NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 103

COURSE TITLE: ELEMENTARY MATHEMATICS III

COURSE GUIDE

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Course Guide

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Introduction

MTH 103 - Elemetary Mathematics III is designed to teach you how mathematics could be used in solving problems in the contemporary Scientific world. Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in Sciences,technology and Engineering Professions.

What you will learn in this Course

You will be taught the basis of mathematics required in solving scientific problems.

Course Aim

There are eleven study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include:

(i) To introduce you to the words and concepts in Elementary mathematics

(ii) To familiarize you with the peculiar characteristics in Elementary mathematics.

(iii) To expose you to the need for and the demands of mathematics in the Science world.

(iv) To prepare you for the contemporary Science world.

Course Objectives

The objectives of this course are:

* To inculcate appropriate mathematical skills required in Science and Engineering.

* Educate learners on how to use mathematical Techniques in solving real life problems.

* Educate the learners on how to integrate mathematical models in Sciences, technology and Engineering.

Working through this Course

{ You have to work through all the study units in the course. There are two modules and ten study units in all.

Course Materials

Major components of the course are:

1. Course Guide

2. Study Units

3. Textbooks

4. CDs

5. Assignments File

6. Presentation Schedule

Study Units

The breakdown of the four modules and eleven study units are as follows:

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Recommended Texts

- Larson Edwards Calculus: An Applied Approach. Sixth Edition.
- Blitzer. Algebra and Trigonometry Custom. 4th Edition
- K.A Stroud. Engineering Mathematics. 5th Edition
- Pure Mathematics for Advanced Level By B.D Bunday H Mulholland1970.
- Godman and J.F Talbert. Additional Mathematics

Assignment File

{ In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and later in this *Course Guide* in the section on assessment.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on tutor-marked assignments (TMAs) and a final examination which you will write at the end of the course.

Exercises TMAS

{ Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment. }

Final Examination and Grading

{ At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course marking Scheme

This table shows how the actual course marking as it is broken down.

Assessment	Marks
Assignments	Four assignments, Best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Most from This Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition do the following:

1. Read this Course Guide thoroughly, it is your first assignment.

2. Organise a Study Schedule. Design a Course Overview " to guide you through the Course". Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the Semester is available from the study centre. You need to gather all the information into one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3. Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4. Turn to Unit 1, and read the introduction and the objectives for the unit.

5. Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6. Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7. Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8. Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the Assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments

your tutor might make on your assignments as these will help in your progress. Make sure that assignments reach your tutor on or before the due date.

Your tutorials are important therefore try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading.

Summary

This course would train you on the concept of multimedia, production and utilization of it.

Wish you the best of luck as you read through this course

MAT 103
ELEMENTARY MATHEMATICS III
(VECTORS, GEOMETRY AND DYNAMICS)

Edited By

Prof. U. A. Osisiogu

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VECTORS, GEOMETRY AND DYNAMICS

UNIT 1

Vectors

1.1 Introduction

Vectors are objects that has both magnitude and direction e.g. displacement, velocity, forces etc. Vectors defined this way are called free vectors.
Examples of everyday activities that involve vectors include:
Breathing (your diaphragm muscles exert a force that has a magnitude and direction).
Walking (you walk at a velocity of around 6km/h in the direction of the bathroom).

1.2 Objectives

In this unit, you shall learn the following:

i Physical quantities

ii Geometric representation of vectors in 1 - 3 dimension.

iii Two equal vector.

iv Types of vector.

1.3 Main Contents

1.3.1 Physical Quantities:

Physical quantities can be classified into two main classes, namely:

i Scalar quantities: A scalar quantity is one that is defined by magnitude (size) but no direction. Examples are length, area, volume, mass, time etc.

ii Vector quantities: A Vector quantity is one that has both magnitude and direction in which it operate. Examples are force, velocity, acceleration.

1.3.2 Geometrical Representation of Vectors in 1-3 dimensions

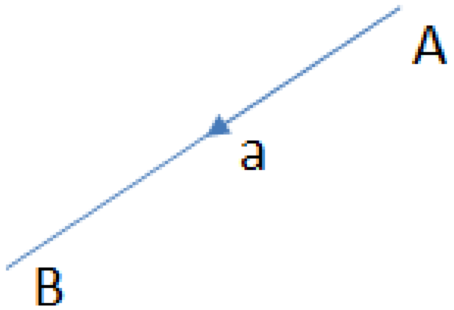
Vector quantity can be represented graphically by lines, drawn such that

i the length of the line denotes the magnitude of the quantity, according to a stated vector scale.

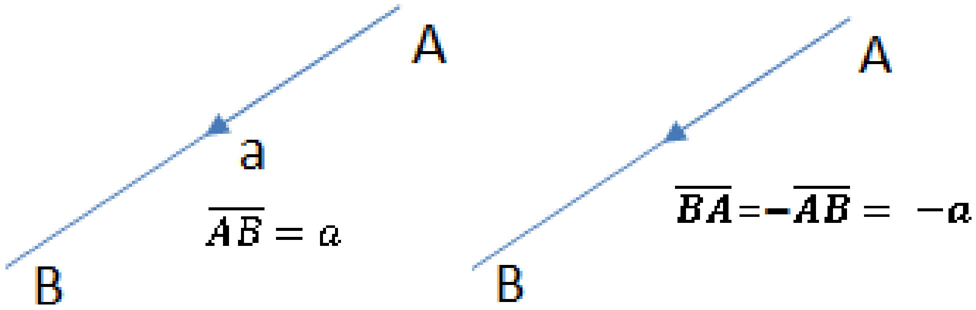
ii the direction of the line represent the direction in which the vector quantity acts. The position of the direction is indicated by an arrow head.

Example 1.3.1 *A* vertical force of 20N acting downward would be indicated by a line as shown below:

If the chosen vector scale were 1cm equivalent to 10N, the line would be 2.0cm long. Vector quantity AB is referred to as $\overline{AB}$ or a.
The magnitude of the vector quantity is written $|\overline{AB}|$ $|a|$ or simply as AB or a.



It is very important to note that $\overline{BA}$ would represent a vector quantity of the same magnitude but of opposite direction with $\overline{AB}$



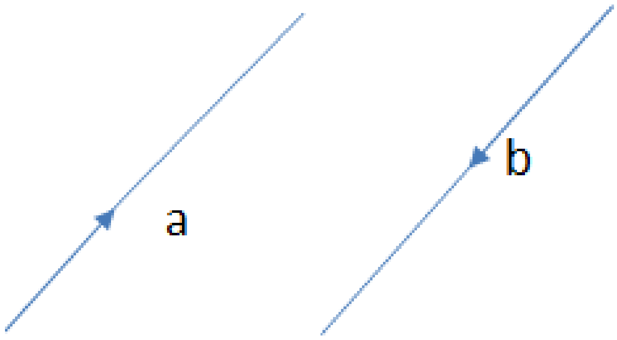
1.3.3 Two Equal Vectors

Two vectors a and b are said to have the same magnitude and the same direction if and only if they are equal. If $a = b$, then

i they have equal magnitude.

ii *a* move in the same direction as *b*.

