

The essence of this unit is to comprehend the idea of inferences and how to deduce them correctly. This is generally seen as the method of natural deduction. We must recall that the truth-table technique is a decision-procedure. Though, as we have seen, it has its limitations. The method of *reductio ad absurdum* may also become tedious. Moreover, it is not foolproof. If an argument is valid, there will be an inconsistency or contradiction in our truth-assignments. However, as we have seen, if the truth-assignments are not done cautiously, an inconsistency may occur when the argument is in fact, invalid. Therefore, the truth-table, long or short, needs to be supplemented by an alternative method of proving the validity of arguments composed of truth-functional statements. The method is that of natural deduction.

The method of natural deduction hinges on two principles, namely:

(i) If an argument is valid, then any argument having that form is valid; in this connection, we have seen some basic argument-forms and logically equivalent formulas.

(ii) If an argument is valid, then its conclusion must be derivable from its premises, either directly or through intermediate stages; in doing this derivation, the basic argument-forms referred to in (i) serve as rules of inference, while the logically equivalent formulas serve as the rule of replacement.

It is obvious that if an argument is valid, then its conclusion must be derivable from its premises. However, the fact that a conclusion is not derived from the premises of an argument does not mean that the argument in question is invalid. The fault may be elsewhere for example, with the student. The method of natural deduction is therefore not a decision-procedure.

Let us look at the following argument and see if we can derive the conclusion from the premises:

(1) If the rains continue, then the floods will increase

(2) If the floods increase, then many homes will be swept away.

(3) If many homes are swept away, then a single drainage channel is not sufficient for the street.

(4) If a single drainage channel is not sufficient for the street, then the town planning engineers have made a mistake.

Therefore, if the rains continue, then the town planning engineers have made a mistake.

Quite clearly, the conclusion can be derived through a number of intermediate steps, as follows:

**Step 1:** (1) If the rains continue, then the floods will increase

(2) If the floods increase, then many homes will be swept away.

Therefore, if the rain continues, then many homes will be swept away.

This argument has the form of a valid hypothetical syllogism, discussed earlier.

The argument in step 2 also has the form of a valid hypothetical syllogism. Note that the conclusion of the argument in step 1 is the first premise of the argument in step 2.

**Step 2:** (1) If the rains continue, then many homes will be swept away.

(2) If many homes are swept away, then a single drainage channel is not sufficient for the street.

Therefore, if the rains continue, then a single drainage is not sufficient for the street.

**Step 3:** (1) If the rains continue, then a single drainage channel is not sufficient for the street.

(2) If a single drainage channel is not sufficient for the street, then the town planning engineers have made a mistake.

Therefore, if the rains continue, then the town planning engineers have made a mistake.

This last argument also has the form of a valid hypothetical syllogism. Note that its conclusion is the same as the conclusion of the original argument. Note also that the conclusion of the argument in step 2 is the first premise of the argument in step 3, the second premise being the same as the fourth premise of the original argument. This means that the original argument is valid, since its conclusion is derivable from its premises, albeit in three steps, not directly.

The procedure followed in this proof can be seen more clearly if we symbolise the argument as follows:

Dictionary: R: The rains continue

F: The floods will increase
H: Many homes will be swept away
G: A single drainage channel is not sufficient for the street
M: The town planning engineers have made a mistake

The argument, symbolised, is:

$R \supset F$
$F \supset H$
$H \supset M$
$M \supset G$
$\therefore \quad R \supset G$

The proof involves:

Step 1: $R \supset F$
$F \supset H$
$\therefore$ $R \supset H$ (Hypothetical Syllogism)

Step 2: $R \supset H$
$H \supset M$
$\therefore$ $R \supset M$ (Hypothetical Syllogism)

Step 3: $R \supset M$
$M \supset G$
$\therefore$ $R \supset G$ (Hypothetical Syllogism)

The process of deriving the conclusion of an argument from its premises is called a "proof". Every step of a proof is either a given premise or a derivation in accordance with a rule of inference, which is a valid argument-form or (as we shall see later) a rule of replacement which is a logically equivalent formula.

## 1.2 Learning Outcomes

Learners will be able to:

1. Engage in the capacity to make logically sound deductions in a natural manner
2. Identify and understand how the rules of replacement and inference function in the process of natural deduction.
3. Identify and apply rules of indirect proof and tautologies

## 1.3 Rules of Inference

Following is a list of the rules of inference consisting of the basic valid argument-forms re-stated as rules:

**1. Simplification (Simp.)**

(i) p . q or p . q
∴ p ∴ q

The Rule of Simplification (abbreviated 'Simp') enables us to infer 'p' if 'p . q' is given. In other words, if we have 'p . q' (or if 'p . q' is true). We can infer 'p'. The alternative formulation of the rule indicates that the inferred conjunct may be the second one. Each of the following arguments is a substitution instance of, or derives its conclusion in accordance with, the Rule of Simplification:

(1) (A v B) . (C v D)
∴ A v B

(2) (X . Y) . (P ⊃ ~Q)
∴ X . Y

(3) [A ⊃ (B v C)] . [D ⊃ (E v F)]
∴ D ⊃ (E v F)

**2. Conjunction (Conj.)**

p
q
∴ p . q

The Rule of Conjunction (or 'conj.' for short) enables us to infer the conjunction 'p . q' if 'p' and 'q' are given as premises. In other words, if 'p' and 'q' are given or true, we can infer 'p . q'. Each of the following arguments is licensed by the Rule of Conjunction:

(1) ~ (P . Q)
R ⊃ ~ S
∴ ~ (P . Q) . (R ⊃ ~ S)

(2) A ⊃ (B v C)
D ⊃ (B v E)
∴ [(A ⊃ (B v C)] . [D ⊃ (B v E)]

(3) C v D
A ⊃ B
∴ (C v D) . (A ⊃ B)

**3. Addition (Add.)**

(i) p

$\therefore$ p v q

or

(ii) p

$\therefore$ q v p

The Rule of Addition enables us to infer ‘p v q’ or ‘q v p’, if ‘p’ is given. In other words, if we have ‘p’ we can infer either ‘p v q’ or ‘q v p’. In each of the following arguments, the conclusion is derived from the premise in accordance with the Rule of Addition:

1. $A \supset B$

   $\therefore (A \supset B) \vee (C \supset D)$

2. $P . Q$

   $\therefore (P . Q) \vee ({\sim}P . {\sim}Q)$

3. $(C \equiv D) \supset (Q \equiv R)$

   $\therefore [(C \equiv D) \supset (Q \equiv R)] \vee [(M \equiv {\sim}Q) \vee (Q \equiv R)$

**4. Disjunctive Syllogism (DS)**

(i) p v q or (ii) p v q

~p ~q

$\therefore$ q $\therefore$ p

The Rule of Disjunctive Syllogism (or DS for short) enables us to infer ‘q’ if ‘p v q’ and ‘~p’ are given. In other words, if ‘p v q’ and ‘~p’ are true, then we can infer ‘q’. The alternative formulation of the rule of DS indicates that the disjunct negated in the second premise may be the second one, in which case the first disjunct will be inferred. Each of the following arguments is licensed by the Rule of Disjunctive Syllogism.

1. $(H \vee I) \vee {\sim} (J . {\sim} P)$

   ${\sim}{\sim} (J . {\sim} P)$

   $\therefore H \vee I$

2. $(P \equiv Q) \vee [(R . S) \vee (T . U)]$

   ${\sim} (P \equiv Q)$

   $\therefore (R. S) \vee (T . U)$

3. $(U. V) \vee {\sim} (W \supset X)$

   ${\sim} (U . V)$

$\therefore \sim (W \supset X)$

**4. Modus Ponens (MP)**

$p \supset q$
p
$\therefore$ q

The rule of *Modus Ponens* enables us to infer 'q' if 'p ⊃ q' and 'p' are given. In other words, if 'p ⊃ q' and 'p' are true, we can derive 'q'. Each of the following arguments is an instance of the application of *modus ponens*:

1. (P . Q) ⊃ (R . S)
   P . Q
   ∴ R. S

2. ~ (P ⊃ (Q ⊃ R) ⊃ ~ (S ⊃ T)
   ~ [P ⊃ (Q ⊃ R)]
   ∴ ~ (S ⊃ T)

3. {[P ⊃ ~ (Q v R)] . [A ⊃ ~ (B v C)]} ⊃ [(S v T) v U]
   [P ⊃ ~ (Q v R)] . [A ⊃ ~ (B v C)]
   ∴ (S v T) v U

**6. Modus Tollens (MT)**

p ⊃ q
~q
~p

The rule enables us to derive '~p' if 'p ⊃ q' and '~q' are given. In other words, if 'p ⊃ q' and '~q' are true, we can derive '~p'. Each of the following arguments is an instance of the use of *Modus Tollens*:

1. (P . Q) ⊃ (S . T)
   ~(S . T)
   ∴ ~(P . Q)

2. ~[P ⊃ (Q ⊃ R)] ⊃ ~(S ⊃ T)
   ~~(S ⊃ T)
   ∴ ~~[P ⊃ (Q ⊃ R)]

3. {(P ⊃ ~(Q v R)}. [P ⊃ ~(Q v R)]} [(S v T) v U]
   ~[(S v T) v U]

∴ ~{[P ⊃ ~(Q v R)] . [P ⊃ ~(Q v R)]}

**7. Hypothetical Syllogism (HS):**

p ⊃ q
q ⊃ r
∴ p ⊃ r

The rule enables us to infer 'p ⊃ r' if 'p ⊃ q' and 'q ⊃ r' are given. In other words, if 'p ⊃ r' and 'q ⊃ r' are true, we can derive 'p ⊃ r'. Each of the following arguments is an instance of *Hypothetical Syllogism*:

1. (A ⊃ B) ⊃ (C v D)
   (C v D) ⊃ (A . D)
   ∴ (A ⊃ B) ⊃ (A . D)
2. (P ≡ Q) ⊃ ~(R . ~S)
   ~(R. ~S) ⊃ (T ⊃ U)
   ∴ (P ≡ Q) ⊃ (T ⊃ U)
3. [(P ⊃ Q) ⊃ R] ⊃ ~(S v T)
   (S v T) ⊃ [(P ⊃ Q) ⊃ R]
   ∴ (S v T) ⊃ ~(S v T)

**8. Rule of Constructive Dilemma (CD):**

(p ⊃ q) . (r ⊃ s)
p v r
∴q v s

The rule enables us to infer 'q v s' if '(p ⊃ q) (r ⊃ s)' and 'p v r' are given. In other words, if

'(p ⊃ q) . (r ⊃ s)' and 'p v r' are true, we can infer 'q v s'. Each of the following arguments is an instance of the use of the Rule of *Constructive Dilemma*:

(1) [(P . ~Q) ⊃ R] . [(Q . ~P) ⊃ S]
    (P . ~Q) v (Q . ~P)
    ∴ R v S
(2) (T ⊃ U) . (F ⊃ H)
    T v F
    ∴ U v H
(3) [(A ⊃ ~B) ⊃ (R ≡ S)] . [(A v B) ⊃ (C ⊃ D)]
    (A ⊃ ~B) v (A v B)
    ∴ (R ≡ S) v (C ⊃ D)

**9. Destructive Dilemma (DD)**

(p ⊃ q) . (r ⊃ s)
~q v ~s

∴ ~p v ~r

The rule enables us to infer '~p v ~r' if '(p ⊃ q) . (r ⊃ s)' and '~q v ~s' are given. In other words, if '(p ⊃ q) . (r ⊃ s)' and '~q v ~s' are true, we can infer '~p v ~r'. Each of the following arguments is an example of *Destructive Dilemma*:

(1) [(P . Q) ⊃ (S v T)] . [(W v X) ⊃ Y]
~(S v T) v ~Y
∴ ~(P . Q) v ~(W v X)

(2) (E ⊃ F) . (T ⊃ V)
~F v ~V
∴ ~E v ~T

(3) [P ⊃ (Q v R)] . (B ⊃ (A v C)]
~(Q v R) v ~(A v C)
∴ ~P v ~B

Now, to revert to the argument we started with, the proof will be written out as follows:

1. R ⊃ F Pr. (for 'premise')
2. F ⊃ H Pr.
3. H ⊃ M Pr.
4. M ⊃ G Pr. / ∴ R ⊃ G
5. R ⊃ H 1, 2, HS
6. R ⊃ M 5, 3, HS
7. R ⊃ G 6, 4, HS

As indicated, lines 1 – 4 are premises; this is shown by writing the letters 'Pr', after each formula. The slash (that is, '/') is used in indicating the conclusion. Line 5 is derived from lines 1 and 2, in accordance with the Rule of Hypothetical Syllogism. Line 6 is derived from lines 5 and 3, again in accordance with the Rule of Hypothetical Syllogism. Lastly, line 7 is derived from lines 6 and 4, also in accordance with the Rule of Hypothetical Syllogism.

Note that each line of the proof which is not a premise requires a 'justification', which is provided by naming the line or lines from which the formula is derived and the rule in accordance which it is derived. A rule is said to 'license' a move. Note also that the conclusion is written in such a way as to stand out from the rest of the argument. The proof ends when the conclusion is derived, as in line 7 above. Note further that all the rules of inference given above apply to whole lines not to parts of line, though some have two premises while some have only one premise.

Let us now apply the rules in constructing proofs for one or two more arguments. Let us take the following argument:

E ⊃ (F . ~G)

(F v G) ⊃ H
E
∴ H

To construct a proof of the argument, we arrange it as follows:

1. E (F . ~G) Pr
2. (F v G) ⊃ H Pr
3. E Pr / ∴ H

In constructing a proof it is useful to have a 'strategy'. One strategy is to proceed to derive whatever is derivable, bearing in mind one's need at every step. Thus, in the above examples, knowing that we can derive 'F . ~G' from 'E ⊃ (F . ~G)' in line 1 and 'E' (line 3), using the Rule of *Modus Ponens* (MP), we make that our first move. Next, we observe that from 'F . ~G' we can derive 'F' as line 5, by Simplification. Having got 'F', we know that we can derive 'F v G' as line 6 by the Rule of Addition (Add.). Lastly, having derived 'F v G' in line 6, and taken along with '(F v G) ⊃ H' (in line 2), we know that we can derive the conclusion, that is, 'H', using the Rule of *Modus Ponens* (MP). This brings the proof to an end.

Another strategy is to work backwards from the conclusion, keeping in mind one's need at every step. Thus, in the above example, we observe that 'H' occurs in line 2, and that given 'F v G' we can derive it, using the rule of MP. Now, to get 'F v G', we need 'F' and we observe that there is 'E ⊃ (F . ~G)' in line 1, from which we can derive 'F . ~G' if we have 'E' using the rule of MP and there is 'E' in line 3, which is where we start the proof. Thus, the proof will proceed as follows:

| | | |
|---|---|---|
| 1. | E ⊃ (F . ~G) | Pr. |
| 2. | (F v G) ⊃ H | Pr. |
| 3. | E | Pr. / ∴ H |
| 4. | F . ~G | 1, 3, MP |
| 5. | F | 4, Simp. |
| 6. | F v G | 5, Add. |
| 7. | H | 2, 6, MP |

It must be admitted that using any of the strategies requires a good knowledge of the rules, since it is only through such knowledge that we can know what is derivable from what.

Let us take a more complex argument:

H ⊃ (I . J)
H ⊃ [(K ⊃ L) . (M ⊃ N)]
(I . J) v [(~N ⊃ K) . (~N ⊃ M)]
~ (I . J) . ~(N . K)
∴ L v N

Its proof will proceed as follows:

| | | |
|---|---|---|
| 1. | $H \supset (I \,.\, J)$ | Pr. |
| 2. | $\sim H \supset [(K \supset L) \,.\, (M \supset N)]$ | Pr. |
| 3. | $(I \,.\, J) \vee [(\sim H \supset K) \,.\, (\sim H \supset L)]$ | Pr. |
| 4. | $\sim(I \,.\, J) \,.\, \sim(N \,.\, K)$ | Pr/ $\therefore$ L v N |
| 5. | $\sim(I \,.\, J)$ | 4, Simp. |
| 6. | $\sim H$ | 1, 5, MT |
| 7. | $(K \supset L) \,.\, (M \supset N)$ | 2, 6, MP |
| 8. | $(\sim H \supset K) \,.\, (\sim H \supset M)$ | 3, 5, DS |
| 9. | $\sim H \vee \sim H$ | 6, Add. |
| 10. | $K \vee M$ | 8, 9, CD |
| 11. | $L \vee N$ | 7, 10, CD |

Note that in constructing a proof a rule of inference may be applied as often as required, provided the rule is correctly applied. In other words, there is no limit to how many times a rule of inference may be applied, provided each application is legitimate.