You illustrate this with the diagram below:



If two vectors are as shown above, **it** is said to have equal magnitude but different in parallel movement.

1.3.4 Types of Vectors

There are different types of vectors which are as follows:

i Position Vector: For position vector $\overline{AB}$ to occurs, point A has to be fixed.

ii A Line Vector: A line vector is one that can shift or slide along its line of action e.g. mechanical force acting on a body.

iii Free Vector: It is a vector which is not restricted in any way. It is defined by its magnitude and direction and can be drawn as any one of a set of equal length parallel lines.

1.4 Conclusion

In this unit, you have studied the concept of vector quantities and its diagrammatic representations. You were also introduced to the the different types of vectors.

1.5 Summary

Having gone through this unit, you have learnt that:

i. Vectors are objects that has both magnitude and direction.

ii. A Scalar quantity has magnitude but no direction.

iii. Vector quantity can be represented by lines, drawn such that the length of the line denotes the magnitude of the vector, while the direction of the line represent the direction of the vector.

iv. Two vectors are said to be equal if and only if they have the same magnitude and direction.

v. The types of vectors includes: position vector, line vector and free vector.

1.6 Exercise

1. What is free and localized vector?
2. Briefly explain the term Zero Vector.
3. Write short notes on vectors in opposite direction.
4. What is Unit Vector?

1.7 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 2

Addition of Vectors

2.1 Introduction

Mathematically, the only operations that vectors posses are those of addition and scalar multiplication.

Objectives

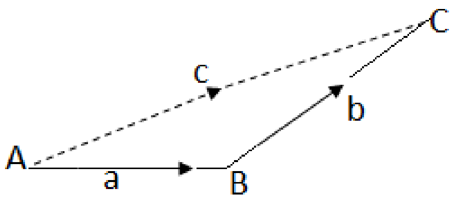
In this unit, you shall learn the following:

i. Addition of vectors

ii. Sum of a number of vectors

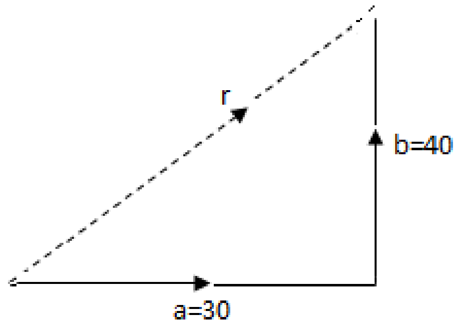
2.2 Main Content

The sum of two vector, $\overline{AB}$ and $\overline{BC}$ is defined as the single or equivalent or resultant vector $\overline{AC}$



i.e $\overline{AB} + \overline{BC} = \overline{AC}$.
When finding the sum of two vectors a and b, we have to draw them in chain, starting the second where the first ends: the sum c is giving by the single vector joining the start of the end of the second.

Example 2.2.1 *If* $a =$ a force of 30N, acting in the east direction. b =a force of 40N, acting in the north direction.
Then the magnitude of the vector sum r of these forces will be 50N.



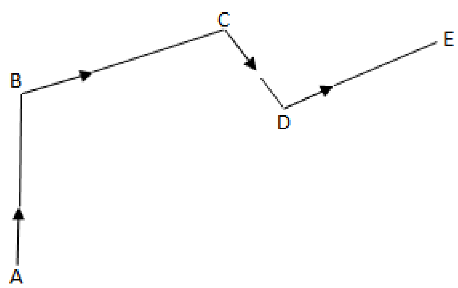
Using the Pythagoras theorem of right angle triangle:
hypotenuse^2 = opposite^2 + adjacent^2.

$r^2 + 40^2 + 30^2 = 1600 + 900 = 2500$
$r^2 = 2500 \Longrightarrow r = 50N$

2.2.1 Sum of a Number of Vectors

The vector sum is given by single equivalent vector joining the beginning of the first vector to the end of the last.
Hence, if vector diagram is a closed figure, the end of the last vector join or meet with the beginning of the first, so the resultant sum is a vector with no magnitude.

Example 2.2.2 *Sum* of numbers a, b, c, d, can be illustrated in the diagram bellow:



$a + b = \overline{AC}$
$\overline{AC} + c = \overline{AD}$
$a + b + c = \overline{AD}$
$\overline{AD} + d = \overline{AE}$
$a + b + c + d = \overline{AE}$
The sum of all vectors a, b, c, d is given by the single vector joining the start of the first to the end of the last- in this case, $\overline{AE}$. This follows directly from our previous definition of the sum of two vectors.

Example 2.2.3 *Find* $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF}$

Solution

Without drawing a diagram, we can see that vectors are arranged in chain, each beginning where the previous one left off. The sum is therefore given by the vectors joining the beginning of the first vectors to the end of the last. So that the sum is $\overline{AF}$

Example 2.2.4 *Find* $\overline{AK} + \overline{KL} + \overline{LP} + \overline{PQ}$

Solution

Since it is arrange in chain, each beginning where the previous one left off, you have that the Sum is $\overline{AQ}$.

Example 2.2.5 *Find* the sum $\overline{BC} - \overline{DC} + \overline{DE} + \overline{FE}$ i.e $\overline{BC} - \overline{DC} + \overline{DE} - \overline{EF}$

Solution

You must take notice of the negative vectors. Remember that $-\overline{DC} = \overline{CD}$ i.e they both have same magnitude and but direction in the opposite form.

Also $-\overline{FE} = \overline{EF}$

Therefore, $\overline{BC} - \overline{DC} + \overline{DE} - \overline{FE} = \overline{BC} + \overline{CD} + \overline{DE} + \overline{EF} = \overline{BF}$

2.3 Conclusion

In this unit, you have studied addition of vectors with examples.

2.4 Summary

Having gone through this unit, you have learnt the following:

i. The vector sum is given by single equivalent vector joining the beginning of the first vector to the end of the last.

ii. If vector diagram is closed, then the end of the last vector, join or meet with the beginning of the first, so that the resultant sum is a vector with no magnitude.

2.5 Exercises

Find the sum of the following vectors.

a. $\overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST}$

b. $\overline{AC} + \overline{CL} - \overline{ML}$

c. $\overline{GH} + \overline{HJ} + \overline{JK} + \overline{KL} + \overline{LG}$

d. $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DB}$

2.6 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 3

Components of a Given Vector

3.1 Introduction

Components of a given vector deals with the resolution of a vector with its direction (i.e. horizontal and vertical components) and also using a single vector to replace any number of component vector as long as they form chain in the vector diagram.