## 1.4 Rules of Replacement

Another set of the rules needed to prove some additional truth-functional arguments derive from the logically equivalent formulas discussed earlier. The rules license some kinds of inferences, and are all grouped together under the rubric of Rule of Replacement. A rule of replacement may apply to a line or part of a line. In this respect, these rules are different from our rules of inference which apply only to whole lines, never part of a line in a proof. The underlying idea of the rule of replacement is that if two propositions are logically equivalent then they may replace each other in a proof. This is because, as we have seen, when two propositions are logically equivalent, they are either both true or both false, thus having the same logical force. The following are the logically equivalent formulas:

**10. Double Negation (DN):**

$$p \equiv \sim\sim p$$

Each of the following inferences is licensed by the rule of Double Negation:

(i) $\sim\sim[D \vee (E \vee F)]$
$\therefore D \vee (E \vee F)$

(ii) $D \supset (\sim\sim E \supset F)$
$\therefore D \supset (E \supset \sim F)$

In (i) the rule applies to the whole line, while in (ii) it applies to only part of the line.

**11. De Morgan's Theorems (DMT)**

(a) $\sim(p . q) \equiv (\sim p \vee \sim q)$

(b) $\sim(p \vee q) \equiv (\sim p . \sim q)$

Each of the following inferences appeals to DeMorgan's Theorem (a):

(i) $\sim[(E . F) . (G \vee H)]$

$\therefore \sim(E . F) \vee \sim(G \vee H)$

(ii) $D \supset \sim(E . \sim F)$

$\therefore D \supset (\sim E \vee \sim\sim F)$

Whereas in (i) the rule is applied to the whole line, in (ii) it is applied to only part of the line. Similarly, each of the following inferences appeals to DeMorgan's Theorem (b):

(i) $\sim[(E \vee F) \vee (G . H)]$

$\therefore \sim(E \vee F) . \sim(G . H)$

(ii) $D \supset \sim(E \vee \sim F)$

$\therefore D \supset (\sim E . \sim\sim F)$

In (i), the rule is applied to the whole line, while in (ii) the rule is applied to only part of the line.

**12. Commutation (Com.)**

(a) $(p \vee q) \equiv (q \vee p)$

(b) $(p . q) \equiv (q . p)$

Each of the following inferences in a conformity with the rule of Commutation (a):

(1) $(D . E) \vee (E \vee F)$

$\therefore (E \vee F) \vee (D . E)$

(2) $(J . K) \supset [(M . O) \vee (J . M)]$

$\therefore (J . K) \supset [(J . M) \vee (M . O)]$

In (1) the rule is applied to the whole line, while in (2) it is applied to only part of the line. Similarly, both

(1) $(R \supset \sim S) . (S \supset T)$

$\therefore (S \supset T) . (R \supset \sim S)$

(2) $P \supset [(Q . R) \supset S]$

$\therefore P \supset [(R . Q) \supset S]$

employ the rule of commutation (b). In (1), the rule is applied to the whole line while in (2) the rule is applied to part of the line.

**13. Association (Ass.)**

(a) $[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$

(b) $[p . (q . r)] \equiv [(p . q) . r]$

Inferences exemplifying the application of Association (a) are:

(1) (E v ~F) v ~G

∴ E v (~F v ~G)

where the rule applies to the whole line, and

(2) [F v (G v H)] v [(J v J) v K]

∴[F v (G v H)] v [J v (J v K)]

where the rule applies to only part of the line. Similarly, in

(1) (E . ~F) . ~G

∴ E . (~F . ~G)

the rule applies to the whole line, whereas in

(2) [F . (G . H)] v [(J . J ) . K]

∴ [F . (G . H)] v [J . (J . K)]

the rule applies to only part of the line.

**14. Tautology (Taut.)**

(a) p ≡ (p v p)

(b) p ≡ (p . p)

Inferences in which the rule of Tautology (a) is applied are:

(1) [D ⊃ (E v F)] v (D ⊃ (E v F)]

∴ D ⊃ (E v F)

where the rule applies to the whole line, and

(2) (T . ~U) ⊃ [(V v V) ⊃ (V ⊃ W)]

∴ (T . ~U) ⊃ [V ⊃ (V ⊃ W)]

where the rule applies to only part of the line. Similarly, in

[D . (E v F)]. [(D . (E v F)]

∴ D . (E v F)

the rules applies to the whole line, while in

[T ⊃ ~U) ⊃ [(V . V) ⊃ (V ≡ W)]

∴ (T ⊃ ~U) ⊃ [V ⊃ (V ≡ W)]

it applies to only part of the line.

**15. Distribution (Dist.)**

(a) [p . (q v r)]  [(p . q) v (p . r)]

(b) [p v (q . r)]  [(p v q) . (p v r)]

The following inferences are both licensed by the rule of Distribution.

(1) (E v F) . (G v H)

∴ [(E v F) . G] v [(E v F) . H]

(2) (E . F) v (G . H)

(E . F) v G] . [(E . F) v H]

**16. Transposition (Trans.)**

(p ⊃ q) ≡ (~q ⊃ ~p)

The following inference exemplifies the application of the rule:

$(G . H) \supset \{J . [K . (L . M)]\}$
$\therefore \sim\{J . [K . (L . M)]\} \supset \sim(G . H)$

**17. Material Implication (MI):**

$(p \supset q) \equiv (\sim p \vee q)$

The following inference:

$[(A . B) . C] \supset (D \equiv \sim E)$
$\therefore \sim[(A . B) . C] \vee (D \equiv \sim E)$

is licensed by the rule of Material Implication (MI)

**18. Exportation (Exp.)**

$[(p . q) \supset r) \equiv [p \supset (q \supset r)]$

The following inference illustrates the application of the rule:

$\sim F \supset \{G \supset [\sim(H . I) \supset \sim J]\}$
$\therefore \sim F \supset \{[G . \sim(H . I)] \supset \sim J\}$

**19. Material Equivalence**

(a) $(p \equiv q) \equiv [(p \supset q) . (q \supset p)]$

(b) $(p \equiv q) \equiv [(p . q) \vee (\sim p . \sim q)]$

Each of the following inferences illustrates one of the two varieties of the rule of Material Equivalence:

(a) $\{(E \supset F) . [(G \supset H) . (H \supset G)]\} \supset (H \supset I)$
$\therefore [(E \supset F) . (G \equiv H)] \supset (H \supset I)$

(b) $(G . \sim H) \supset (I \equiv \sim J)$
$\therefore (G . \sim H) \supset [(I . \sim J) \vee (\sim I . \sim\sim J)]$

Let us now construct some proofs of validity requiring the RULE of replacement. Let us take the argument:

$F \supset E$
$F \vee E$
$\therefore E$

Seeing that if we have '~E ⊃ E', we can derive '~~E v E' (using MI) from which we can derive 'E v E' (using DN), and thence 'E' (using Taut.), the proof will proceed as follows:

1. $F \supset E$ Pr
2. $F \vee E$ Pr / $\therefore E$
3. $\sim E \supset \sim F$ 1, Trans.
4. $\sim\sim F \vee E$ 2, DN
5. $\sim F \supset E$ 4, MI
6. $\sim E \supset E$ 3, 5, HS
7. $\sim\sim E \vee E$ 6, MI
8. $E \vee E$ 7, DN

9. E 8, Taut.

Let us take the following slightly more complex example:

(E ⊃ (F . G)

(F v G) ⊃ H

∴ E ⊃ H

The proof will proceed as follows:

| | | |
|---|---|---|
| 1. | E ⊃ (F . G) | Pr |
| 2. | (F v G) ⊃ H | Pr/ ∴ E ⊃ H |
| 3. | ~E v (F . G) | 1, MI |
| 4. | (~E v F) . (~E v G) | 3, Dist. |
| 5. | ~(F v G) ⊃ H | 2, MI |
| 6. | (~F . ~G) v H | 5, DMT |
| 7. | H v (~F . ~G) | 6, Comm. |
| 8. | (H v ~F) . (H v ~G) | 7, Dist. |
| 9. | ~E v F | 4, Simp. |
| 10. | E ⊃ f | 9, MI |
| 11. | H v ~F | 8, Simp. |
| 12. | ~F v H | 11, Comm. |
| 13. | F ⊃ H | 12, MI |
| 14. | E ⊃ H | 10, 13, HS |

The student will have noticed that unlike the truth-table, constructing a proof of validity s not a mechanical process. It requires knowledge of the rules and a lot of ingenuity. It is therefore hazardous to conclude, from the fact that one cannot construct a proof for an argument, that the argument is invalid. One way to obviate this hazard is to cross-check the validity of the argument concerned with the method of *reductio ad absurdum* studied earlier, wherever feasible.

## 1.5 Rule of Conditional Proof

Our proof apparatus can be strengthened by the addition of the rule of Conditional Proof (CP) to the nineteen we already have. The basic assumptions of the rule of conditional proof are that:

(1) If an assumed formula, along with given premises and derived formulas, implies a desired formula, then that assumed formula can be made part of the premises, and

(2) If an assumed formula implies another formula, then the result can be written as a conditional whose antecedent is the assumed formula, and whose consequent is the

implied formula. Thus, if we assume ‘p’ and derive ‘q’, then ‘p’ implies ‘q’ (written ‘$p \supset q$’).

A proof involving the rule of Conditional Proof is called a conditional proof.

The rule of Conditional Proof (CP) allows us to assume any formula, provided every assumption is ‘discharged’. An assumption is discharged when the assumed formula and the last implied formula are written out as an explicit conditional, stating that the assumed formula implies the derived formula. Thus, if in a proof we assume ‘p’ and derive ‘q’, the assumption will be discharged. The conditional proof of the following argument-form:

$p \supset r$

$r \supset s$

$s \supset q$

$\therefore p \supset q$

will run as follows:

1. $p \supset r$ Pr
2. $r \supset s$ Pr
3. $s \supset q$ Pr / $\therefore$ $p \supset q$

4. p
5. r 1, 4, MP
6. s 2, 5, MP
7. q 3, 6, MP
8. p ⊃ q 4-7, CP

The arrow is used in indicating the assumed formula. The vertical line from the arrow to the last derived formula indicates the 'scope' of the assumption. The scope here extends from 'p' (the assumed formula) to 'q' (the last derived formula), which has a line drawn below it. The assertion that 'p' implies 'q' (written 'p ⊃ q') follows immediately below the line. The 'discharging conditional' requires a justification, which is the rule of Conditional Proof (CP), citing all the lines within the scope of the assumption, starting with the assumed formula and ending with the last derived formula. Note that once an assumption is discharged, no line within its scope can again be invoked. Thus, in the above example, lines 4 to 7 cannot be invoked after line 8 even if the proof continues beyond that line. For practice, let us take the following argument.

E ⊃ (F ⊃ G)
F ⊃ (G ⊃ H)
∴ E ⊃ (F ⊃ H)

Its proof will go as follows:

| | | |
|---|---|---|
| 1. | E ⊃ (F ⊃ G) | Pr |
| 2. | F ⊃ (G ⊃ H) | Pr/ ∴ E ⊃ (F ⊃ H) |
| 3. | E . F | |
| 4. | E | 3, Simp. |
| 5. | F ⊃ G | 1, 4, MP |
| 6. | F | 3, Simp |
| 7. | G | 5, 6, MP |
| 8. | G ⊃ H | 2, 6 MP |
| 9. | H | 8, 7 MP |
| 10. | (E . F) ⊃ H | 3-9, CP |
| 11. | E ⊃ (F ⊃ H) | 10, Exp. |

Since the rule of Conditional Proof allows us to assume any formula, provided that the assumption is discharged, the only wise thing to do is to assume what we need. In the above example, the assumption of 'E . F' is to enable us to derive the conclusion, as shown above. However, note that we could have assumed 'E' first and then 'F'. Thus, requiring us to discharge two times, thus:

| | | |
|---|---|---|
| 1. | E ⊃ (F ⊃ G) | Pr |
| 2. | F ⊃ (G ⊃ H) | Pr/ ∴ E ⊃ (F ⊃ H) |
| 3. | E | |

| | | |
|---|---|---|
| 4. | $F \supset G$ | 1, 3, MP |
| 5. | $F$ | |
| 6. | $G$ | 4, 5, MP |
| 7. | $G \supset H$ | 2, 5, MP |
| 8. | $H$ | 7, 6, MP |
| 9. | $F \supset H$ | 5-8, CP |
| 10. | $E \supset (F \supset H)$ | 3-9, CP |

The question arises as to how to know which formula to assume. A rule of thumb is that, in general, if the conclusion is a conditional proposition, then it is advisable to assume the antecedent of the conditional. If the conclusion is not a conditional proposition, then it is wise to assume whatever formula will yield the conclusion. Thus, the conditional proof of the following argument:

$F \supset (G \cdot H)$
$(G \vee I) \supset J$
$I \vee F$
$\therefore J$

whose conclusion is not a conditional, will proceed as follows:

| | | |
|---|---|---|
| 1. | F ⊃ (G . H) | Pr |
| 2. | (G v I) ⊃ J | Pr |
| 3. | I v F | Pr / ∴ J |
| 4. | ~ J | |
| 5. | ~ (G v I) | 2, 4, MT |
| 6. | ~ G . ~I | 5, DMT |
| 7. | ~ I | 6, Simp |
| 8. | F | 3, 7, DS |
| 9. | G . H | 1, 8, MP |
| 10. | G | 7, Simp |
| 11. | G v I | 8, Add |
| 12. | J | 2, 3 MP |
| 13. | ~J ⊃ J | 4-12, CP |
| 14. | ~~J v J | 13, MI |
| 15. | J v J | 14, DN |
| 16. | J | 15, Taut. |

Note that it would have been wrong to stop the proof in line 12, having derived the conclusion ‘J’, because the assumption would not have been discharged.

The rule of Conditional Proof, as we have seen above, and like any other rule of inference, may be applied more than once in a proof, provided that each assumption is discharged. The scope of each assumption, as usual, will be indicated by the arrow and the lines. Thus, in proving the following argument:

(G ⊃ H) . (I ⊃J)
(H v J) ⊃ {[K ⊃ (K v L)] ⊃ (G . I)
∴ G ≡ I

there will be need for more than one application of the rule of CP, thus requiring that assumptions be made and discharged more than once, as follows:

| | | |
|---|---|---|
| 1. | (G ⊃ H) . (I ⊃ J) | Pr |
| 2. | (H v J) ⊃ {[K ⊃ (K . L)] ⊃ (G . I)} | Pr / ∴ G ≡ I |
| 3. | G v I | |
| 4. | H v J | 1, 3, CD |
| 5. | [K ⊃ (K v L)] ⊃ (G . I) | 2, 4, MP |
| 6. | K | |
| 7. | K v L | 6, Add |
| 8. | K ⊃ (K v L) | 6-7, CP |
| 9. | G . I | 5, 8, MP |
| 10. | (G v I) ⊃ (G . I) | 3-9 CP |

| | | |
|---|---|---|
| 11. | ~ (G v I) v (G . I) | 10, MI' |
| 12. | (~G . ~I) v (G . I) | 11, DMT |
| 13. | (G . I) v (~G . ~I) | 12, Com. |
| 14. | G ≡ I | 13 ME |

## 1.6 Rule of Indirect Proof

The last rule to strengthen our proof apparatus is the rule of Indirect Proof (IP), also called the rule of *Reduction ad Absurdum*. Like its counterpart in the truth-table method, the rule starts with the assumption that the argument is invalid and that, therefore, the conclusion is false. The first step in the application of the rule is thus the negation of the conclusion, whether it is an atomic (simple) or a molecular (compound) proposition, which then serves as an additional premise. The justification for the conclusion thus negated is the rule of Indirect Proof (IP). The proof is then continued until an explicit contradiction is derived. The contradiction may consist in the conclusion and its negation, or any other formula and its negation. The proof ends once such an explicit contradiction has been derived. Any proof which employs the rule of IP is called an indirect proof.

As an example, let us take the following argument:

(J v K) ⊃ (L . M)
(L v O) ⊃ (~P . Q)
(P v R) ⊃ (J . S)
∴ P

The indirect proof of the argument will go thus:

| | | |
|---|---|---|
| 1. | (J v K) ⊃ (L . M) | |
| 2. | (L v O) ⊃ (~P . Q) | |
| 3. | (P v R) ⊃ (J . S) / ∴ ~P | |
| 4. | ~~P | IP |
| 5. | P | 4, DN |
| 6. | P v R | 5, Add |
| 7. | J . S | 3, 6, MP |
| 8. | J | 7, Simp |
| 9. | J v K | 8, Add |
| 10. | L . M | 1, 9, MP |
| 11. | L | 10, Simp |
| 12. | L v O | 11, Add |
| 13. | ~P . Q | 2, 12, MP |
| 14. | ~P | 13, Simp |
| 15. | P . ~P | 5, 14, Conj. |

Next, let us take the following example whose conclusion is a molecular or compound proposition:

| | | |
|---|---|---|
| 1. | (A ⊃ ~B) . (C ⊃ D) | |
| 2. | (~B ⊃ E) . (D ⊃ ~P) | |
| 3. | (E ⊃ ~Q) . (~P ⊃ R) | |
| 4. | A . C / ∴ ~Q . R | |
| 5. | ~(~Q . R) | IP |
| 6. | A ⊃ ~B | 1, Simp |
| 7. | ~B ⊃ E | 2, Simp |
| 8. | A ⊃ E | 6, 7, HS |
| 9. | E ⊃ ~Q | 3, Simp |
| 10. | A ⊃ ~Q | 8, 9, HS |
| 11. | A | 4, Simp |
| 12. | C ⊃ D | 1, Simp |
| 13. | D ⊃ ~P | 2, Simp |
| 14. | C ⊃ ~P | 12, 13, HS |
| 15. | ~P ⊃ R | 3, Simp |
| 16. | C ⊃ R | 14, 15, HS |
| 17. | C | 4, Simp |
| 18. | R | 16, 17, MP |
| 19. | ~Q | 10, 11, MP |
| 20. | ~Q . R | 19, 18, Conj. |
| 21. | (~Q . R) . ~(~Q . R) | 20, 5, Conj. |

## 1.7 Proof of Tautologies

A tautology, as we have seen, can be determined using the truth-table method. Using that method, a tautology is defined as a formula which is true under all interpretations of its constituent letters, or true in all possible worlds. Similarly, a tautology can be verified by constructing a zero-premise conditional or indirect proof for it. Such proofs may be said to be ‘categorical’ or zero-premise proofs since they do not have any premise. So, there are two basic ways in which proofs of tautologies can be constructed. The first way, as suggested above, is to construct a conditional proof by invoking the rule of conditional proof, along with other rules. Thus, the tautology:

[D ⊃ (E ⊃ F)] ⊃ [(D ⊃ E) ⊃ (D ⊃ F)]

can be proved as follows:

| | | |
|---|---|---|
| 1. | $D \supset (E \supset F)$ | |
| 2. | $D \supset E$ | |
| 3. | $D$ | |
| 4. | $E$ | 2, 3, MP |
| 5. | $E \supset F$ | 1, 3, MP |
| 6. | $F$ | 5, 4, MP |
| 7. | $D \supset F$ | 3-6, CP |
| 8. | $(D \supset E) \supset (D \supset F)$ | 2-7, CP |
| 9. | $[D \supset (E \supset F)] \supset [(D \supset E) \supset (D \supset F)]$ | 1-8, CP |

As can be seen, the proof ends once we derive the tautology. Lines 1, 2 and 3 re all assumptions licensed by the rule of Conditional Proof. Line 1 is the antecedent of the whole formula. Line 2 is the antecedent of the consequent of the original expression. Line 3 is the antecedent of the consequent of the original formula. These applications of the rule of CP are correct since they have each been discharged. Let us look at another example:

$(A \supset B) \supset [\sim(B \,.\, C) \supset \sim(C \,.\, A)]$

| | | |
|---|---|---|
| 1. | $A \supset B$ | |
| 2. | $\sim(B \,.\, C)$ | |
| 3. | $C$ | |
| 4. | $\sim B \vee \sim C$ | 2, DMT |
| 5. | $\sim\sim C$ | 3, DN |
| 6. | $\sim B$ | 4, 5, DS |
| 7. | $\sim A$ | 1, 6, MT |
| 8. | $C \supset \sim A$ | 3-7, CP |
| 9. | $\sim C \vee \sim A$ | 8, MI |
| 10. | $\sim(C \,.\, A)$ | 9, DMT |
| 11. | $\sim(B \,.\, C) \supset \sim(C \,.\, A)$ | 2-10, CP |
| 12. | $(A \supset B) \supset [\sim(B \,.\, C) \supset \sim(C \,.\, A)]$ | 1-11, CP |

As can be clearly seen, our proof is greatly facilitated if we assume the antecedents of the conditional propositions in the formula, beginning with the conditional sign with the widest scope. Line 3 needs some explanation. Though '$\sim(C \,.\, A)$' is not a conditional proposition it can be turned into one using the rule of DMT which makes it $\sim C \vee \sim A$ and the rule of MI which turns it into $C \supset \sim A$, thus making C the antecedent to be assumed.

We may also construct an indirect proof of tautologies, invoking the rule of Indirect Proof, along with others. Since the tautology is to be assumed to be a non-tautology, the whole expression is assumed to be false and thus negated. The first line of the proof will therefore be the negation of the expression to be proved. As in the application of the rule of IP for ordinary arguments, the proof ends whenever an explicit contradiction is derived. Thus, the following tautology:

(S ⊃ T) v (T ⊃ U)

can be proved as follows:

| | | |
|---|---|---|
| 1. | ~[(S ⊃ T) v (T ⊃ U)] | IP |
| 2. | ~(S ⊃ T) . ~(T ⊃ U) | 1, DMT |
| 3. | ~(S ⊃ T) | 2, Simp |
| 4. | ~(~S v T) | 3, MI |
| 5. | ~~S . ~T | 4, DMT |
| 6. | ~(T ⊃ U) | 2, Simp |
| 7. | ~(~T v U) | 6, MI |
| 8. | ~~T . ~U | 7, DMT |
| 9. | ~T | 5, Simp |
| 10. | ~~T | 8, Simp |
| 11. | ~T . ~~T | 9, 10, Conj. |

Since the proof has no premise, the negation of the tautology to be proved becomes the first line of the proof. We then apply other rules until we encounter an explicit contradiction, consisting in the conjunction of any formula and its negation. Let us take another example, as follows:

| | | |
|---|---|---|
| 1. | ~{P ≡ [P v (P . Q)} | IP |
| 2. | ~{P . [P v (P . Q)]} v {~P . ~[P v (P . Q)]} | 1, ME |
| 3. | ~{P . [P v (P . Q)]} . ~ {~P . ~ [P v (P . Q)]} | 2, DMT |
| 4. | ~{P . [P v (P . Q)]} | 3, Simp. |
| 5. | ~P v ~[P v (P . Q)] | 4, DMT |
| 6. | ~P v [~P . ~(P . Q)] | 5, DMT |
| 7. | ~P v [~P . (~P v ~Q)] | 6, DMT |
| 8. | ~P v [(~P . ~P) v (~P . ~Q)] | 7, Dist. |
| 9. | ~P v [~P v (~P . ~Q)] | 8, Taut. |
| 10. | (~P v ~P) v (~P . ~Q) | 10, Assoc |
| 11. | ~P v (~P . ~Q) | 11, Taut |
| 12. | (~P v ~P) . (~P v ~Q) | 12, Dist |
| 13. | ~P . (~P v ~Q) | 13, Taut |
| 14. | ~P | 14, Simp |
| 15. | ~{~P . ~[P v (P . Q)]} | 3, Simp. |
| 16. | ~~P v ~~[P v (P . Q)] | 16, DMT |
| 17. | P v ~~[P v (P . Q)] | 17, DN |

| | | |
|---|---|---|
| 18. | P v [P v (P . Q)] | 18, DN |
| 19. | (P v P) v (P . Q) | 19, Assoc |
| 20. | P v (P . Q) | 20, Taut |
| 21. | (P v P) . (P v Q) | 21, Dist |
| 22. | P . (P v Q) | 22, Taut |
| 23. | P | 23, Simp. |
| 24. | P . ~P | 24, 15, Conj. |

As can be seen, the same principles are at play, whether the proof is a short or a long one. We start with the negation of the tautology to be proved. We then attempt to complete the proof, using other rules, as necessary. The proof ends when an explicit contradiction is generated. The contradiction consists in the conjunction of any formula and its negation.

## 1.8 Redundancy of Some of our Rules

The twenty-one rules discussed above provide us with a powerful proof apparatus for truth-functional logic. The rules taken together are sufficient to prove the validity of most valid truth-functional arguments. However, it appears that we have an excess of rules over what is absolutely necessary. The reason is that some of the rules can be proved using some of the other rules, thus making such 'provable' rules redundant. For example, the rule of destructive Dilemma (DD) can be proved by applying the rule of Conditional Proof (CP) or the rule of Indirect Proof (IP), along with others, thus:

| | | |
|---|---|---|
| 1. | (p ⊃ q) . (r ⊃ s) | Pr |
| 2. | ~q v ~s | Pr./ ∴ ~p v ~r |
| → 3. | p | |
| 4. | p ⊃ q | 1, Simp |
| 5. | q | 4, 3, MP |
| 6. | ~~q | 5, DN |
| 7. | ~s | 2, 6, DS |
| 8. | r ⊃ s | 1, Simp |
| 9. | ~r | 8, 7, MT |
| 10. | p ⊃ ~r | 3-9, CP |
| 11. | ~p v ~r | 10, MI |

Similarly, the rule of Transposition (Trans) can be proved using other rules, as follows:

| | | |
|---|---|---|
| → 1. | p ⊃ q | |
| → 2. | ~q | |
| 3. | ~p | 1, 2, MT |
| 4. | ~q ⊃ ~p | 2-3, CP |

5. (p ⊃ q) ⊃ (~q ⊃ ~p)   1-4, CP

6. ~q ⊃ ~p

7. p

8. ~~p   7, DN

9. ~~q   6, 8, MT

10. q   9, DN

11. p ⊃ q   7-10, CP

12. (~q ⊃ ~p) ⊃ (p ⊃ q)   6-12, CP

13. [(p ⊃ q) ⊃ (~q ⊃ ~p)] . [(~q ⊃ ~p) ⊃ (p ⊃ q)]   5, 13, Conj.

14. (p ⊃ q) ≡ (~q ⊃ ~p)   14, ME

This means that the rules of Destructive Dilemma and Transposition are not absolutely necessary for our proof apparatus, and are therefore redundant. Their retention in our apparatus is merely to have shorter proofs.

## 1.9 Summary

In this unit, we have been able to look at some of the means of natural deduction in rules of inference and replacement. We also looked at indirect proof, conditional proof and the rule of tautologies.

## 1.10 References and Further Readings


Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

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## 1.11 Unit Exercises

I. Each of the following is a proof of validity for the indicated argument. State the ‘justification’ for each line that is not a premise.

1.
1. P . Q
2. (P v R) ⊃ S / ∴ P . S
3. P
4. P v R
5. S
6. P . S

2.
1. (P v Q) . (R v T)
2. (P ⊃ S) . (Q ⊃ T)
3. ~S / ∴ T
4. P v Q

5. S v T
6. T

3. 1. A ⊃ B
2. B ⊃ C
3. E ⊃ D
4. A v E / ∴ C v D)
5. A ⊃ C
6. (A ⊃ C) . (E ⊃ D)

4. 1. (A v M) ⊃ R
2. (L v R) . ~R
3. ~(C . D) v (A v M) / ∴ ~(C . D)
4. ~R
5. ~(A v M)
6. ~(C . D)

5. 1. C
2. A ⊃ B
3. C ⊃ D
4. D ⊃ E / ∴ E v B
5. C ⊃ E
6. C v A
7. (C ⊃ E) . (A ⊃ B)
8. E v B

II. Construct a proof of validity fort each of the following arguments:

1. P ⊃ Q
P v (S . T)
~Q . ~R
∴

2. (P ⊃ Q) . (S ⊃ T)
N ⊃ O
(P v N) . (S v M)
∴Q v O

3. (A . D) ⊃ ~C
(R v S) ⊃ (A . D)

$\sim C \supset \sim(A \,.\, D)$
$\therefore (R \vee S) \supset \sim(A \,.\, D)$

4. $(A \vee \sim C) \supset B$
$A$
$(A \vee \sim D) \supset (R \,.\, S)$
$\therefore (R \,.\, S) \,.\, B$

5. $[\sim A \,.\, \sim(D \,.\, E)] \supset (B \supset \sim E)$
$\sim(D \,.\, E) \,.\, \sim R$
$E \supset F$
$\sim A \vee (D \,.\, E)$
$\sim(D \,.\, E) \supset (B \vee E)$
$\therefore E \vee \sim F$

III. Construct a proof of validity for each of the following arguments using your own abbreviation:

1. If either the Super Eagles or the Black Stars win, then both Leventis United and Rangers International lose, the super Eagles win. Therefore, Leventis United loses.
2. If Solarin joins, then the party's social prestige will rise; and if Abiola joins then the party's financial position will be more secure. Either Solarin or Abiola will join. If the party's social prestige rises then Abiola will join; and if the party's financial position becomes more secure then Mbakwe will join. Therefore, either Abiola or Mbakwe will join.
3. If Dikko received the package then he took the train; and if he took the train, then he will not be late for the meeting. If the package was incorrectly addressed, then Dikko will be late for the meeting. Either Dikko received the package or the package was incorrectly addressed. Therefore, either Dikko took the train or he will be late for the meeting.
4. If Lagos state Government takes the loan, then a housing estate will be constructed, whereas if Kano State takes the loan, then it will grow more wheat. If Rivers State takes the loan, it will construct more fish ponds; and if more fish ponds are constructed, then the Ministry of Agriculture will offer to lease them. Either Lagos or Rivers will take the loan. Therefore, either a housing estate or more fish ponds will be constructed.
5. If the rains continue, then the floods increase. If the floods increase, then many homes will be swept away. If many homes will be swept away then a single gutter is not sufficient for the street. The rains continue, and either a single gutter is

sufficient for the street or the two planning engineers have made a mistake. Therefore, the town planning engineers have made a mistake.

I. Each of the following is a proof of validity for the indicated argument. State the 'justification' for each line that I not a premise:

1.
1. P ⊃ Q
2. R ⊃ ~Q/ ∴ P ⊃ ~R
3. ~~Q ⊃ ~R
4. Q ⊃ ~R
5. P ⊃ ~R

2.
1. (P . Q) ⊃ R
2. (P ⊃ R) ⊃ S / ∴ Q ⊃ S
3. (Q . P) ⊃ R
4. Q ⊃ (P ⊃ R)
5. Q ⊃ S

3.
1. (P . Q) ⊃ R
2. Q / ∴ P ⊃ R
3. (Q . P) ⊃ R
4. Q ⊃ (P ⊃ R)
5. P ⊃ R

4.
1. ~P v ~Q / ∴ (P . Q) ⊃ ~Q
2. ~(P . Q)
3. ~(P . Q) v P
4. (P . Q) v P
5. P ⊃ ~Q
6. (P . Q) ⊃ ~Q

5.
1. (P . Q) v R / ∴ ~P ⊃ R
2. R v (P . Q)
3. (R v P) . (R v Q)
4. R v P
5. ~~R v P
6. ~R ⊃ P
7. ~P ⊃ ~~R
8. ~P ⊃ R

II. Using the rules, prove that the following arguments are valid:

1. $(P \;.\; Q) \supset R$
   $P$
   $\therefore Q \supset R$

2. $(P \;.\; Q) \vee (R \;.\; S)$
   $\sim P$
   $\therefore R$

3. $(P \;.\; Q) \supset R$
   $P \;.\; \sim P$
   $\therefore \sim Q$

4. $P \supset \sim Q$
   $\sim(R \;.\; \sim P)$
   $\therefore R \supset \sim Q$

5. $[(P \;.\; Q) \;.\; R] \supset S$
   $L \supset [(R \;.\; P) \;.\; Q]$
   $\therefore \sim L \vee S$

III. Construct a proof of validity for each of the following arguments:

1. Either the President did not consider the possible effects of the raids or else he approved of them. He did consider the effects of the raids all right. So, he must approve of them.
2. It is not the case that either the commander forgot or was not able to see the enemy troops. Therefore, the commander was able to see the enemy troops.
3. If the chemical was poisonous, then the laboratory test would give positive results. Hence, if the chemical was poisonous, then either the laboratory test would give positive results or something was wrong somewhere.
4. If the United Nations' resolutions are fair and their enforcement is strict then Iraq will withdraw. If strict enforcement of the United Nations' resolutions will make Iraq to withdraw, then our problem is a practical one. The United Nations' resolutions are fair. Therefore, our problem is a practical one.
5. Samuel Doe is to be condemned if he usurped power that was not rightfully his own. Either Samuel Doe was a legitimate ruler or else he usurped power

that was not rightfully his own. Samuel Doe was not a legitimate ruler. So Samuel Doe is to be condemned.