3.2 Objectives

In this unit, you shall learn the following:

i. Components of a given vector

ii. Proves and its applications

3.3 Main Content

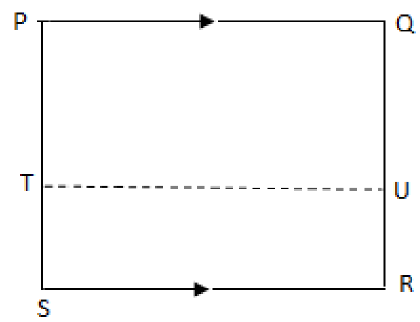
3.3.1 Component of an given vector.

A single vector MN can be used to replace any number of component vectors so long as they form a chain in the vector, as shown above which begins with M and end with N, just like as $\overline{AB} + \overline{BC} + \overline{CD} + \overline{DE}$ is been replaced by $\overline{AE}$.

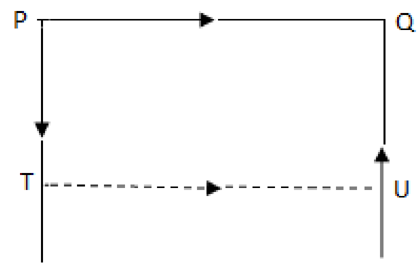
Example 3.3.1 *ABCD* is a square (quadrilateral) with T and U the mid points of SP and QR respectively. Show that $\overline{PQ} + \overline{RS} = 2\overline{TU}$

Solution

Consider the diagram bellow:



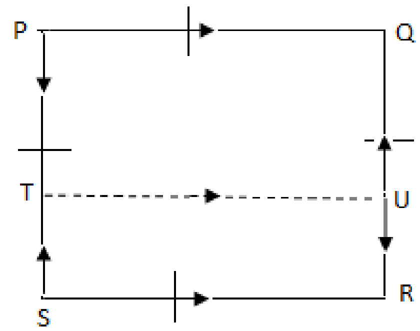
Reconstructing this diagram, you will have the following:



With the lettering, you could replace vector $\overline{PQ}$ by any chain of vectors so long as they start at P and end at Q. For Instance: $\overline{PQ} = \overline{PT} + \overline{TU} + \overline{UQ}$

Similarly, you could say, $\overline{SR} = \overline{ST} + \overline{TU} + \overline{UR}$

Therefore, you have:



$\overline{PQ} = \overline{PT} + \overline{TU} + \overline{UQ}$

$\overline{SR} = \overline{ST} + \overline{TU} + \overline{UR}$

Therefore, $\overline{PQ} + \overline{SR} = \overline{PT} + \overline{TU} + \overline{UQ} + \overline{ST} + \overline{TU} + \overline{UR}$

Collecting like terms, you will obtain:

$\overline{PQ} + \overline{SR} = \overline{TU} + \overline{TU} + (\overline{PT} + \overline{ST}) + (\overline{UQ} + \overline{UR})$

$$= 2\overline{TU} + (\overline{PT} + \overline{ST}) + (\overline{UQ} + \overline{UR}) \qquad (1)$$

From the diagram above, you have that $\overline{PT}$ and $\overline{ST}$ are equal in length but of opposite sides or direction. This implies;

$$\overline{PT} = -\overline{ST} \tag{2}$$

Similarly, you have that;

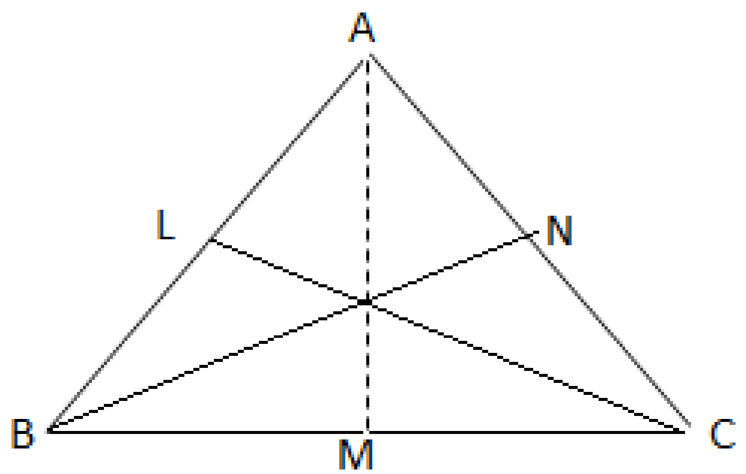
$$\overline{QU} = -\overline{UR} \tag{3}$$

Substituting equation (2) and (3) into equation (1), you get that
$\overline{PQ} + \overline{ST} = 2\overline{TU} + (-\overline{ST} + \overline{ST}) + (-\overline{UR} + \overline{UR})$
$\overline{PQ} + \overline{ST} = 2\overline{TU}$

Example 3.3.2 *In* triangle ABC, the points L, M, N are the midpoints of the sides AB, BC and CA respectively. Show that: $2\overline{AB} + 3\overline{BC} + \overline{AC} = 2\overline{LC}$

Solution

To prove this, you construct the diagram below;



With the lettering above, you have that

$$2\overline{AB} + 3\overline{BC} + \overline{CA} \quad (1)$$

Note that $\overline{AB} = 2\overline{AL}$, $\overline{BC} = \overline{BL} + \overline{BC}$, $\overline{CA} = \overline{CL} + \overline{LA}$ (2)

Substituting (2) into (1), you have

$$2.2\overline{AL} + 3(\overline{BL} + \overline{LC}) + \overline{CL} + \overline{LA}$$

This implies $4\overline{AL} + 3\overline{BL} + 3\overline{LC} + \overline{CL} + \overline{LA}$ (3)

Again, $\overline{BL} = -\overline{AL}$, $\overline{CL} = -\overline{LC}$, $\overline{LA} = -\overline{AL}$ (4)

Substituting equation (4) into equation (3), you will obtain that:

$$\overline{2AB} + \overline{3BC} + \overline{CA} = \overline{4AL} - \overline{3AL} + \overline{3LC} - \overline{LC} - \overline{AL}$$

Collecting like terms, you obtain that;

$$\begin{aligned} \overline{2AB} + \overline{3BC} + \overline{CA} &= \overline{4AL} - \overline{3AL} - \overline{AL} + \overline{3LC} - \overline{LC} \\ &= \overline{LC} - \overline{LC} \\ &= 2\overline{LC} \end{aligned}$$

3.4 Conclusion

In this unit, you have learnt the concept of vector components. You also studied the examples in this unit.

3.5 Summary

Having gone through this unit, you have learnt that a single vector can be used to replace any number of component vectors so long as they form a chain in the vector. I.e., $AB + BC + CD + DE = AE$

3.6 Exercises

1. MNOP is quadrilateral in which C and D are the mid-point of the diagonals MO and NP respectively. Show that $\overline{MN} + \overline{MP} + \overline{ON} + \overline{OP} = 4\overline{CD}$

2. Prove by vectors that the line joining the midpoint of two sides of a triangle is parallel to third side and half its length.

3.7 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 4

Components of a Vector in terms of Unit vectors and Vectors in Space

4.1 Introduction

Component of a vector in terms of unit vector is a vector as a way of determining the magnitude and direction in its coordinate of x and y axis.
Vector in space is a collection of objects called vectors which may be add together and multiplied by numbers. They are the subject of linear algebra and are well characterized by their dimension.