Construct a proof, using the rule of conditional proof, among others, for each of the following arguments:

1. $(P \supset Q) \,.\, (R \supset S)$
   $\therefore (P \,.\, R) \supset (Q \,.\, S)$

2. $P \supset Q$
   $(P \,.\, R) \supset S$
   $(Q \,.\, S) \supset T$
   $\therefore P \supset (R \supset T)$

3. $(P \,.\, Q) \equiv R$
   $P \supset Q$
   $\therefore P \equiv R$

4. $[(A \vee B) \,.\, C] \supset D$
   $(C \supset D) \supset (E \supset F)$
   $E$
   $\therefore A \supset F$

5. $M \supset (N \,.\, O)$
   $(N \vee O) \supset P$
   $\therefore M \supset P$

For each of the following arguments, construct a proof which employs, among others, the rule of indirect proof:

1. $(A \vee B) \supset (C \supset D)$
   $(\sim D \vee E) \supset (A \,.\, C)$
   $\therefore D$

2. $D \vee (E \,.\, F)$
   $D \supset F$
   $\therefore F$

*3. $P \supset (Q \,.\, R)$
   $(Q \vee S) \supset T$
   $S \vee P$
   $\therefore T$

4. $(P \cdot Q) \vee R$
   $\sim R \vee Q$
   $\therefore P \supset Q$

5. $(P \cdot Q) \supset (R \cdot S)$
   $Q \supset \sim S$
   $\therefore \sim P \vee \sim Q$

I. Use the rule of conditional proof of verify that the following are tautologies:

1. $(P \supset Q) \supset [(P \supset (P \cdot Q)]$
2. $(Q \supset R) \supset [(P \vee Q) \supset (R \vee P)]$
3. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$
4. $(P \supset Q) \supset [(P \cdot R) \supset (Q \cdot R)]$
5. $[(P \vee Q) \supset R] \{[(R \vee S) \supset T] \supset (P \supset R)\}$

II. Use the rule of indirect proof to verify that the following are tautologies:

1. $(P \supset Q) \vee (\sim P \supset Q)$
2. $(P \supset Q) \vee (Q \supset P)$
3. $P \vee (P \supset Q)$
4. $\sim[(P \supset \sim P) \cdot (\sim P \supset P)]$
5. $A \equiv \sim\sim A$

Prove the following rules using any of the others:

*1. Modus Tollens*
2. Disjunctive Syllogism
3. Constructive Dilemma
4. Hypothetical Syllogism
5. Exportation

**Module 3: Predicate Calculus I**

Unit 1: Introducing Predicate logic

Unit 2: Symbolising Propositions in Predicate Logic

Unit 3: Truth and Falsity in Predicate Logic

**Unit 1: Introducing Predicate logic**



## 1.1 Introduction

In this unit, we are going to begin our discussions on predicate logic. Although not entirely new or strange to you at this point, you will enjoy the unique feature of predicate logic as well as the rules involved.

## 1.2 Learning Outcomes

At the end of this unit, the learners will be able to:

1. Identify the essential features of predicate logic
2. Apply the rules and understand the symbols peculiar to symbolic logic

## 1.3 Essentials of Predicate Logic

In the last few units, we discussed the analysis and evaluation of arguments involving truth-functional compounds. The principles and techniques developed in those units are not appropriate for evaluating arguments involving categorical or singular propositions. Thus, the categorical syllogism,

All Nigerians are Africans
All indigenes of Oyo State are Nigerians
Therefore, all indigenes of Oyo State are Africans.

though valid, cannot be proven valid using techniques appropriate for arguments involving truth-functional propositions, similarly, the argument:

All human beings are mortal beings
Socrates is a human being
Therefore, Socrates is a mortal being

though valid, cannot be proven valid using techniques used for proving the validity of truth-functional arguments. At best, each of the above arguments will be symbolised as follows

A
B
∴ C

which does not represent a valid argument. The reason for this may be either that there is a fault with the proof apparatus developed in the previous unit or that these propositions are of a logically different kind and thus require a different analysis. It is the latter suggestion that is favoured and will be pursued here. The tools for analysing and evaluating the above types of arguments involving non-compound propositions are provided by quantification theory or predicate logic.

Let us start by giving elements of the symbolic apparatus needed for the analysis of propositions like:

All human beings are animals
Some human beings are animal
No human beings are animals
Some human beings are not animals
Everything is animate
Something is animate
Nothing is animate
Something is not animate
Plato is a philosopher
Wolfhounds and terries are hunting dogs
If any bananas are yellow then they are ripe, and
Every daughter has a father but not every father has a daughter.

The elements of our symbolic apparatus are:

(1) Our five truth-functional connectives, namely:
   '∼' (wave or tilde) for negation
   '.' (dot) for conjunction
   'v' (vee or wedge) for disjunction
   '⊃' (horseshoe) for conditional, and
   '≡' (triple bar) for biconditional
(2) Upper-case letters 'A' to 'Z' shall be used to represent predicates, and are called predicate letters or predicate constants.
(3) Greek letters 'ϕ' (phi) and 'ψ' (psi) shall erve as predicate variables.
(4) Lower-case letters 'a' to 't' shall serve as individual or subject letters or constants.
(5) Lower-case letters 'u' to 'z' and the Greek letters μ (mu) and v (nu) shall serve as individual or subject variables.
(6) Quantifiers:
   (i) Existential Quantifier:
      (∃x) or (∃y), that is, an inverted 'E' accompanied by a subject variable, enclosed within parentheses.
   (ii) Universal Quantifier:
      (x) or (y), that is, an individual variable within parentheses

We shall soon see how these symbols are used. For now, let us look at the logical types and structures of propositions involved in predicate logic.

All the propositions to be considered are either:

(i) Singular propositions, or

(ii) General propositions

A *singular* proposition says of an individual person, place or thing, that it or she or he has a property, characteristic or attribute. For example, each of the following is a singular proposition:

1. Plato is a philosopher
2. Plato is not a statesman.
3. If Plato is Greek, then he is a philosopher.
4. Plato is the teacher of Aristotle.
5. The University of Ibadan hosts the 2023 NUGA games.
6. If the President visits Abeokuta, he will receive a tumultuous welcome.
7. Ilorin is to the north-west of Ibadan.

Each of the above sentences speaks about an individual and says of that individual that it or he or she has one attribute or other. For example, 1-4 speak of Plato, (1) that he is a philosopher, (2) that he is not a statesman (3) that if he is Greek, then he is a philosopher, and (4) that he is the teacher of Plato. Similarly, (5) says that the University of Ibadan (which is a corporate individual) hosts the 2022 NUGA games, (6) says that if the President visits Abeokuta, he will receive a tumultuous welcome. Last, (7) says of Ilorin that it is to the north-west of Ibadan. An attribute, characteristic or property is also called a predicate.

## 1.4 Summary

What we have done in this unit is to expose some of the main features of predicate logic and the connectives and what they mean. These are essential in making us to now proceed to other important crucial features such as how to symbolise and make propositions valid or otherwise.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

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## Unit 2: Symbolising Propositions in Predicate Logic



### 1.1 Introduction

In this unit, we are going to consider the various ways of using symbols and connectives in predicate logic to examine the validity of propositions. We shall consider how to symbolize simple propositions and complex propositions

### 1.2 Learning Outcomes

By the end of this unit, the learners must be able to:

1. Identify and apply the relevant connectives for predicate logic
2. Develop the ability to symbolize simple and compound propositions

### 1.3 How to Symbolise in Predicate Logic

Since a singular proposition involves an individual and its or her or his attribute or predicate, it means that to symbolise a singular proposition we need a predicate letter, to represent the predicate and an individual letter, to represent the individual. Thus, the singular proposition.

Plato is a philosopher

will be symbolised as

Pp

'P' representing the predicate expression '…..is a philosopher' and 'p' representing the individual 'Plato'; the convention is to write the predicate letter to the left of the individual letter. Similarly, the singular proposition:

Plato is not a statesman

will be symoblised as:

~Sp

'S' standing for the predicate expression '…is a statesman', and the negation sign modifying the predicate letter. Similarly, the singular proposition,

Plato is the teacher of Aristotle

will be represented as:

Tp

if 'T' is used to stand for the predicate expression '….is the teacher of Aristotle'. Again, the singular proposition:

The University of Ibadan hosts the 2023 NUGA games will be symbolised as:

Hu

If we use 'u' to represent 'the University of Ibadan' and 'H' to represent '…hosts the 2002 NUGA games'.

More complex propositions will be symbolised using the same principles. Thus, the compound proposition:

If Plato is Greek, then he is a philosopher

will be represented as:

$Gp \supset Pp$

If we use 'p' for 'Plato' 'G' for '…is Greek' and 'P' for '…is a philosopher'. 'He' in the consequent of the above conditional proposition obviously refers to 'Plato' in the antecedent. Similarly, the compound proposition:

The President visits Abeokuta and receives a tumultuous welcome

will be represented as:

$Vp \,.\, Rp$

If 'p' stands for 'President', 'V' stands for '…visits Abeokuta' and 'R' for '…receives a tumultuous welcome'. Quite clearly, the 'President' that occurs in the first conjunct is implied in the second conjunct.

**General Propositions**

The following propositions are examples of general propositions:

1. Everything is eternal.
2. Nothing is eternal.
3. Something is eternal.
4. Something is not eternal.
5. A few of them are teachers.
6. Many of them are lecturers or consultants.
7. All lecturers are professors.
8. No lecturers are professors.
9. Some lecturers are professors.
10. Some lecturers are not professors.
11. If anyone is a member, then he will be admitted.
12. People are eligible for the Shell Essay Competition if and only if they are on the academic staff of a University.
13. If Plato is philosopher, then all philosophers are statesmen.
14. If some philosophers are statesmen, then all statesmen are thinkers.
15. If all philosophers are statesmen, then some statesmen are thinkers.
16. If all philosophers are statesmen, then all statesman are thinkers.
17. If some philosophers are statesmen, then some statesmen are thinkers.

As is evident from the examples above, a proposition is general if it says something (or affirms a predicate) of some or all individuals of a certain kind. For example, each of propositions (1) to (4) says of all or some individuals either that they are eternal or that they are not eternal. A general proposition is also called a quantified proposition. When a general proposition is about every individual (in a universe of discourse) it is *universally* quantified; if a general proposition is about one or more, but not all, individuals (in a universe of discourse) then it is *existentially* quantified.

The notion of a *universe of discourse* is important here, since even in real life we seldom talk or write about absolutely everything (plants, animals, minerals, artefacts, human beings, etc., all together). The notion of universe of discourse is introduced to enable us to limit the class of individuals to which a quantifier refers. Thus, in saying 'All lecturers are professors', we must be understood to be talking about groups of human beings, or more specifically, academics, one designated by the term 'lecturers' and another by the term 'professors'. Similarly, in 'If anyone is a member, then he will be admitted' we must be understood to be talking about people, not tables or chairs. Therefore, when we talk about 'everything' or 'something', it is useful to be able to specify what individuals or types of individuals are intended. In the above examples, 1, 2, 7, 8, 11 and 12 are universally quantified propositions, while 3, 4, 5, 6, 9, and 10 are existentially quantified propositions.

There is also a distinction, among quantified propositions, between singly general propositions and multiply general propositions. Singly general propositions contain only one quantifier, while multiply general propositions contain more than one quantifier. Among the above examples, 14 to 17 are multiply general propositions. Example 13 is a combination of a singular with a general proposition.

**Symbolising universally quantified propositions containing one quantifier.** Universally quantified propositions can only be symbolised using the sign for the universal quantifier '(x)', that is, the variable 'x' enclosed in parentheses, and read 'for all x' or 'for any x'. (Note that the variable could have been 'u' 'v' 'w' 'y' or 'z', in which case the universal quantifier would be written (y) or (z), read 'for all y'' or 'for any y', or 'for all z' or 'for any z'. This reading of the symbol, '(x)' suggests, correctly, that all universally quantified propositions must be 'translated' or rewritten in such a way that the quantifier, the predicate and the individual variable will be evident. Thus, to say:

(1) Everything is eternal

is to say,

for all x, x is eternal

symbolised

(x) Ex

That is, using 'Ex' for 'x is eternal'. Similarly:

(2) Nothing is eternal

Translates for all x, x is not eternal

Symbolised;

(x)~Ex

Again, to say:

(7) All lecturers are professors

is to say:

For all x, if x is lecturer, then x is a professor

If we use ‘Lx’ for ‘x’ is a lecturer’ and ‘Px’ for ‘x is a professor’, this will be symbolised

(x) (Lx ⊃ P x)

Similarly, to say:

(8) No lecturers are professors

is to say:

For all x, if x is a lecturer, then x is not a professor

Symbolised:

(x) Lx ⊃ ~Px)

Similarly, the compound proposition

(11) If anyone is a member, then he will be admitted

translates into

For all x, if x is a member, then x will be admitted.

Using ‘Mx’ for ‘x is a member’, and ‘Ax’, for ‘x will be admitted’, then the proposition can be symbolised:

(x) (Mx ⊃ Ax)

Similarly, the compound proposition:

(12) People are eligible for the Shell Essay Competition if and only if they are on the academic staff of a University

translates into:

For all x, x is eligible for the Shell Essay Competition if and only if x is on the academic staff of a University

If ‘Ex’ represents ‘x is eligible for the Shell Essay Competition’ and ‘Ax’ represent ‘x is on the academic staff of a University’, the proposition is symbolised:

(x)(Ex ≡ Ax)

**Scope of a quantifier:** Every quantifier is said to have a scope. A quantifier is placed immediately to the left of the formula or schema that falls within its scope. In the above examples (7), (8), (11) and (12), we introduced parentheses to indicate the scope of the quantifier. In each case, to leave out the parentheses will render the expression an incorrect representation of the original proposition, since it will exclude from the scope of the quantifier what properly falls within its scope. Thus, the expression:

(a) (x)Lx ⊃ Px

is logically different from the expression

(b) (x) Lx ⊃ Px)

If “Lx” represents x is a lecturer and Px represents ‘x is a professor’, what (a) says is that if everything is a lecturer then x is a professor; (b) on the other hand, says that all lecturers are professors.

**Symbolising Existentially Quantified Propositions:** Existentially quantified propositions can only be symoblised using the symbol for the existential quantifier ‘(∃x)’, that is, an inverted ‘E’ and the variable ‘x’ enclosed in parentheses, and read: ‘there exists an x’, or ‘there is an x’ (Note that the variable could have been ‘u’, ‘v’, ‘w’, ‘y’ or ‘z’. This reading of the symbol ‘(∃x)’ suggests that an existentially quantified proposition presupposes:

(1) the *existence* of the entity being described, and

(2) at least one such entity.

In other words, existentially quantified propositions have existential import. All existentially quantified propositions must be ‘translated’ in such a way that the quantifier, the predicate and the individual variable will be evident. However, though a true existentially quantified proposition presupposes the existence of at least one entity, it is different from a singular proposition in that it does not name a specific individual.

Thus, to say:

(3) Something is eternal

translates into:

There is an x (such that) x is eternal

If ‘Ex’ represents ‘x is eternal’, this is symbolised:

(∃x)Ex

Similarly, the proposition:

(4) Something is not eternal

translates into:

There is an x (such that) x is not eternal

which is symoblised:

(∃x)~Ex

Again, the proposition:

(5) A few of them are teachers

translates into

There is an x (such that) x is a teacher

If ‘Tx’ represents ‘x is a teacher’ this is symbolised:

(∃x)Tx

Similarly, the proposition;

(6) Many of them are lecturers or consultants

translates into:

There is an x (such that) x is either a lecturer or a consultant.

If ‘Lx’ represents ‘x is a lecturer’ and Cx’ represents ‘x is a consultant the proposition is symbolised:

$(\exists x)\ (Lx \vee Cx)$

Further, the proposition:

(9) Some lecturers are professors

translates into

There is an x (such that) x is a lecturer and x is a professor.