4.2 Objectives

In this unit, you shall learn the following:

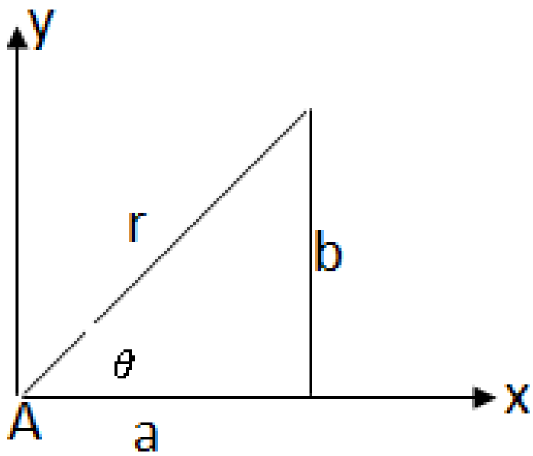
i Components of a vector in terms of unit vectors.

ii Vectors in Space

4.3 Main Contents

4.3.1 Components of a vector in terms of Unit Vectors

Vector $\overline{AB}$ is defined by its magnitude (r) and its angle (θ). From the diagram, **it** is defined by **t**wo components in the AY (horizontal) and AY (vertically) direction



i.e. $\overline{AB}$ is equivalent to a vector a in the AX direction + a vector b in the AY direction.

$$\overline{AB} = a + b \tag{1}$$

If we now define $\boldsymbol{i}$ as a unit vector in horizontal direction.

$$a = ai \tag{2}$$

Similarly, if we define $\boldsymbol{j}$ as a unit vector in the vertical direction, then

$$b = bj \qquad (3)$$

So that the vector $\overline{AB}$ can be written as: $\overline{AB} = \boldsymbol{r}$
Substituting (2) and (3) into (1), yields;
$\boldsymbol{r} = ai + bj$
Where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the horizontal directions.

Example 4.3.1 *If* $Z_1 = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $Z_2 = 7\boldsymbol{i} + 3\boldsymbol{j}$. Find $Z_1 + Z_2$

Solution
$Z_1 + Z_2 = 3\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{i} + 3\boldsymbol{j} = (3 + 7)\boldsymbol{i} + (5 + 3)\boldsymbol{j} = 10\boldsymbol{i} + 8\boldsymbol{j}$

Example 4.3.2 *If* $Z_1 = 2\boldsymbol{i} - 4\boldsymbol{j}$, $Z_2 = 2\boldsymbol{i} + 6\boldsymbol{j}$, $Z_3 = 3\boldsymbol{i} - \boldsymbol{j}$. Find

a. $Z_1 + Z_2 + Z_3$

b. $Z_1 - Z_2 - Z_3$

Solution

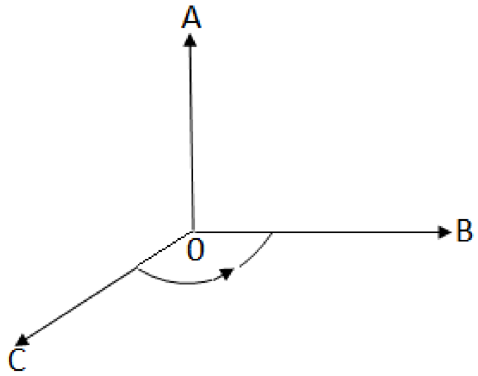
a.

$$\begin{aligned} Z_1 + Z_2 + Z_3 &= 2\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{i} + 6\boldsymbol{j} + 3\boldsymbol{i} - \boldsymbol{j} \\ &= (2 + 2 + 3)\boldsymbol{i} + (6 - 4 - 1)\boldsymbol{j} \\ &= 7\boldsymbol{i} + \boldsymbol{j} \end{aligned}$$

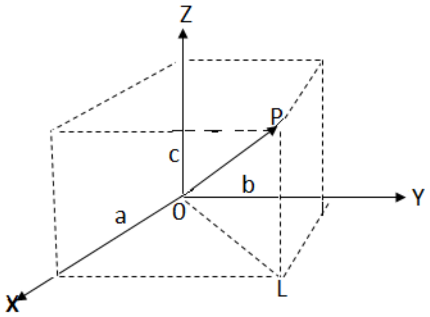
b.

$$\begin{aligned} Z_1 - Z_2 - Z_3 &= (2\boldsymbol{i} + 4\boldsymbol{j}) - (2\boldsymbol{i} + 6\boldsymbol{j}) - (3 - \boldsymbol{j}) \\ &= (2 - 2 - 3)\boldsymbol{i} + (-4 - 6 + 1)\boldsymbol{j} \\ &= -3\boldsymbol{i} - 9\boldsymbol{j} \end{aligned}$$

4.3.2 Vectors in Space



According to the right hand rule, the axes, OC, OB, OA form a right handed corkscrew action along the positive direction OA. Similarly, rotation from OB to OA gives right-hand corkscrew action along the positive direction of OC.



Vectors $\overline{OP}$ is defined by its components:

a. along OX

b. along OY

c. along OZ

Let $\boldsymbol{i}$ =unit vector in OX direction.
$\boldsymbol{j}$ = unit vector in OY direction.
k = unit vector in OZ direction.
So $\overline{OP} = ai + bj + ck.$
Also $OL^2 = a^2 + b^2$ and $OP^2 = OL^2 + C^2.$
$OP^2 = a^2 + b^2 + c^2$
So, if $\boldsymbol{r} = ai + bj + ck$, then $\boldsymbol{r} = \sqrt{a^2 + b^2 + c^2}$

Example 4.3.3 *Find* the magnitude of a vector expressed in terms of the unit vectors. That is, if $\overline{PQ} = 5\boldsymbol{i} + 2\boldsymbol{j} + 4k$, then find

$\overline{|PQ|}$.

Solution.

$\overline{PQ} = 5\boldsymbol{i} + 2\boldsymbol{j} + 4k$

$\Longrightarrow \quad |\overline{PQ}| = \sqrt{5^2 + 2^2 + 4^2} = \sqrt{25 + 4 + 16} = \sqrt{45} = 3\sqrt{5}$

4.4 Conclusion

In this unit, you have studied in depth, the notion of vector components in terms of unit vectors and vector in space.