If ‘Lx’ represents ‘x is a lecturer’ and ‘Px’ represents ‘x is a professor’, then the proposition is symbolised:

$(\exists x)\ (Lx\ .\ Px)$

Similarly, the proposition:

(10) Some lecturers are not professors

translates into:

There is an x (such that) x is a lecturer and x is not a professor

which is symbolised:

$(\exists x)(Lx\ .\ {\sim}Px)$

Care must be taken in symbolising propositions containing the particles ‘a’, ‘an’ and ‘the’. For example, the proposition:

(18) A tiger escaped from the University of Ibadan Zoo

translates into:

There is an x (such that) x is a tiger and x escaped from the University of Ibadan Zoo

If ‘Tx’ represent ‘x is a tiger’ and ‘Ex’ represents ‘x escaped from the University of Ibadan Zoo’, then the proposition is symbolised:

$(\exists x)\ (Tx\ .\ Ex)$

However, the proposition:

(19) A tiger is a carnivore

translates into:

For all x, if x is a tiger, then x is a carnivore

If ‘Tx’ represents ‘x is a tiger’ and ‘Cx’ represents ‘x is a carnivore’, then the proposition is symbolised:

$(x)\ (Tx \supset Cx)$

The reason for the difference is that (18) is talking about a specific tiger, that is, the one that escaped from the University of Ibadan Zoo, whereas (19), in spite of its form, is talking about all tigers.

Similarly, the proposition:

(20) A hijacked passenger plane crashed into the Pentagon building.

translates into:

There is an x (such that) x is a hijacked passenger plane and x crashed into the Pentagon building.

If ‘Hx’ translates ‘x is a hijacked passenger plane’ and ‘Cx’ represents ‘x crashed into Pentagon building’, then the proposition is symbolised;

$(\exists x)(Hx \,.\, Cx)$.

However, the proposition:

(21) A supersonic aeroplane travels faster than the speed of sound

translates into

For all x, if x is supersonic aeroplane, then ‘x travels faster the speed of sound

If ‘Sx’ represent ‘x is a supersonic aeroplane, and ‘Tx’ represents ‘travels faster than the speed of sound’, then the proposition is symbolised

$(x)(Sx \supset Tx)$

Furthermore, the proposition

(22) The glass door is broken

translates into

There is an x (such that) x is a glass door and x is broken

If we use ‘Fx’ for ‘x is a glass door’ and ‘Bx’ for ‘x is broken’, then the proposition will be symbolised:

$(\exists x)(Gx \,.\, Bx)$

However, the proposition:

(23) The giraffe is herbivorous

translates into

For all x, if x is a giraffe, then x is herbivorous

Using ‘Gx’ for ‘x is a giraffe’ and ‘Hx’ for ‘x is herbivorous’, we symbolise the proposition as follows:

$(x)\ (Gx \supset Hx)$

Some propositions containing the particle ‘and’ are correctly translated using the sign of disjunction rather than the sign of conjunction. For example, the proposition:

(24) Architects and engineers are members of professional organisations.

may be translated either as:

(i) for all x, if x is an architect, then x is a member of a professional organisation, and for all x, if x is an engineer, then x is a member of a professional organisation

or else:

(ii) for all x, if either x is an architect or x is an engineer, then x is a member of a professional organisation.

If ‘Ax’ is used for ‘x is an architect’, ‘Ex’ is used for ‘x is an engineer’, and ‘Mx’ is used for ‘x is a member of a professional organisation’, then (i) will be symbolised:

$[(x)\ (Ax \supset Mx)] \,.\, [(x)\ (Ex \supset Mx)]$

and (ii) will be symbolised:

$(x)[(Ax \vee Ex) \supset Mx]$

Note that it will be incorrect to translate the proposition as:

For all x, if x is an architect and an engineer, then x is a member of a professional organisation,

which will be symbolised:

$(x)[(Ax \,.\, Ex) \supset Mx]$

This is because the two propositions have different meanings and truth-conditions. Thus,

$(x)[(Ax \vee Ex) \supset Mx]$

will be true just in case x is either an architect or an engineer, whereas for

$(x)[(Ax \,.\, Ex) \supset Mx]$

to be true, x must be both an architect and an engineer. At the least, the latter symbolisation says more than the original proposition. In other words, it does not convey the correct sense of the original proposition:

Propositions which do not have explicit quantifiers must also be treated with caution. For example, the proposition:

Academic staff are present

should correctly be translated:

There is an x such that x is an academic staff and x is present which is symbolised, as follows, if we use 'Ax' for 'x is an academic staff' and 'Px' for 'x is present':

$(\exists x)(Ax \,.\, Px)$

Note that the proposition cannot be translated

For all x, if x is an academic staff then x is present

which is symbolised

$(x)(Ax \supset Px)$

which says more than is intended by the original proposition. However, the proposition:

oranges are fruits

must be translated:

For all x, if x is an orange, then x is a fruit

If we use 'Ox' for 'x is an orange' and Fx' for 'x is a fruit' then the proposition will be symbolised:

$(x)\ (Ox \supset Fx)$

since the original proposition is intended to refer to all oranges.

Finally, propositions such as:

Some policemen have excellent credentials

cannot be translated:

There is an x (such that) if x is a policeman, then x has excellent credentials

Symbolised (using 'Px' for 'x is a policeman' and 'Ex' for 'x has excellent credentials'):

$(\exists x)\ (Px \supset Ex)$

but must be translated:

There is an x (such that) x is a policeman and x has excellent credentials and symbolised:

$(\exists x)\ (Px\ .\ Ex)$

The two formulas have completely different truth-conditions. The former will be true if anything whatever exists, provided that it is not a policeman. The latter, however, will be true only if there is at least one policeman who has excellent credentials.

## 1.4 Summary

What we have done in this unit is to identify the various ways or means through which propositions can be symbolised in predicate logic.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 1.6 Unit Exercises

Symbolise each of the following statements, each case using the suggested nations;

1. Reporters are present. (Rx, Px)
2. Nurses are always considerate. (Nx, Cx)
3. Snake bites are sometimes fatal. (Sx, Fx)
4. Only pacifists are Rotarians. (Px, Rx)

*5. To be a swindler is to be a thief. (Sx, Tx)

6. Doctors and lawyers are professional people. (Dx, Lx, Px,)
7. Any authors are successful if and only if they are well read. (Ax, Sx, Wx)
8. A horse is gentle only if it has been well trained (Hx, Gx, Tx)
9. Bees and wasps sting if they are either angry or frightened. (Bx, Wx, Ax, Fx)

*10. A professor is a good lecturer if and only if he is both well informed and entertaining. (Px, Gx, Wx, Ex)

## Unit 3: Truth and Falsity in Predicate Logic



### 1.1 Introduction

In this unit, we focus on an important issue in any aspect of logic – the ways of determining truth-values. So, in the present context, the agenda is to consider the various ways of deducing truth and false propositions in predicate logic.

### 1.2 Learning Outcomes

In this unit, we should have learned to:

1. Identify the various ways or means of attaining truth-values in predicate logic
2. Identify the various connectives and what truth-values they imply

### 1.3 How to Identify Truth and Falsity in Predicate Logic

1. **Singular Propositions:** A singular proposition, like an atomic or simple proposition in truth-functional logic, is either true or false. Thus, for example, the singular proposition represented as

Hs

(Read: 'Socrates is human') is either true or false, since it ascribes an attribute '…is human' to a specific individual, Socrates. The proposition is true if Socrates is indeed human. Similarly, the singular proposition;

Ha

is either true or false since, like the one above, it ascribes a definite attribute say, 'x is human' to a specific individual, say, Aristotle. The same can be said for 'Hb', 'Hc', 'Hd' etc.

If the singular proposition is compound the truth-value will be determined in accordance with the truth-conditions of the connective involved. Thus, the proposition:

$Hs \supset Ms$

(Read: if Socrates is human then he is mortal) is false just in case Socrates is human but is not mortal. In other words, it is false only if the antecedent is true and the consequent false; it is otherwise true. Similarly, the singular proposition represented by:

Hs . Ms

interpreted as: Socrates is human and Socrates is mortal is true just in case Socrates is both human and mortal; that is, 'Hs . Ms' is true only if both conjuncts are true; it is false if any or both of the conjuncts are false.

2. **Quantified Propositions:** Universally quantified propositions as well as existentially quantified propositions can be either true or false. Thus, for example, the universally quantified proposition:

(x)Hx

(Interpreted as: 'Everything is human') is either true or false. It is true if everything is indeed human, Thus, if '(x)Hx' is true it means that 'Ha', 'Hb', 'Hc', 'Hd' … 'Hn' are all true. In other words, the truth of '(x)Hx' implies the truth of a compound conjunction of 'Ha', 'Hb' 'Hc' 'Hd' … 'Hn', which are all *instantiations* of the *matrix* of the universally quantified proposition, '(x)Hx'.

Similarly, the existentially quantified proposition

'(∃x)Hx'

(Interpreted as: 'Something is human') is either true or false, '(∃x)Hx' is true if at least one thing is human. Thus '(∃x)Hx' is true if 'Ha' or 'Hb' or 'Hc' or 'Hd' etc is true. In other words, the truth of '(∃x)Hx' implies the truth of a disjunction of 'Ha', 'Hb', 'Hc', 'Hd' … 'Hn' which are all *instantiations* of the *matrix* of the existentially quantified proposition, '(∃x)Hx'.

If the matrix of the quantified proposition is compound, the truth-value of each instantiation of the matric will be determined in accordance with the truth-conditions of the connective involved. Thus, the universally quantified proposition:

(x)(Hx ⊃ Mx)

(Interpreted as; 'All human beings are mortal') is true just in case the conjunction of 'Ha ⊃ Ma', 'Hb ⊃ Mb', 'Hc ⊃ Mc', 'Hd ⊃ Md' etc, is true. Any of the instantiations is false just in case the antecedent is true and the consequent is false.

Similarly, the existentially quantified proposition:

(∃x)(Nx . Ax)

(Interpreted as: 'Some Nigerians are Africans') is true just in case the disjunction of 'Na . Aa', 'Nb . Ab', 'Nc . Ac' etc, is true. Now, any of the disjuncts is true if both its constituent conjuncts are true, that is, just in case there is one person who is both a Nigerian and an African.

3. **Propositional Functions:** A propositional function is any formula which contains a free occurrence of a variable. A variable occurs free in a formula if it is not bound by a quantifier. A variable is bound by a quantifier if the variable falls within the scope of the quantifier. The scope of a quantifier is indicated by punctuation marks, namely, parentheses, brackets and/or braces. The following formulas are all examples of propositional functions:

Hx

Hx ⊃ Mx

Hx . Ma

(x)Hx ⊃ Mx

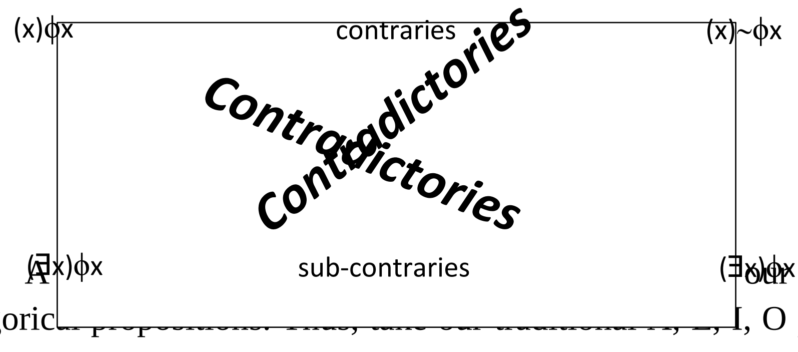
(x)(Hx ⊃ Mx) ≡ Hx

(∃x)Hx . Mx

(∃x)(Hx . Mx) . Ax

A propositional function is neither true nor false. The reason is obvious. For example, ‘Hx’ read ‘x is human’ does not say anything about any specifiable individual; so its truth-value is indeterminable. A propositional function can be turned into a proposition either by replacing the variable with an individual constant, or by quantifying over the free variable. Thus, the propositional function, ‘Hx’, can be turned into a proposition by replacing ‘x’ with ‘a’, making it ‘Ha’, or by quantifying over it, having ‘(x)Hx’ or ‘(∃x)Hx’. Conversely, a proposition can be turned into a propositional function either by replacing an individual constant with an individual variable, or by dropping a quantifier, or by removing a part of a formula from the scope of a quantifier.

**Squares of Oppositions Involving Quantified Propositions:** A relation of opposition holds between different sets of the following propositions: ‘(x)Hx’, ‘(x)~Hx’, ‘(∃x)Hx’ and ‘(∃x)~Hx’. In a universe containing at least one individual, ‘(x)Hx’ and ‘(x)~Hx’ are *contraries,* that is, they cannot both be true, though they may both be false. In such a universe ‘(∃x)Hx’ and ‘(∃x)~Hx’ are *sub-contraries,* that is, they cannot both be false and may both be true. Similarly, ‘(x)Hx’ and ‘(∃x)~Hx’ are *contradictories,* as are ‘(x)~Hx’ and ‘(∃x)Hx’, that is, members of each set cannot both be true, nor can they both be false. In other words, the truth of one member of the set implies the falsity of the other, and the falsity of one member of the set implies the truth of the other. Lastly, in a universe containing at least one individual, the truth of (x)Hx implies the truth of ‘(∃x)Hx’, just as the truth of ‘(x)~Hx’ implies the truth of ‘(∃x)~Hx’. If we use our predicate variable ‘ϕ’, we can generalise the above results in the following square array:




A [illegible] our traditional standard-form categorical propositions. Thus, take our traditional A, E, I, O propositions symbolised as follows:

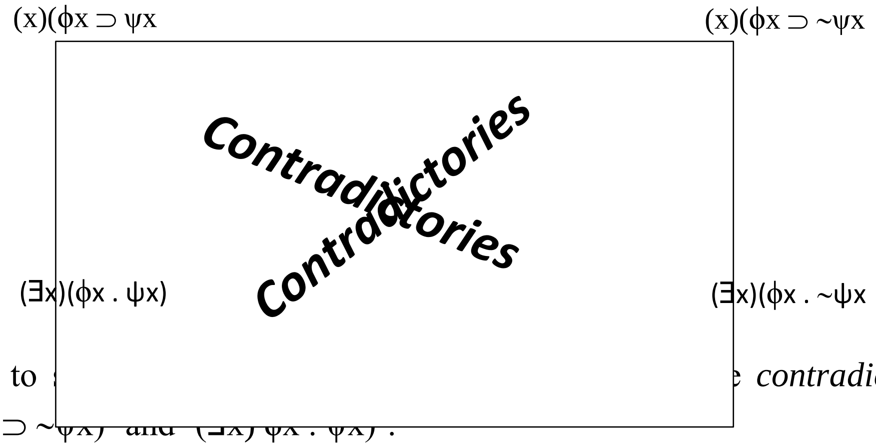
(x) (Hx ⊃ Mx) read: All human beings are mortal beings

(x) (Hx ⊃ ~Mx) read: No human beings are mortal beings

(∃x) (Hx . Mx) read: Some human beings are mortal beings

(∃x) (Hx . ~Mx) read: Some human beings are mortal beings

In a universe containing one individual, say, ‘a’, if ‘Ha’ is false and ‘Ma’ is true, then, (x) (Hx ⊃ Mx) is true, while ‘(∃x) (Hx . ~Mx)’ is false. Similarly, in such a universe ‘(x) (Hx ⊃ ~Mx)’ is true while ‘(∃x) (Hx . Mx)’ is false. However, none of the other relations of the traditional square of opposition holds. Thus, if ‘Ha’ is false and ‘Ma’ is true, then in a universe containing at least one individual, both ‘$(x)\ (Hx \supset Mx)$ and $(x)\ (Hx \supset {\sim}Mx)$’ are true, thus not being contraries. Similarly, both ‘$(\exists x)\ (Hx\ .\ Mx)$’ and ‘$(\exists x)\ (Hx\ .\ {\sim}Mx)$’ will be false, thus not being sub-contraries. To generalise from all this, using our predicate variables ‘$\phi$’ and ‘$\psi$’, what remains of the ‘traditional’ square of opposition is the following:




That is to [illegible] *contradictories*, just as are ‘$(x)(\phi x \supset {\sim}\psi x)$’ and ‘$(\exists x)\ \phi x\ .\ \psi x)$’.

Quite clearly, these squares of opposition afford us some inferences. Thus, from the truth of ‘$(x)\phi x$’ we may infer the falsity of ‘$(x){\sim}\phi x$’, since they are contraries. Similarly, from the falsity of ‘$(\exists x)\phi x$’ we may infer the truth of ‘$(\exists x){\sim}\phi x$ since they are sub-contraries, just as we may infer the falsity of ‘$(\exists x)(\phi x\ .\ \psi x)$’ from the truth of ‘$(x)\ (\phi x \supset {\sim}\psi x)$’, since they are contradictories.