4.5 Summary

Having gone through this unit, you have learnt the following:

i. Vector $\overline{AB}$ is defined by its magnitude (r) and its angle (θ)

ii. If r is the Hypothenus of a right-angled triangle with sides a and b, then $\boldsymbol{r} = ai + bj$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors in the horizontal and vertical directions.

iii. The symbols $\boldsymbol{i}$, $\boldsymbol{j}$ and k represent unit vector in the directions OX, OY, OZ respectively. If $\overline{OP} = ai + bj + ck$, then $|\overline{OP}| = \boldsymbol{r} = \sqrt{a^2 + b^2 + c^2}$

4.6 Exercise

Determine the magnitude of a vector expressed in terms of a unit vector in the following bellow:

a. $\overline{AB} = 3\boldsymbol{i} + 4\boldsymbol{j} - 2k$

b. $\overline{BC} = 4\boldsymbol{i} + 3\boldsymbol{j} + 5k$

c. $\overline{PQ} = 2\boldsymbol{i} + 3\boldsymbol{j} + 6k$

4.7 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 5

Direction Cosines

5.1 Introduction

Direction Cosines in analytic geometry of vector are the cosines of angles between the vector and the three coordinate axes. Or equivalently it is the component contributions of the basis to the unit vector.
It refers to the cosine of the angle between any two vectors. They are useful for forming direction cosine matrices that expresses one set of orthonormal basis vector in terms of another set, or for expressing a known vector in a different basis.

5.2 Objectives

In this unit, you shall study the following:

i. Direction cosines

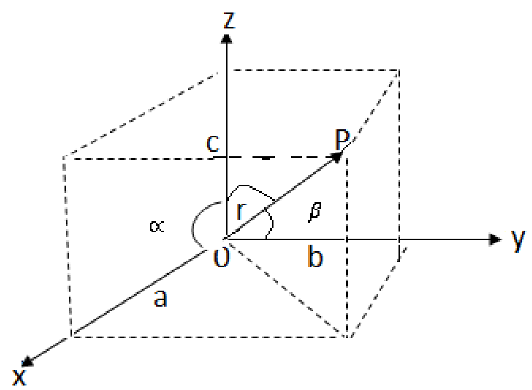
ii. Application of direction cosines

5.3 Main Content

5.3.1 Direction Cosines

Direction cosines are the cosines of the angles between a line and the coordinate axes. The direction of a vector in three dimensions is determined by the angles which the vector makes with the three axes of reference.

Let $\overline{OP} = r = ai + bj + ck$



Then

$\frac{a}{r} = \cos\alpha \implies a = r\cos\alpha$

$\frac{b}{r} = \cos\alpha \implies b = r\cos\beta$

$\frac{c}{r} = \cos\gamma \implies c = r\cos\gamma$

Also $a^2 + b^2 + c^2 = r^2$

$\implies r^2\cos^2\alpha + r^2\cos^2\beta + r^2\cos^2\gamma = r^2$

Dividing both side by r^2, you have:
$\frac{r^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{r^2} = \frac{r^2}{r^2}$
$\cos^2\alpha + \cos^2\beta + \cos^2 r = 1$
If $l = \cos x,\ \ m = \cos\beta,\ \ n = \cos\gamma$
Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Longleftrightarrow l^2 + m^2 + n^2 = 1$
Then [l,m,n] is called the direction cosines of the vector OP and is the value of the cosines of the angles which the vector makes with the three axes of reference.
Hence for vector $r = ai + bj + ck$
$l = \frac{a}{r},\ \ m = \frac{b}{r},\ \ n = \frac{c}{r}$ and of course $r = \sqrt{a^2 + b^2 + c^2}$

Example 5.3.1 *Find* the direction cosine [l,m,n] of the vector $r = 2i + 4j - 3k$
Solution
Let a=2, b=4, c=-3, then
$r = \sqrt{a^2 + b^2 + c^2} = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$
$l = \frac{a}{r} = \frac{2}{\sqrt{29}}$
$m = \frac{b}{r} = \frac{4}{\sqrt{29}}$
$n = \frac{c}{r} = \frac{-3}{\sqrt{29}}$
$\Longrightarrow\ [l, m, n] = \left[\frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}\right]$

5.4 Summary

Having gone through this unit, you have learnt the following:

i. Direction cosines are the cosines of the angles between a line and the coordinate axes.

ii. The direction cosines [l, m, n] are the cosines of the angles between the vector and the axes OX, OY, OZ respectively.

iii. For the vector $r = ai + bj + ck$, where i, j, k are unit vectors, you have that $l = \frac{a}{r}$, $m = \frac{b}{r}$, $n = \frac{c}{r}$, $l^2 + m^2 + n^2 = 1$ and
$r = \sqrt{a^2 + b^2 + c^2}$

5.5 Exercises

1. Find the direction cosines of the vector joining the two points (4,2,2) and (7,6,14).
2. If A is the point (1,-1,2), B is the point (-1,2,2) and C is the point (4,3,0), find the direction cosines of $\overline{BA}$ and $\overline{BC}$.
3. If $a = 3i - j + 2k$, $b = j + 3j - 2k$, determine the magnitude and direction cosines of the product vector (a x b)

5.6 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 6

Scalar Product of two vectors

6.1 Introduction

A scalar quantity is one that has magnitude but no direction. Scalar product is one of the two ways of multiplying vectors, which has most applications in physics, engineering and astronomy.

6.2 Objectives

In this unit, you shall study the following

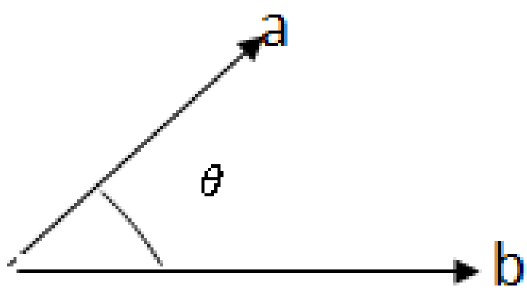
i. Scalar product of two vectors

ii. Expressing scalar product of two vectors in terms of the unit vector i, **j** and k

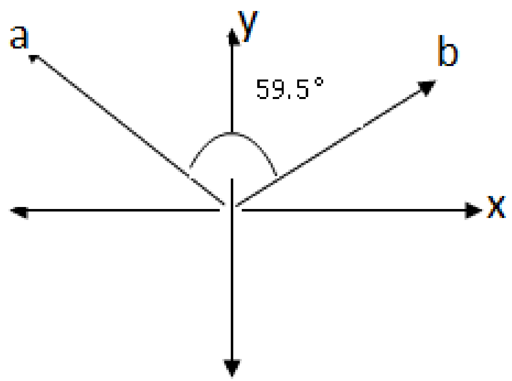
6.3 Main Content

6.3.1 Scalar product of two vectors

The scalar product of a and b is defined as the scalar (number) $ab\cos\theta$. i.e $a.b\,|a|\times|b|\cos\theta$.
where $|a|$ is the magnitude of the vector *a*.
$|b|$ is the magnitude of vector *b*.
θ the angle between *a* and *b*.
So we multiply the length of *a* times the length of *b*, then multiply by the cosine of the angle between *a* and *b*.
The scalar product is denoted *a.b* (called the dot product).