**Quantifier Equivalence**: Every universally quantified statement can be expressed in terms of an existentially quantified statement. Similarly, every existentially quantified statement can be expressed in terms of a universally quantified statement. Thus, to say

$(x)\ Hx$ (read: Everything is human)

is to say:

${\sim}(\exists x){\sim}Hx$ (read: It is not the case that anything exists that is not human).

Similarly, to say

$(\exists x)Hx$ (read: Something is human)

is to say:

${\sim}(x){\sim}Hx$ (read: It is not true that nothing is human).

Again, to say that:

$(x){\sim}Hx$ (read: Nothing is human)

is to say:

${\sim}(\exists x)Hx$ (read: It is false that anything exists which is human).

Similarly, to say:

$(\exists x){\sim}Hx$ (read: Something is not human)

is to say:

${\sim}(x)Hx$ (read: It is not true that everything is human).

These results can be generalised into the following logically equivalent formulas:

$(x)\ \Phi x \equiv \sim(\exists x)\sim\Phi x$

$(x)\sim\Phi x \equiv \sim(\exists x)\ \Phi x$

$(\exists x)\ \Phi x \equiv \sim(x)\sim\Phi x$

$(\exists x)\sim\Phi x \equiv \sim(x)\Phi x$

More complex formulas can be handled in the same way. Thus, to say:

$(x)(Hx \supset Mx)$ (read: All humans are mortal)

is to say:

$\sim(\exists x)(Hx\ .\ \sim M)$ (read: There is no human that is not mortal).

Similarly, to say:

$(\exists x)\ (Hx\ .\ Mx)$ (read: Some humans are mortal)

is to say:

$\sim(x)\ (Hx \supset \sim Mx)$ (read: It is not true that nothing human is mortal),

and so on. We can generalise these logically equivalent formulas as follows:

$(x)(\Phi x \supset \psi x) \equiv \sim(\exists x)\ (\Phi x\ .\ \sim\psi x)$

$(x)(\Phi x \supset \sim \psi x) \equiv \sim(\exists x)\ (\Phi x\ .\ \psi x)$

$(\exists x)\ (\Phi x\ .\ \psi x) \equiv \sim(x)\ (\Phi x \supset \sim \psi x)$

$(\exists x)\ (\Phi x\ .\ \sim\psi x) \equiv \sim(x)\ (\Phi x \supset \psi x)$

The correctness of these equivalences can be seen from the square of opposition. Thus, for example, since '$(x)\Phi x$' is the contradictory of '$(\exists x)\sim\Phi x$', the addition of the negation sign to any of them will guarantee their equivalence. Thus, the addition of the negation sign to '$(x)\Phi x$', that is '$\sim(x)\Phi x$' will make it equivalent to '$(\exists x)\sim\Phi x$' and vice versa. Similarly, since '$(x)(\Phi x \supset \psi x)$' is the contradictory of '$(\exists x)(\Phi x\ .\ \sim\psi x)$', the addition of the negation sign to any of them will guarantee their equivalence. These equivalences can be adopted as rules, called *Quantifier Negation* (QN).

## 1.4 Summary

This unit has been able to examine the various means through which truth and falsity can be deduced in predicate logic.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

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**Module 4: Predicate Calculus II**

Unit 1: Validity in Predicate Logic

Unit 2: Invalidity in Predicate Logic

## Unit 1: Validity in Predicate Logic



### 1.1 Introduction

Units in previous modules have been dedicated to the various ways of becoming familiar with the nitty-gritty of predicate logic. In this module, we take this bold step even further as we now turn to see how validity can be determined in predicate logic.

### 1.2 Learning Outcomes

In this unit, the learners are expected to:

Identify the means of attaining validity in predicate logic

Solve relevant exercises that will deepen understanding

### 1.3 Determining Validity in Predicate Logic

The validity of arguments in predicate logic can be demonstrated by using the method of natural deduction. As we remarked in our discussion of this method in truth-functional logic, if an argument is valid, then it will be possible to produce a proof of its validity. However, our inability to produce a proof in a particular instance does not necessarily imply that the argument is not valid; it may have something to do with the student ability.

Before we discuss some of the rules needed to prove arguments in predicate logic, it is necessary to clarify a point about the distinction we made in between a proposition and a propositional function. A proposition, whether singular or general, may be either true or false. A propositional function, on the other hand, is neither true nor false. However, in constructing proofs of validity in predicate logic, both propositions and propositional functions will have to be treated *as if* they are either true or false. This assumption will be found to be innocuous, but necessary, if we are to be able to make inferences from propositions to propositional functions, and from propositional function. It is only to be hoped that only propositions and no propositional functions, will occur as premises and conclusion, propositional functions occurring only as intermediate derivations between those premises and conclusion.

**Rules of Inference in Predicate Logic**

1. All the rules discussed in truth-functional logic also apply in predicate logic. These include the rules of inference, the rule of replacement, the rule of

conditional proof and the rule of indirect proof. These rules apply to the *matrices* (or matric functions) of quantified statements; the *matrix* of a quantified statement consists in the formula without the quantifier. For example, the matrix of the proposition.

$(x)\ (Hx \supset Mx)$

Is the propositional function

$Hx \supset Mx$

whose substitution instances include ‘Ha ⊃ Ma’, ‘Hb ⊃ Mb’, ‘Hc ⊃ Mc’… ‘Hn ⊃ Mn’. Both the matrix and its substitution instances are amenable to the rules mentioned above. Thus, the conjunction of ‘Ha ⊃ Ma’ and ‘Ha’ implies ‘Ma’ in accordance with the rule of *modus ponens*. Similarly, ‘Ha . Ma’ implies ‘Ha’ in accordance with the rule of *simplification*. Again, from ‘Ha’ we can infer’~~Ha’ in accordance with rule of double negation.

However, it is possible to have a quantified proposition as part of a formula which undergoes transformation according to those rules. Thus, for example, the conjunction of ‘(x)Hx ⊃ Hs’ and ‘(x)Hx’ implies ‘Hs’ in accordance with the rule of *modus ponens*. Similarly, ‘(x) Hx’ implies ‘(x)Hx v (x)Mx’ according to the rule of addition. Note that only the antecedent of ‘(x)Hx ⊃ Hs’ is a quantified statement, since the quantifier does not ‘scope’ over the consequent. Similarly, the disjunction ‘(x)Hx v (x)Mx’ is a disjunction of two quantified propositions.

2. Another set of the rules that apply to propositions in predicate logic are the Quantifier Negation (QN) rules. They are as follows:
   (i) $(x)\Phi x \equiv \sim(\exists x)\sim\Phi x$
   (ii) $(\exists x)\sim \Phi x \equiv \sim(x)\sim\Phi x$
   (iii) $(x)\sim\Phi x \equiv \sim(\exists x)\ \Phi x$
   (iv) $(\exists x)\sim\Phi x \equiv \sim(x)\ \Phi x$

If we take these rules along with other rules, we can show that, ‘~(x)(Φx . ~Ψx)’, can be rewritten as ‘(x)(Φx ⊃ Ψx)’ thus:

1. $(x)\sim(\Phi x\ .\ \sim\Psi x)$ by QN(iii)
2. $(x)(\sim\Phi x \vee \sim\sim\Psi x)$ by DMT from (1)
3. $(x)(\Phi x \supset \sim\sim\Psi x)$ by MI from (2)
4. $(x)(\Phi x \supset \Psi x)$ by DN from (3)

This set of rules, as we have noted earlier, enables us to rewrite a universal quantifier in terms of (the negation of) an existential quantifier and vice versa, and to rewrite and existential quantifier in terms of (the negation of) a universal quantifier and vice versa.

3. The last set of rules to be discussed enables us to dd a quantifier to a formula or to drop a quantifier from a formula. Thus, there will be four such rules, two to add

quantifiers, either universal or existential, and two to drop quantifiers, either universal or existential. Like our rules of inference for truth-functional logic, these rules apply to whole lines only, never to parts of a line. The rules are:

(i) Universal Instantiation (UI)
(ii) Existential Generalisation (EG)
(iii) Existential Instantiation (EI), and
(iv) Universal Generalisation (UG)

Let us discuss each rule in turn:

**(i) Rule of Universal Instantiation (UI):** The import of the rule of universal instantiation is that if everything (in a universe of discourse) has a certain property then anything in that universe has the property. The rule enables us to make transition from a universally quantified proposition to any substitution instance of its matrix or matric function. In other words, the rule allows us to drop a universal quantifier, leaving its matrix or a substitution instance thereof. It is useful in this connection to make a distinction between the variable of quantification and the variable (or constant) of instantiation. The rule allows us to retain the variable of quantification as the instantiating variable, for example:

1. (x)Fx
   ∴ Fx

2. (x)(Fx ⊃ Gx)
   ∴ Fx ⊃ Gx

3. (x) [Fx .(Gx v Hx)]
   ∴ Fx . (Gx v Hx)

or to replace the variable of quantification with another variable, for example,

1. (x)Fx
   ∴ Fy

2. (x)(Fx v Gx)
   ∴ Fz v Gz

3. (x) [Fx ⊃ (Gx v Hx)]
   ∴ Fy ⊃ (Gy v Hy)

or to replace the variable of quantification with an individual constant, for example,

1. (x)Fx

$\therefore$ Fa

2. (x)(Fx $\supset$ Gx)
$\therefore$ Fa $\supset$ Ga

3. (x)(Fx $\supset$ Gx)
$\therefore$ Fb $\supset$ Gb

4. (x)(Fx $\supset$ Gx)
$\therefore$ Fc $\supset$ Gc

or to replace the occurrence of the variable of quantification with another variable which occurs free in the quantified proposition, for example,

1. (x)(Fx $\supset$ Gy)
$\therefore$ Fy $\supset$ Gy

2. (y)(Fx $\supset$ Gy)
$\therefore$ Fx $\supset$ Gx

3. (x) [(Fx . Gy) $\supset$ Hx]
$\therefore$ (Fy . Gy) $\supset$ Hy

4. (z)[Fx $\supset$ (Gy . Hz)]
$\therefore$ Fx $\supset$ (Gy . Hy)

or to replace the variable of quantification with a new variable, for example,

1. (y)(Fy $\supset$ Gb)
$\therefore$ Fx $\supset$ Gb

2. (x)(Fx $\supset$ Gy)
$\therefore$ Fz $\supset$ Gy

or to replace the variable of quantification with a constant that occurs in the premises, for example,

1. (z)(Fz $\supset$ Gb)
$\therefore$ Fb v Gb

2. (z)(Fz $\supset$ Gb)
$\therefore$ Fb $\supset$ Gb

or to replace the variable of quantification with a fresh constant, for example,

1. (y)(Fy $\supset$ Gb)
   $\therefore$ Fa $\supset$ Fb

2. (x)(Fx $\supset$ Gy)
   $\therefore$ Fc $\supset$ Gy

However, the Rule of Universal Instantiation does not permit any of the following inferences:

1. (x)(Fx $\supset$ Gx)
   $\therefore$ Fx $\supset$ Gy

2. (x)(Fx $\supset$ Gx)
   $\therefore$ Fa $\supset$ Gx

3. (x)(Fx $\supset$ Gx)
   $\therefore$ Fb $\supset$ Gx

4. (x) (Fx $\supset$ Gx)
   $\therefore$ Fz $\supset$ Gx

5. (x) [Fx $\supset$ (Gx v Hx)]
   $\therefore$ Fz $\supset$ (Gy v Hy)

because none is a 'consistent' or 'uniform' substitution of the variable of quantification with either another variable or with a constant. A substitution is consistent or uniform if it replaces every occurrence of the variable of quantification with the same variable of instantiation or constant throughout. However, in our example (1) the first occurrence of 'x' is replaced with 'x' while the second is replaced with 'y'. Similarly, in (3) the first occurrence of 'x' is replaced with 'b' while the second is replaced with 'x'. Again, in (5) the first occurrence of the variable 'x' is replaced with 'z' whereas the last two are replaced with 'y'.

Similarly, the rule does not permit the following inferences:

1. (x)(Fx $\supset$ Gy)
   $\therefore$ Fx $\supset$ Gx

2. (x)(Fx $\supset$ Gb)
   $\therefore$ Fx $\supset$ Gx

because the rule does not allow us to change any variable or constant other than the variable of quantification. In other words, constants and variables that occur free in a matric function cannot be changed; we are free to change only the variable of quantification.

Again, the rule does not allow any of the following inferences:

1. $(x)Mx \supset Mc$
$\therefore Mc \supset Mc$

2. $(x)Fx \supset Ga$
$\therefore Fa \supset Ga$

because the rule is not applied to the whole line but to only the part of the line that is quantified.

Lastly, the rule does not permit any of the following types of inferences:

1. $\sim(x)Fa$
$\therefore \sim Fa$

2. $\sim(x)(Hx \supset Mx)$
$\therefore \sim(Hx \supset Mx)$

because the rule cannot be applied to the negation of a general proposition.

It is now possible to state the Rule of Universal Instantiation, along with its restrictions:

From '(x)Φx' it is permissible to infer either 'Φa' or 'Φy' provided that:

(1) every 'x' in Φx is replaced by the constant 'a' in 'Φa' or by the variable 'y' in 'Φy'.

(2) no other variable or constant in 'Φx' is changed or its status (for example, freedom) is tampered with.

(3) '(x)Φx' is the whole expression and not just part of a more complex expression.

(4) '(x)' in '(x)Φx' is not preceded by the sign of negation.

For the purpose of this rule all the following expressions are '(x)Φx':

$(x)Fx$

$(x)(Fx \supset Hx)$

$(x)(Fx \supset Gy)$

$(x)(Fy \supset Gx)$

$(x)[Fx \supset (Gx \vee Hx)]$

'Φa' and 'Φy' are any substitution instances involving either a constant or a variable, no matter how complex.

The following proof illustrates the use of the Rule of Universal Instantiation.

| | | |
|---|---|---|
| 1. | (x)(Ax ⊃ Rx) | Pr. |
| 2. | ~Rs | Pr./ ∴ ~As |
| 3. | As ⊃ Rs | I, UI |
| 4. | ~As | 3,2, MT. |

(ii) **Rule of Existential Generalization (EG):** The Rule of Existential Generalisation enables us to make a transition from a singular proposition such as 'Fa', 'Fy', 'Fy. 'Gy', 'Fa . Gb' to its existential quantification. What this amount to is that if an individual has a property, then it is right to conclude that there exists an individual who has that property. The rule allows us to existentially quantify over an individual constant, for example,

1. Fa
   ∴ (Ǝy)Fx

2. Fa
   ∴ (Ǝy)Fy

3. Ga . Fa
   ∴ (Ǝy)(Gx . Fx)
4. Fa v Gb
   ∴ (Ǝx)(Fx v Gb)

5. Fa ⊃ Gb
   ∴ (Ǝy)(Fa ⊃ Gy)

6. Ba ⊃ Le
   ∴ (Ǝy)(Bx ⊃ Le)

7. Ga . Fb
   ∴ (Ǝx)(Gx . Fb)

8. Kd ⊃ (Md v Nd)
   ∴ (Ǝx)[Kx ⊃ (Mx v Nx)]

9. Fa . Ga

$\therefore$ (∃x)(Fx . Ga)

10. Fx

$\therefore$(∃y)Fy

11. Fx ⊃ Gy

$\therefore$(∃y)(Fx ⊃ Gy)

12. Fx . Gx

$\therefore$(∃y)(Fy . Gx)

13. ~Gx

$\therefore$(∃x)~Gx

14. Fy . (Gy . Hy)

$\therefore$(∃x) [Fx . (Gx . Hx)

However, the Rule of Existential Generalisation does not allow inferences of the following types:

1. Fx . Gy

$\therefore$(∃z)(Fz . Gz)

2. Fx . Gy

$\therefore$(∃z)(Fz . Gw)

3. Fx ≡ ~Fy

$\therefore$(∃x)(Fx ≡ ~Fx)

4. Px . Ba

$\therefore$(∃x) (Px . Bx)

because two different individual symbols (constant or variable) have been changed at once, using the same variable of quantification; in (1), (2), and (3), 'x' and 'y', in (4) 'x' and 'a'. Similarly, the rule does not allow inferences of the following forms:

1. Ba ⊃ Le

$\therefore$(∃x)Bx Le

2. ~Ad

$\therefore$~(∃x) Ax

because the quantification is applied to only part of the line, not the whole time.