Example 6.3.1 *Calculate* the dot product of vectors a and b in the diagram below:

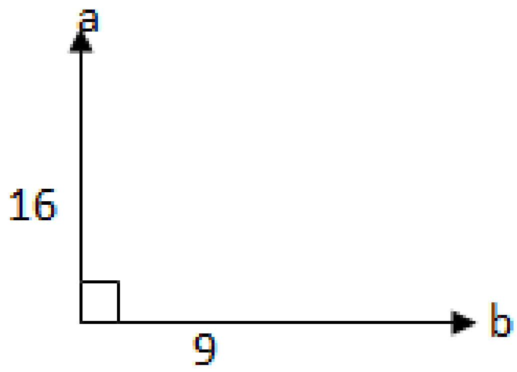


Solution

$$
\begin{aligned}
a.b &= |a| \text{ x } |b| \text{ x } \cos\theta \\
&= 10 \text{ x } 13 \cos 59.5^o \\
&= 10 \text{ x } 13 \text{ x } 0.5075 \\
&= 65.98
\end{aligned}
$$

Remark: When two vectors are at right angles to each other the dot product is zero.

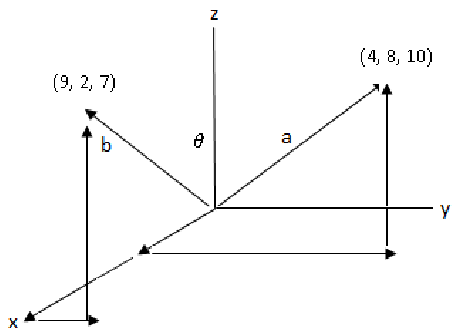
Example 6.3.2 *Calculate* the Dot product for the diagram below:



Solution

$$
\begin{aligned}
a.b &= |a| \times |b| \times \cos\theta \\
&= 16 \times 9 \times \cos 90^o \\
&= 16 \times 9 \times 0 = 0
\end{aligned}
$$

Example 6.3.3 *Uche* has measured the end points of two poles and wants to know the angle between them. What angle will he get?



Solution

From the diagram above, you have three dimensions: $a.b = a_x.b_x+$

$a_y.b_y + a_z.b_z = 9 \text{ x } 4 + 2 \text{ x } 8 + 7 \text{ x } 10$
$\Longrightarrow a.b = 122$
You have to find $|a|$ and $|b|$ i.e the length or magnitude of the vector a and b respectively.
Using Pythagoras theorem, you will obtain:
$|a| = \sqrt{4^2 + 8^2 + 10^2} = \sqrt{180}$
Similarly, $|b| = \sqrt{9^2 + 2^2 + 7^2} = \sqrt{134}$
You now apply the formula $a.b = |a| \text{ x } |b| \text{ x } \cos\theta$, to obtain θ
$122 = \sqrt{180} \text{ x } \sqrt{134} \text{ x } \cos\theta$
$\Longrightarrow \cos\theta = \frac{122}{\sqrt{180}\sqrt{134}}$
$\Longrightarrow \theta = \cos^{-1}(0.7855)$
$\Longrightarrow \theta = 38.2^o$

6.3.2 Expressing Scalar Product Of Two Vectors In Terms Of The Unit Vector I, j, k.

If $a = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k$
$b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k$
Then $a.b = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k).(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k) = a_1b_1\boldsymbol{i}.\boldsymbol{i} + a_1b_2\boldsymbol{i}.\boldsymbol{j} + a_1b_3\boldsymbol{i}.k + a_2b_1\boldsymbol{j}.k + a_2.b_2\boldsymbol{j}.\boldsymbol{j} + a_2b_3\boldsymbol{j}.k + a_3b_1k.\boldsymbol{i} + a_3b_2k.\boldsymbol{j} + a_3k.k$
Recall that $\boldsymbol{i}.\boldsymbol{i} = (1)(1)\cos 0^o = 1$.
Simplifying the above, you will obtain

$$\boldsymbol{i}.\boldsymbol{i} = 1, \quad \boldsymbol{j}.\boldsymbol{j} = 1, \quad k.k = 1 \qquad (a)$$

Similarly, $\boldsymbol{i}.\boldsymbol{j} = (1)(1)\cos 90^o = 0$. This implies that;

$$-\boldsymbol{i}.\boldsymbol{j} = 0, \; \boldsymbol{j}.k = 0, \quad k.i = 0 \qquad (b)$$

Now using the result (a) and (b), you get $a.b = a_1b_1 + a_2b_2 + a_3b_3$

Example 6.3.4 *If* $a = 3\boldsymbol{i} + 4\boldsymbol{j} + 5k$ and $b = 2\boldsymbol{i} + \boldsymbol{j} + 7k$. Find $a.b$

Solution

$a.b = 3 \text{ x } 2 + 4 \text{ x } 1 + 5 \text{ x } 7 = 45$

6.4 Conclusion

In this unit, you have learnt the concept of scalar product of two vectors and its expression in terms of unit vectors.

6.5 Summary

Having through this unit, you have studied the following:

i. A scalar quantity is a quantity that has magnitude but no direction.

ii. The scalar product (dot product) a.b, is equal to $ab\cos\theta$ and θ is the angle between a and b.

iii. If $a = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k$ and $b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k$, then $a.b = a_1b_1 + a_2b_2 + a_3b_3$.

iv. The dot product of two vectors at right angle to each other is zero.

6.6 Exercises

1. If $a = 2\boldsymbol{i} + 2\boldsymbol{j} - k$ and $b = 3\boldsymbol{i} - 6\boldsymbol{j} + 2k$. Find the scalar product of a and b

2. If $a = 5\boldsymbol{i} + 4\boldsymbol{j} + 2k$, $b = 4\boldsymbol{i} - 5\boldsymbol{j} + 3k$, $c = 2\boldsymbol{i} - \boldsymbol{j} - 2k$. Where i, $\boldsymbol{j}$, k are unit vectors. Determine (a) the value of *a.b* (b) the value of *a.c* (c) the value of *b.c*
3. If $a = 2\boldsymbol{i} + 4\boldsymbol{j} - 3k$ and $b = \boldsymbol{i} + 3\boldsymbol{j} + 2k$, determine the scalar product.
4. Find the scalar product (a.b) when:
   (a) $a = \boldsymbol{i} + 2\boldsymbol{j} - k$, $b = 2\boldsymbol{i} + 3\boldsymbol{j} + k$
   (b) $a = 2\boldsymbol{i} + 3\boldsymbol{j} + 4k$, $b = 5\boldsymbol{i} - 2\boldsymbol{j} + k$

6.7 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 7

Vector Product of Two Vectors

7.1 Introduction

The vector product is one of the two ways of multiplying vectors which has numerous applications in physics and astronomy.
Geometrically, the vector product is useful as a method for constructing a vector perpendicular to a plane if you have two vectors in the plane.
Physically, it appears in the calculation of torque and in the calculation of the force on a moving charge.

7.2 Objectives

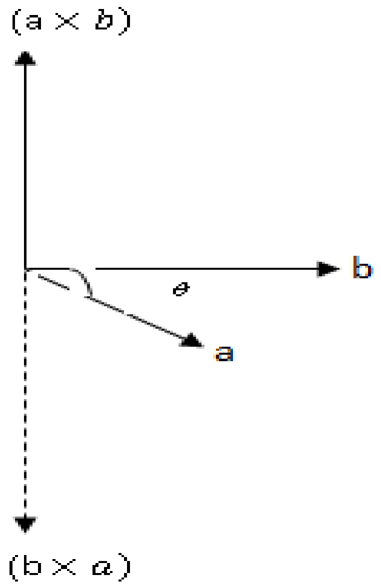
In this unit, you shall learn the following:

i. Vector product of two vectors

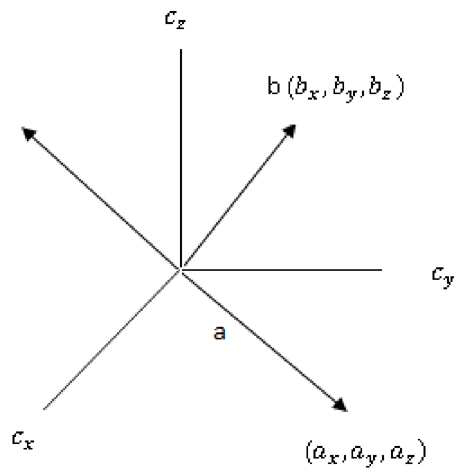
ii. Vector product in determinant form vector product calculation

7.3 Main Content

7.3.1 Vector Product of Two Vectors



Vector product of a and b is represented by a x b. **It** is also known as the Cross Product of a and b. **It** is a vector having magnitude $|a||b|\sin\theta$. Where $|a|$ is the magnitude (length) of vector a, $|b|$ is the magnitude (length) of vector b, and θ is the angle between a and b.



From the diagram above, when a and b start at the origin (0, 0 ,0), the cross product will be:
$c_x = a_y b_z - a_z b_y$
$c_y = a_z b_x - a_x b_z$
$c_z = a_x b_y - a_y b_x$

Example 7.3.1 *Find* the cross product of $a = (2, 3, 4)$ and $b = (5, 6, 7)$

Solution

$c_x = a_y b_z - a_z b_y = 3 \text{ x } 7 - 4 \text{ x } 6 = -3$
$c_y = a_z b_x - a_x b_z = 4 \text{ x } 5 - 2 \text{ x } 7 = 6$
$c_z = a_x b_y - a_y b_x = 2 \text{ x } 6 - 3 \text{ x } 5 = -3$
$a \text{ x } b = (-3, 6, -3)$

7.3.2 Expressing Vector Product of two Vectors in Terms of the Unit Vectors i,j and k.

If $a = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k$ and $b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k$
Then:
$a \times b = (a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k)(b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k) = a_1b_1\boldsymbol{i} \times \boldsymbol{i} + a_1b_2\boldsymbol{i} \times \boldsymbol{j} + a_1b_3\boldsymbol{i} \times k + a_2b_1\boldsymbol{j} \times \boldsymbol{i} + a_2b_2\boldsymbol{j} \times \boldsymbol{j} + a_2b_3\boldsymbol{j} \times k + a_3b_1k \times \boldsymbol{i} + a_3b_2k \times \boldsymbol{j} + a_3b_3k \times k$
But $|\boldsymbol{i} \times \boldsymbol{i}| = (1)(1)(\sin 0^o$

$$\Longrightarrow \boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = k \times k = 0 \qquad (a)$$

Also $|\boldsymbol{i} \times \boldsymbol{j}| = (1)(1)(\sin 90^o) = 1$

$$\boldsymbol{i} \times \boldsymbol{j} = k \quad \boldsymbol{j} \times k = \boldsymbol{i} \quad k \times \boldsymbol{i} = \boldsymbol{j} \qquad (b)$$

Also remember that
$\boldsymbol{i} \times \boldsymbol{j} = -(\boldsymbol{j} \times \boldsymbol{i}) \quad \boldsymbol{j} \times k = -(k \times \boldsymbol{j}) \quad k \times \boldsymbol{i} = -(\boldsymbol{i} \times k) \Longrightarrow a \times b =$
$a_1b_1.0 + a_1b_2.k + a_1b_3(-\boldsymbol{j}) + a - 2b_1(-k) + a_2b_2.0 + a_2b_3.\boldsymbol{i} + a_3b_1.\boldsymbol{j} +$
$a_3b_2(-\boldsymbol{i}) + a_3b_3.0$
$\Longrightarrow a \times b = a_1b_2.k + a_1b_3(-\boldsymbol{j}) + a - 2b_1(-k) + a_2b_3.\boldsymbol{i} + a_3b_1.\boldsymbol{j} +$
$a_3b_2(-\boldsymbol{i}) = (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)k$

OR

$a \times b = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 -$

$a_2b_1)k$

Method 2

In this method, you have to put the vectors in a matrix, so that the determinant of the matrix becomes the vector product (cross product), as shown below

$$a \times b = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)k$$

This is the easiest way to write out the vector product of two vectors.

Example 7.3.2 *If* $a = 2\boldsymbol{i} + 4\boldsymbol{j} + 3k$ and $b = \boldsymbol{i} + 5\boldsymbol{j} - 2k$. Find the vector product of a and b.

Solution

$$a \times b = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ 2 & 4 & 3 \\ 1 & 5 & -2 \end{vmatrix} = \boldsymbol{i}(-8 - 15) - \boldsymbol{j}(-4 - 3) + k(10 - 4) =$$
$$-23\boldsymbol{i} + 7\boldsymbol{j} + 6k$$

7.4 Conclusion

In this unit, you have studied the concept of vector product, also known as the cross product. You also learnt the methods for computing the vector product of any given two vectors.

7.5 Summary

Having gone through this unit, you have learnt the following:

i. Vector product, also known as cross product of a and b is represented by $a \times b$. Where $a \times b = |a||b| \sin\theta$.

ii. If $a = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3k$ and $b = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3k$, then a x $b =$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\boldsymbol{i} - (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)k$$

7.6 Exercises

1. If $a = 2\boldsymbol{i} + 2\boldsymbol{j} - k$ and $b = 3\boldsymbol{i} - 6\boldsymbol{j} + 2k$. Find a x b.
2. If $a = 3\boldsymbol{i} + 4k - 2k$ and $b = 6\boldsymbol{i} + 3k + 5k$. Find the vector product of a and b.
3. Find the vector product (a x b) when:

   (a) $a = 3\boldsymbol{i} + 2\boldsymbol{j} - k, \quad b = 2\boldsymbol{i} + 3\boldsymbol{j} - k$.