It is now possible to state the rule of existential generalisation with its restrictions, as follows:

From an expression of the form 'Φa' or one of the form 'Φy', it is permissible to infer an expression of the form '(∃x)Φx', provided that:

(1) every 'a' and only 'a' in 'Φa', or every 'y' and only 'y' in 'Φy' is replaced by a corresponding 'x' in 'Φx',

(2) no other variable or constant in 'Φa' or 'Φy' is changed or its status (for example, its freedom) otherwise tampered with;

(3) the existential quantifier governors the whole line, not merely part of the line.

(iii) **Rule of Existential Instantiation (EI):** The rule of existential instantiation is more difficult to describe than any of the previous two. What the rule says is that from an existentially quantified proposition we can derive any of its instances. In other words, if ‘(∃x)Φx’ is true, then ‘Φa’ or ‘Φb’ or ‘Φc’ or ‘Φn’ is true. The problem, however, is that we do not know which substitution instance is true. This must remain essentially so because all we are justified in claiming from the truth of ‘(∃x)Φx’ is that something or other has Φ, but whatever it is must remain unknown.

However, we are free to *assume* that the instantiating individual, which is known, is either ‘x’ or ‘y’ or ‘z’ or ‘a’ or ‘a’, provided that whichever individual we assume is not confused with any known individual. In the context of a proof, what this means is that the individual symbol must not occur free prior to the assumption, or if it is a constant, it must not have a prior occurrence. Moreover, since the substitution instance is an assumption, it must be discharged, as it is the case with all legitimate assumptions to which the rule of conditional proof has made us accustomed.

Let us take the following argument:

Beninois and Ugandans are Africans

There are Beninois.

Therefore, some Beninois are Africans.

Symbolised, we have:

(x) [(Bx v Ux) ⊃ Ax]

(∃x) Bx

∴(∃x) (Bx . Ax)

Its proof will proceed as follows:

| | | |
|---|---|---|
| 1. | (x) [(Bx v Ux) ⊃ Ax] | |
| 2. | (∃x)Bx / ∴ (x) (Bx . Ax) | |
| →3. | By | |
| 4. | (By v Uy) ⊃ Ay | 1, UI |
| 5. | By v Uy | 3, Add |
| 6. | Ay | 4, 5, MP |
| 7. | By . Ay | 3, 6, Conj. |
| 8. | (∃x)(Bx . Ax) | 7, EG |
| 9. | (∃x) (Bx . Ax) | 2, 3-8, EI |

The proof proceeds as follows: As part of the rule of existential instantiation, we assume that 'y' is 'B', the assumption being indicated by the usual arrow. The proof thence proceeds to Line 8 where we have the conclusion. The assumption is discharged by ruling a line under the expression in Line 8 and repeating the formula in Line 9. The justification consists in citing the line from which assumption derives, that is Line 2, and the lines within the scope of the assumption, that is, Lines 3 to 8. It will be noted that the 'variable' in the assumption, the 'unknown' does not occur in the conclusion, thus showing that the assumed unknown can be dispensed with. It should also be noted that our unknown does not have a prior occurrence in the context.

However, each of the following examples contains a wrong application of the Existential Instantiation, thus:

| | | |
|---|---|---|
| 1. | (∃x)Fx | |
| 2. | Ga / ∴ (∃x)(Fx . Gx) | |
| →3. | Fa | |
| 4. | Fa . Ga | 3, 2, Conj. |
| 5. | (∃x) (Fx . Gx) | 4, EG |
| 6. | (∃x) (Fx . Gx) | 1, 3-5, EI |

The assumption here (Line 3) is illegitimate because the instantiating symbol 'a' already occurs in premise 2. Similarly, the application of EI in the following 'proof' is erroneous:

| | | |
|---|---|---|
| 1. | (∃x)Sx | |
| 2. | (∃x)~Sx / ∴ (∃x)(Sx . ~Sx) | |
| →3. | Sy | |
| →4. | ~Sy | |
| 5. | Sy . ~Sy | 3, 4, Conj. |
| 6. | (∃x)(Sx . ~Sx) | 5, EG |
| 7. | (∃x)(Sx . ~Sx) | 2, 4-6, EI |
| 8. | (∃x)(Sx . ~Sx) | 1, 3-7, EI |

The error here is that the instantiating symbol in 4, ‘y’, has already been introduced by EI in Line 3. Again, the application of the rule in the following example is erroneous:

1. (∃y)(Fx ≡ ~Fy) / ∴ (∃x)(Fx ≡ ~Fx)
2. Fx ≡ ~Fx
3. (∃x)(Fx ≡ ~Fx) 2, EG
4. (∃x)(Fx ≡ ~Fx) 1, 2-3, EI

The error here is that ‘x’ the instantiating symbol in Line 2, occurs free in the premise.

Let us now attempt to state the rule of Existential Instantiation, along with its restrictions:

> From an existentially quantified proposition ‘(∃x)Φx’, it is permissible to introduce any of its instances, ‘Φy’ or ‘Φa’ as an assumption. After any subsequent line in which the individual letter in the assumption does not occur free, it is permissible to discharge the assumption and infer that same line below the bar, provided that:

(1) the individual symbol does not occur free in any line preceding the assumption or, if it is a constant, does not have a prior occurrence in the argument;
(2) every ‘x’ in ‘Φx’ is replaced by a corresponding ‘y’ in ‘Φy’, or by a corresponding ‘a’ in ‘Φa’;
(3) a new individual symbol is used for each application of EI;
(4) when both EI and UI are needed, EI must be applied first;
(5) the rule is applied to the whole line of the proof not to part of the line.

(iv) **Rule of Universal Generalisation (UG):** The Rule of Universal Generalisation enables us to introduce a universal quantifier over a propositional function or a singular proposition. What the rule amounts to is that if a universalisable predicate is true of an individual (in the universe of discourse) then every individual in that universe has that predicate. However, since the introduction of a universal quantifier has the effect of universalising the predicate(s), the predicates must be truly universalisable. The predicate must be such as is true of every individual whatsoever (within the universe of discourse). The rule enables us to make the following types of inferences:

1. Fy
   ∴(x)Fx
2. Fa
   ∴(x)Fx
3. Fx ⊃ Ga
   ∴(x)(Fx ⊃ Ga)
4. Fx ⊃ Gy
   ∴(x)(Fx ⊃ Gy)

5. Hy ⊃ ~Py

∴ (x)(Hx ⊃ ~Px)

The rule also allows the following type of inference:

Ba ⊃ Da

∴ (x)(Bx ⊃ Dx)

if 'Ba ⊃ Da' is not a premise and if it was not introduced by the rule of EI.

However, the rule of universal generalisation does not permit any of the following types of inferences:

1. Fx ⊃ Gz

∴ (x)(Fx . Gx)

2. Fx ⊃ Fy

∴ (x)(Fx ⊃ ~Fx)

because the conclusion in each case generalises over two different variables, using the same variable of quantification. Again, the rule does not allow the following type of inference:

Fx ≡ Fx

∴ (y)(Fx ≡ Fy)

Because 'x' occurs free in '(y)(Fx ≡ Fy)'. Some of the errors in the use of the rule of Universal Generalisation are best illustrated in the context of complete 'proofs'. Thus, for example, in the following proofs:

1. (∃x)Sx
2. Sa
3. (x)Sx 2, UG (wrong)
4. (x)Fx 1, 2-3, EI

2. 1. (x)(Fx / ∴ (x)Fx
2. Fy
3. (x)Fx 2, UG (wrong)
4. (x)Fx 1, 2-3, EI

3. 1. (x)(Fx ⊃ Gx)
2. (∃x)Fx / ∴ (x)Gx
3. Fy
4. Fy ⊃ Gy 1, UI
5. Gy 4, 3, MP
6. (x)Gx 5, UG (wrong)
7. (x)Gx 2, 3-6, EI

the uses of UG are erroneous because the variable or constant quantified over in each case was introduced by the rule of EI. Similarly, in the following 'proofs':

A. 1. Lc / ∴ (x)Lx

2. (x)Lx I, UG (wrong)

B. 1. Fa / ∴ (x)Fx

2. (x)Fx I, UG (wrong)

the uses of UG are erroneous, because the constant quantified over in each case occurs in the premise. Lastly, in the following 'proof':

| | | |
|---|---|---|
| 1. | ~(x)Gx / ∴ (x) ~Gx | |
| 2. | Gy | |
| 3. | (x)Gx | 2, UG (wrong) |
| 4. | Gy ⊃ (x)Gx | 2-3, CP |
| 5. | ~Gy | 4, 1, MT |
| 6. | (x) ~Gx | 5, UG |

the first use of UG (in Line 3) is erroneous because the variable quantified over ('y') falls within the scope of an assumption.

Let us now state the rule of Universal Generalisation, along with its restrictions:

From 'Φy' or 'Φa' one may infer '(x)Φx', provided that:

(1) every 'y', and 'y' in 'Φy' or every 'a' and only 'a' in 'Φa' is replaced by a corresponding 'x' in 'Φx'.

(2) the 'y' in 'Φy' or 'a' in 'Φa' did not occur in the premise and was not introduced into the proof by EI and was not introduced into the proof as an assumption;

(3) the 'y' in 'Φy' does not occur free in any premise;

(4) the variable generalised over does not occur free in '(x)Φx'.

Let us see how the rules can be applied in proving the validity of arguments. The following argument:

(x) (Ax ⊃ Bx)

(x) (Bx ⊃ Cx)

∴(x) (Ax ⊃ Cx)

can be proved using the rules of UI and UG, along with others. The proof will proceed as follows:

| | | |
|---|---|---|
| 1. | (x) (Ax ⊃ Bx) | Pr. |
| 2. | (x) (Bx ⊃ Cx) | Pr. / ∴ (x) (Ax ⊃ Cx) |
| 3. | Ay ⊃ By | 1, UI |
| 4. | By ⊃ Cy | 2, UI |
| 5. | Ay ⊃ Cy | 3, 4, HS |
| 6. | (x) (Ax ⊃ Cx) | 5, UG |

Similarly, the following argument:

(∃x) (Ax . ~Bx)

(x)(Ax ⊃ Cx)

∴ (∃x (Cx . ~Bx)

can be proved using, among others, the rules of EI, UI, and EG. Note that where the rules of EI and UI are to be applied, the rule of EI must be applied first. The proof will proceed as follows:

| | | |
|---|---|---|
| 1. | (∃x)(Ax . ~Bx) | Pr. |
| 2. | (x)(Ax ⊃ Cx) | Pr. / ∴ (∃x)(Cx . ~Bx) |
| 3. | Aw . ~Bw | |
| 4. | Aw ⊃ Cw | 2, UI |
| 5. | Aw | 3, Simp |
| 6. | Cw | 4, 5, MP |
| 7. | ~Bw | 3, Simp. |
| 8. | Cw . ~Bw | 6, 7, Conj. |
| 9. | (∃x)(Cx . ~Bx) | 8, EG |
| 10. | (∃x)(Cx . ~Bx) | 1, 3-9, EI |

Again, the rules of UI and UG can be combined with the rule of CP, among others, to produce a proof. For example, the proof of the following argument:

(x)[(Ax v Bx) ⊃ Cx]
(x)(Dx ⊃ Ax)
∴(x)(Dx ⊃ Cx)

will proceed as follows:

| | | |
|---|---|---|
| 1. | (x)[(Ax v Bx) ⊃ Cx] | Pr. |
| 2. | (x)(Dx ⊃ Ax) | Pr. / (x)(Dx ⊃ Cx) |
| 3. | (Ay v By) ⊃ Cy | 1, UI |
| 4. | Dy ⊃ Ay | 2, UI |
| 5. | Dy | |
| 6. | Ay | 4, 5, MP |
| 7. | Ay v By | 6, Add |
| 8. | Cy | 3, 7, MP |
| 9. | Dy ⊃ Cy | 5-8, CP |
| 10. | (x)(Dx ⊃ Cx) | 9, UG |

Let us construct a proof for the following more complex argument; using the rules of EI, UI and EG, among others:

(x)[(Ax . ~Bx) ⊃ Cx]
(x)(Dx ⊃ Ax)
(∃x)(Dx . ~Cx)
∴(∃x)Bx

The proof will proceed as follows:

| | | |
|---|---|---|
| 1. | (x)[Ax . ~Bx) ⊃ Cx] | Pr. |
| 2. | (x)(Dx ⊃ Ax) | Pr. |
| 3. | (∃x)(Dx . Cx) | Pr. / ∴ (∃x) Bx |

| | | |
|---|---|---|
| 4. | Dw . ~Cw | |
| 5. | Dw ⊃ Aw | 2, UI |
| 6. | (Aw . ~Bw) ⊃ Cw | 2, UI |
| 7. | Dw | 4, Simp |
| 8. | Aw | 5, 7, MP |
| 9. | ~Cw | 4, Simp |
| 10. | ~(Aw . ~Bw) | 6, 9, MT |
| 11. | ~Aw v ~~B | 10, DMT |
| 12. | ~~Aw | 8, DN |
| 13. | ~~Bw | 11, 12, DS |
| 14. | Bw | 13, DN |
| 15. | (∃x)Bx | 14, EG |
| 16. | (∃x)Bx | 3, 4-15, EI |

Lastly, let us look at the proof for the following argument involving the rules of UI and UG; among others:

(x)[Ax ⊃ (Bx ⊃ Cx)]

(x)(Cx  (Dx . Ex)]

∴(x)(Ax  (Bx  Dx)]

The proof will proceed as follows:

| | | |
|---|---|---|
| 1. | (x)[Ax ⊃ (Bx ⊃ Cx)] | Pr. |
| 2. | (x)[Cx ⊃ (Dx . Ex)] | Pr. / ∴ (x)[Ax ⊃ (Bx ⊃ Dx)] |
| 3. | Ay ⊃ (By ⊃ Cy) | 1, UI |
| 4. | Cy ⊃ (Dy . Ey) | 2, UI |
| 5. | Ay . By | |
| 6. | (Ay . By) ⊃ Cy | 3, Exp. |
| 7. | Cy | 6, 5, MP |
| 8. | Dy . Ey | 4, 7, MP |
| 9. | Dy | 8, Simp. |
| 10. | (Ay . By) ⊃ Dy | 5-9, CP |
| 11. | Ay ⊃ (By ⊃ Dy) | 10, Exp. |
| 12. | (x)[Ax ⊃ (Bx ⊃ Dx)] | 11, UG. |

## 1.4 Summary

What has been done thus in this unit is to examine the ways through which inferences or propositions can be determined as valid. The various rules of validity and the conditions for their validity have also been disclosed in this unit.

## 1.5 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press
Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited
Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 1.6 Unit Exercises

I. Construct a proof of validity for each of the following arguments:

1. $(x)(Sx \supset Tx)$
   $(x)(Tx \supset Ux)$
   $\therefore (x)(Sx \supset Ux)$
2. $(x)(Gx \supset Hx)$
   $(x)(Ix \supset {\sim}Hx)$
   $\therefore (x)(Ix \supset {\sim}Gx)$
3. $(x)(Bx \supset {\sim}Cx)$
   $(\exists x)(Cx \cdot Dx)$
   $\therefore (\exists x)(Dx \cdot {\sim}Bx)$
4. $(x)(Mx \supset Nx)$
   $(\exists x)(Mx \cdot Ox)$
   $\therefore (\exists x)(Ox \cdot Nx)$
5. $(x)(Ax \supset {\sim}Bx)$
   $(x)(Cx \supset Bx)$
   $\therefore (x)(Cx \supset {\sim}Ax)$

II. Construct a proof of validity for each of the following arguments, using the suggested notations to symbolise the arguments:

1. No contractors are dependable. Some contractors are engineers. Therefore, some engineers are not dependable. (Cx, Dx, Ex)
2. No gamblers are happy. Some idealists are happy. Therefore, some idealists are not gamblers. (Gx, Hx, Ix)

*3. There are no uniforms that are not washable. There are no washable velvets. Therefore, there are no velvet uniforms. (Ux, Wx, Vx)

4. Tigers are fierce and dangerous. Some tigers are beautiful. Therefore, some dangerous things are beautiful. (Tx, Fx, Dx, Bx)
5. Bananas and oranges are fruits. Fruits and vegetables are nourishing. Therefore, bananas are nourishing. (Bx, Ox, Fx, Vx, Nx)

## Unit 2: Invalidity in Predicate Logic



### 1.1 Introduction

Dear students, this is the last unit of this interaction in predicate logic. It is also the second and final unit of this fourth module. What we are going to do is to provide some attention to ways that we can identity invalidity in predicate logic. This will balance our understanding with what was discussed in the preceding section. There are two ways to show the invalidity of an argument in predicate logic, namely, (1) by giving a natural interpretation of the argument, and (2) by giving an interpretation for a model universe. Both methods are based on the oft-repeated principles that if an argument is invalid then any argument with that form is invalid, and that an argument with true premises and a false conclusion is invalid. Both methods can be applied in predicate logic because the validity or invalidity of arguments here, like arguments in truth-functional logic, depends on their forms. Let us discuss each method in some detail.

### 1.2 Learning Outcomes

In this unit, the learners will learn how to:

1. Identify the rules of invalidity in predicate logic
2. Identify how to solve and apply the rules through some practice exercises
3. Identify and apply the determining conditions in the methods of natural interpretation and interpretation for a model universe

### 1.3 Method of Natural Interpretation

It is also known as the *method of refutation by logical analogy):* This method is based on the principle that if an argument in predicate logic is invalid it will be possible to come up with an interpretation of its predicate and individual letters which will show that the premises are true and the conclusion false. The interpretation involves assigning some value to every predicate and individual letter in the argument, provided that every letter is assigned the same value wherever it occurs in the argument and no two letters are assigned the same value. Thus, for example, the argument:

All Nigerians are Africans
All indigenes of Jigawa State are Africans

Therefore, all indigenes of Jigawa States are Nigerians

can be shown to be invalid. First, we symbolise it as follows:

$(x)(Nx \supset Ax)$

$(x) (Bx \supset Ax)$

$\therefore (x)(Bx \supset (Nx)$

Then we assign some (other) values to the letters as follows:

Nx: x is a dog

Ax: x is an animal

Bx: is a cat

The 'invalidating interpretation' becomes:

All dogs are animals

All cats are animals

Therefore, all cats are dogs

which is patently invalid, because the premises are true while the conclusion is false. But this argument is of the same form as the original argument. Therefore, the original argument is invalid, since it has the same form as an invalid argument. This proof of invalidity will be written out as follows:

$(x)(Nx \supset Ax)$

$(x)(Bx \supset Ax)$

$\therefore (x)(Bx \supset Nx)$

shown invalid by Nx: x is a dog

Ax: x is an animal

Bx: x is a cat

Similarly, the following argument:

All soldiers are politicians

Babangida is a politician

Therefore, Babangida is a soldier

can be shown to be invalid. First, let us symbolise the argument, as follows:

(x) (Sx Px)

Pb

$\therefore$ Sb

An invalidating interpretation is:

Sx: x is a Nigerian

Px: x is an African

b: Idi Amin

which turns into the argument;

All Nigerians are Africans

Idi Amin is an African

Therefore, Idi Amin is a Nigerian

which is clearly invalid, since the premises are true and the conclusion false. This proof of invalidity will be written out as follows:

$(x)\ (Sx \supset Px)$

Pb

$\therefore$ Sb

Shown invalid by Sx: x is a Nigeria

Px: x is an African

b: Idi Amin

However, though if an argument is invalid we may be able to come up with an invalidating interpretation, that we are not able to come up with one does not mean that the argument is valid. Moreover, there are no rules for constructing an invalidating interpretation apart from the requirements that the interpretation be consistent, that the interpretation have the same form as the original argument, and that the premises should be *seen* to be true and the conclusion *seen* to be false. (This last requirement is such that unless it is fulfilled the whole exercise may be futile.) Though only some ingenuity is needed to produce an invalidating interpretation, that is not always there. At any rate, once an invalidating interpretation that meets the above requirements is found, it tends to show, conclusively, that the argument in question is invalid.

## 1.4 Method of Interpretation for a Model Universe

Recall that the truth of a universally quantified proposition implies the truth of the *conjunction* of the substitution instances of the matric function of the universally quantified proposition. Thus, the truth of:

$(x)Mx$

implies the truth of the following conjunction:

$Ma \ . \ Mb \ . \ Mc \ . \ Md \ . \ Mn.$

Similarly, the truth of the universally quantified proposition:

$(x)(Hx \supset Mx)$

implies the truth of the following conjunction;

$(Ha \supset Ma) \ . \ (Hb \supset Mb) \ . \ (Mc \supset Mc) \ . \ (Hn \supset Mn)$

Since the truth of a conjunction implies the truth of all its conjuncts, another way of putting this is that the universally quantified statement '(x)Mx' is true only if everything in the universe satisfies the propositional function 'Mx', and the universally quantified proposition '(x) (Hx ⊃ Mx)' is true only if everything in the universe satisfies the propositional function 'Hx ⊃ Mx'.

Recall, similarly, that the truth of an existentially quantified proposition implies the truth of the disjunction of the substitution instances of the matric function of the existentially quantified proposition. Thus, the truth of:

$(\exists x)Mx$

implies the truth of the following disjunction:

Ma v Mb v Mc v Md v Mn

Similarly, the truth of the existentially quantified proposition:

(∃x) (Hx . Mx)

implies the truth of the following disjunction:

(Ha. Ma) v (Hb . Mb) v (Hc . Mc) v (Hn . Mn)

Since the truth of a disjunction implies the truth of at least one of its disjuncts, another way of putting this is that the existentially quantified proposition '(∃x)Mx' is true if at least one thing in the universe satisfies the propositional function 'Mx'. Similarly, the existentially quantified proposition '(x)(Hx . Mx)' is true if at least one thing in the universe satisfies the propositional function 'Hx . Mx'.

However, the real universe contains an infinite number of things (though it need not), and our conjunction and disjunction can in principle be extended infinitely. This is why we need a more manageable universe of discourse or a model universe. Thus, we can talk of a universe containing only one, or two, or three, individuals. A one-member model universe may be indicated by writing the name of the member, say, 'a' in braces, thus: {a} . A two-member universe containing, say, 'a' and 'b', will be indicated as follows: {a,b}, that is, 'a' and 'b' separated by a comma, in braces. Similarly, a three-member universe consisting of say, 'a', 'b' and 'c' will be written thus: {a,b,c}, that is, 'a', 'b', 'c', separated by commas, in braces.

Using this method in testing the invalidity of arguments in predicate logic follows a pattern similar to the *reductio ad absurdum* method in truth-functional logic. It involves the principle that an argument is invalid if it is possible for its premises to be true and its conclusion false. The only addition here is that an argument may be invalid for a universe containing one, two or three individuals. In practice, no more than three individuals are needed to prove the invalidity of any argument in the section, though in theory the number can be much higher. The procedure for testing invalidity involves the following steps:

(1) construct an interpretation for a model universe; it is advisable to start by constructing an interpretation for a universe containing one individual, increasing the number of individuals, if necessary, one at a time, up to three;

(2) on the assumption that the argument is invalid, assign a truth-value to every premise and conclusion, making the premises true and the conclusion false:

(3) try to make true the assumption that the argument is invalid by assigning a value to every remaining connective and component, as usual:

(4) if we succeed in showing that every premise is true and the conclusion is false, then the argument is, as assumed, invalid: we succeed if there is no inconsistency in our truth-assignments:

(5) to summarise the answer, list the truth-value assignments for all the distinct components in the argument.

For examples, let us prove the invalidity of the following argument:

$(x)\ (Ex \supset Fx)$

$(x)\ (Gx \supset Fx)$

$\therefore (x)\ (Ex \supset Gx)$

Assuming a universe containing one individual, 'a' the argument translate into:

$Ea \supset Fa$

$Ga \supset Fa$

$\therefore Ea \supset Ga$

Assuming that the argument is invalid, we make the premises true and the conclusion false, thus:

| | | |
|---|---|---|
| Ea | ⊃ | Fa |
| | T | |
| Ga | ⊃ | Fa |
| | T | |
| ∴Ea | ⊃ | Ga |
| | F | |

We then try to make good our assumption by assigning a truth-value to every component as follows:

| | | |
|---|---|---|
| Ea | ⊃ | Fa |
| T | T | T |
| Ga | ⊃ | Fa |
| F | T | T |
| ∴Ea | ⊃ | Ga |
| T | F | F |

Let us inspect our truth-value assignments to see if there is any inconsistency. Since there is no inconsistency in our truth-value assignments, we can conclude that the argument is indeed invalid. The summary will be:

Shown invalid for {a} by Ea Fa Ga<br>T T F

(read: shown invalid for a universe containing one individual, 'a', and where 'Ea' is true 'Fa' is true and 'Ga' is false.)

Let us look at another argument:

$(x)\ (Hx \supset {\sim}Ix)$

$(\exists x)\ (Jx \,.\, {\sim}Ix)$

$\therefore (x)\ (Hx \supset Jx)$

If we assume one individual, 'a', in our universe, this argument is equivalent to:

$Ha \supset {\sim}Ia$

Ja . ~Ia

∴Ha ⊃ Ja

Assigning truth-values, we have:

| | | |
|---|---|---|
| Ha | ⊃ | ~Ia |
| T | T | T |
| Ja | . | ~Ia |
| F | T | T |
| ∴Ha | ⊃ | Ja |
| T | F | F |

There is an inconsistency in the second premise: the conjunction cannot be true since one of the conjuncts, 'Ja', is false. So, this argument is not invalid for a universe containing one individual, 'a'. Therefore, let us try a universe containing two individuals, 'a' and 'b'. For such a universe, the argument is equivalent to:

(Ha ⊃ ~Ia) . (Hb ⊃ ~Ib)

(Ja . ~Ia) v (Jb . ~Ib)

∴(Ha ⊃ Ja) . (Hb ⊃ Jb)

Assigning truth-values, we have

| | | | | | | |
|---|---|---|---|---|---|---|
| (Ha | ⊃ | ~Ia) | . | (Hb | ⊃ | ~Ib) |
| T | T | T | T | T | T | T |
| (Ja | . | ~Ia) | v | (Jb | . | ~Ib) |
| F | F | T | T | T | T | T |
| ∴(Ha | ⊃ | Ja) | . | (Hb | ⊃ | Jb) |
| T | F | F | F | T | T | T |

On inspection, we find no inconsistency. Thus, the argument is invalid for a universe containing two individuals, 'a' and 'b', summarised as follows:

| | Ha | Hb | Ia | Ib | Ja | Jb |
|---|---|---|---|---|---|---|
| Shown invalid for {a, b} by | T | T | F | F | F | T |

Going through the truth-value assignments, it will be seen that 'Hb' may be assigned the truth-value 'F', rather than 'T', without changing the result.

Lastly, let us take the argument:

(x) (Ax ⊃ Bx)

(∃x) (Cx . Bx)

(∃x) (Cx . ~Bx)

∴(x) (Ax ⊃ Cx)

Assuming a universe containing an individual, ‘a’, the argument and our truth-value assignments will come to:

| | | | |
|---|---|---|---|
| | Aa | ⊃ | Ba |
| | T | T | T |
| | Ca | . | ~Ba |
| | F | T | T |
| | Ca | . | ~Ba |
| | F | T | F |
| ∴ | Aa | ⊃ | Ca |
| | T | F | F |

which contains a number of inconsistencies. For a two-number universe, we have the following:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | (Aa | ⊃ | Ba) | . | (Ab | ⊃ | Bb) |
| | T | T | T | T | F | T | T |
| | (Ca | . | Ba) | v | (Cb | . | ~Bb) |
| | F | F | T | T | T | T | T |
| | (Ca | . | ~Ba) | v | (Cb | . | ~Bb) |
| | F | F | F | T | T | F | F |
| ∴ | (Aa | ⊃ | Ca) | . | (Ab | ⊃ | Cb) |
| | T | F | F | F | F | T | T |

There is still an inconsistency in the third premise; so let us try out a universe containing three individuals, ‘a’ ‘b’, and ‘c’, thus:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | [(Aa | ⊃ | Ba) | . | (Ab | ⊃ | Bb)] | . | (Ac | ⊃ | Bc) |
| | T | T | T | T | F | T | T | T | F | T | F |
| | [(Ca | . | Ba) | v | (Cb | . | ~Bb)] | v | (Cc | . | Bc) |
| | F | F | T | T | T | T | T | T | T | F | F |
| | [(Ca | . | ~Ba) | v | (Cb | . | ~Bb)] | v | (Cc | . | ~Bc) |
| | F | F | F | T | T | F | F | T | T | T | T |
| ∴ | [(Aa | ⊃ | Ca) | . | (Ab | ⊃ | Cb)] | . | (Ac | ⊃ | Cc) |
| | T | F | F | F | F | T | T | F | F | T | T |

On inspection we find no inconsistency. Thus, the argument has been shown to be invalid for a universe containing three individuals, summarised thus:

| | | | |
|---|---|---|---|
| Shown invalid for {a, b, c} by | Aa | Ab | Ac |
| | T | F | F |
| | Ca | Cb | Cc |
| | F | T | T |

| Ba | Bb | Bc |
|---|---|---|
| T | T | F |

Note that 'Ab' may be assigned the value 'T' without any harm.

One final point that needs to be made is that if any of the original formulas in an argument contains a constant or a free variable, such a constant or free variable must be used in naming one of the members of the model universe constructed for the argument. Thus, for example, in constructing an interpretation for the following argument:

(x) (Cx ⊃ ~Dx)
~Cj
∴ Dj

'j' must be one of the letters used in constructing an interpretation, as follows:

| Cj | ⊃ | ~Dj |
|---|---|---|
| F | T | T |

~Cj
T
∴ Dj
F

Shown invalid for {j} by Cj Dj
F F

Similarly, for the argument;

(x) (Fx ⊃ Gx)
Gy
∴ Fy

'y' must be one of the letters used in constructing an interpretation, as follows:

| Fy | ⊃ | Gy |
|---|---|---|
| F | T | T |

Gy
T
∴ Fy
F

## 1.5 Summary

The aim of this unit is to identify the means of noting the ways through which we can determine invalidity in predicate logic. In this unit, two of these have been discussed. It is based on these two that we can notice the grounds upon which invalidity can be noted. It is also on this note that we bring the entire course on Symbolic Logic, PHL301 to a close. Thank you dear students for your time and attention!

## 1.6 References and Further Readings

Bello, A.G.A. (2000). *Introduction to Logic* Ibadan: Ibadan University Press

Copi, I., Cohen, C., & McMahon, K. (2014). *Introduction to Logic.* Harlow: Pearson Education Limited

Offor, F. (2010). *Essentials of Logic.* Ibadan: Book Wright Nigeria Publishers

## 1.7 Unit Exercises

Each of the arguments below is invalid. Symbolise each of them using the suggested notation, and prove its invalidity, using the method of refutation by logical analogy:

1. All crows are birds. All crows are warm-blooded. Therefore, all birds are warm-blooded. (Cx, Bx, Wx)
2. No senators are social democrats. No social democrats are governors. Therefore, no senators are governors. (Sx, Dx, Gx)
3. All politicians are liars. All politicians are wealthy. Therefore, all liars are wealthy. (Px, Lx, Wx)
4. Some paediatricians are not specialists in surgery, so some general practitioners are not paediatricians, since some general practitioners are not specialists in surgery. (Px, Sx, Gx).
5. No intellectuals are successful politicians, because no outspoken people are successful politicians, and some intellectuals are outspoken people. (Ix, Sx, Px, Ox)

I. Prove the invalidity of the following:

1. $(x)(Ax \supset {\sim}Bx)$
   $(x)(Bx \supset Cx)$
   $\therefore (x)(Cx \supset {\sim}Ax)$
2. $(x)(Ax \supset Bx)$
   $(x)(Ax \supset Cx)$
   $\therefore (x)(Cx \supset Bx)$
3. $(x)(Ax \supset {\sim}Bx)$
   $(x)(Bx \supset Cx)$
   $\therefore (\exists x)(Cx \,.\, {\sim}Ax)$
4. $(\exists x)(Ax \,.\, Bx)$
   $(\exists x)(Cx \,.\, {\sim}Bx)$
   $\therefore (\exists x)(Cx \,.\, {\sim}Ax)$
5. $(\exists x)(Ax \,.\, {\sim}Bx)$
   $(x)(Cx \,. \supset {\sim}Bx)$

$\therefore (x)\ (Dx \supset Bx)$

II. Prove the invalidity of the following arguments in each case using the suggested notations in symbolising the arguments:

1. All generals like pepper-soup. Some intellectuals like pepper-soup. Therefore, some generals are intellectuals. (Gx, Px, Ix)
2. Some journalists are not responsible. Some responsible people are not interesting. Therefore, some journalists are not interesting. (Jx Rx, Ix)
3. Some politicians are orators. Some orators are not leaders. Therefore, some leaders are not politicians (Px, Ox, Lx)
4. All extremists are bearded. All opponents of the government are bearded. Therefore, all extremists are opponents of the government. (Ex, Bx, Ox)
5. If anything is metallic, then it is breakable. There are breakable ornaments. Therefore, there are metallic ornaments. (Mx, Bx, Ox)