   (b) $a = 8\boldsymbol{i} + 3\boldsymbol{j} + 4k, \quad b = 6\boldsymbol{i} + 3\boldsymbol{j} - k$

7.7 References

Blitzer. *Algebra and Trigonometry custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 8

Loci

8.1 Introduction

In order to free geometry from the use of diagrams through the use of algebraic expression, Descartes wished to give meaning to algebraic operations by interpreting them geometrically.
The two basic ideas made up in the concept of Locus:

i. If the condition or description of a locus are given to find the algebraic expression (equation) of the Locus .e.g. the Locus of points at a distance of 3 from the point (0, 0) is given by the equation $x^2 + y^2 = 9$.

ii. If an algebraic formula of a locus is given to find its geometric (graphic) representation or the description in words. For instance, the locus of a point that satisfies the equation $x^2 +$

$y^2 = 9$ lies on a circle with centre at (0, 0) and the radius of 3.

8.2 Objectives

In this unit, you shall learn the following:

i. The meaning of straight line

ii. The distance between two points and midpoint of two points

iii. The angle of slope, equation of a straight line

iv. Angle between two lines

8.3 Main Content

8.3.1 Straight Line

A line is said to be straight if and only if it has a constant gradient. That is, if the gradients between any two points on the line are equal.

8.3.2 Cartesian Coordinates

In Cartesian Coordinates, the coordinates is given by the position of a point in a plane. That is, the sign distances of the points from two perpendicular axes OX, OY. The x- coordinates is called the Abscissa and the coordinate, the Ordinate. Figure 1 below illustrates Cartesian coordinate abscissa and ordinates.

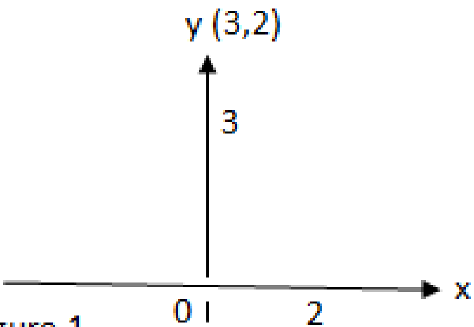


Figure 1

The diagram illustrate the abscissa and ordinate .i.e. (x, y) is (2, 3) respectively.

8.3.3 Distance between two Points and Midpoint of two Points

8.3.3.1 Distance between two Points

Consider two points A and B with coordinate (x_1, y_1), (x_2, y_2) respectively on Cartesian plane as shown figure 2:

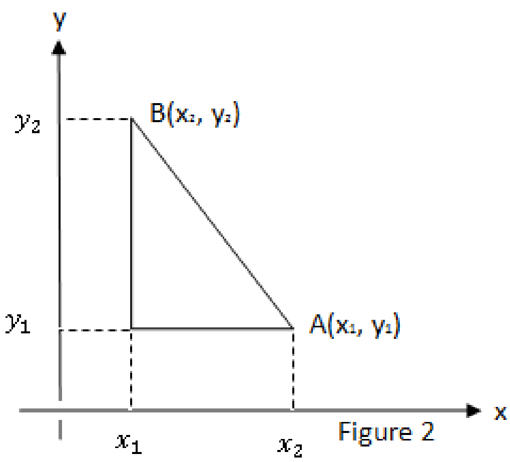


Figure 2

Using Pythagoras theorem, you will have

$$\overline{AB}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \qquad (1)$$

Making $\overline{AB}^2$ subject of the formula, you will have

$$\overline{AB}^2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The equation shown above is the equation for distance between two points.

Example 8.3.1 *Find* the distance between the points A(4, 3) and B(6, 5).

Solution

Using the equation for distance between two points: Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

If $A(4, 3)$, $B(6, 1)$ and $A(x_1, y_1)$, $B(x_2, y_2)$, then $x_1 = 4$, $y_1 = 3$ and $x_2 = 6$, $y_2 = 5$

Substituting the value of x_1, x_2, y_1 and y_2 into the equation of distance between two points.

$\Longrightarrow$ Distance= $\sqrt{(6 - 4)^2 + (5 - 3)^2} = 2\sqrt{2}$

8.3.3.2 Midpoint of two Points

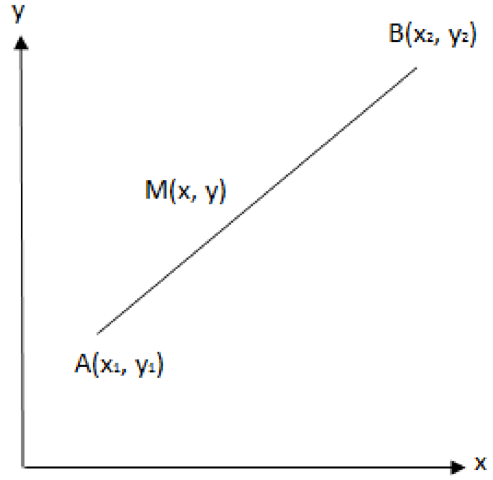


Figure 3

In Analytical geometry, the midpoint of line segment, given the coordinates of the endpoints is very important and useful.

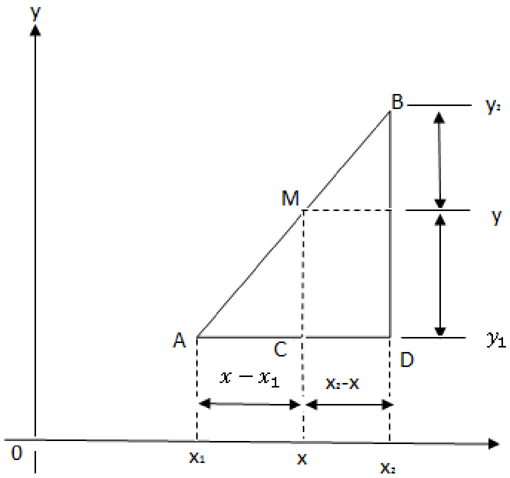


Figure 4

Let M be the midpoint of figure 4 above, so $L:ABD /// L:AMC$, so you have: $\frac{AD}{AC} = \frac{BD}{MC} = 2$
$AD = 2AC$
$x_2 - x_1 = 2x - 2x_2 \implies 2x = 2x_2 - x_2 + x_1 \implies x = \frac{1}{2}(x_1 + x_2)$
Similarly, $BD = 2MC$
$y_2 - y_1 = 2(y - y_1) \implies y_2 - y_1 = 2y - 2y_1 \implies y = \frac{1}{2}(y_1 + y_2)$
The coordinates of the midpoint of a line segment is the average of the coordinates of the endpoints .i.e. $[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)]$

Example 8.3.2 *If A is* $(3, 6)$ and B is $(4, 8)$. Find the coordinate of the midpoint of AB.

Solution

Using the midpoint of a line segment formula, $[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)]$. You have that, $x_1 = 3$, $x_2 = 4$, $y_1 = 6$, $y_2 = 8$
This implies that the coordinate of the midpoint of AB is
$[\frac{1}{2}(3 + 4), \frac{1}{2}(6 + 8)] = [3\frac{1}{2}, 7]$

8.3.4 Gradient (gradient, angle of slope, and equation of a straight line).

8.3.4.1 Gradient

The gradient of a line is defined as the ratio of the vertical distance the line rises or falls to the horizontal distance. It is represented by lettered m.

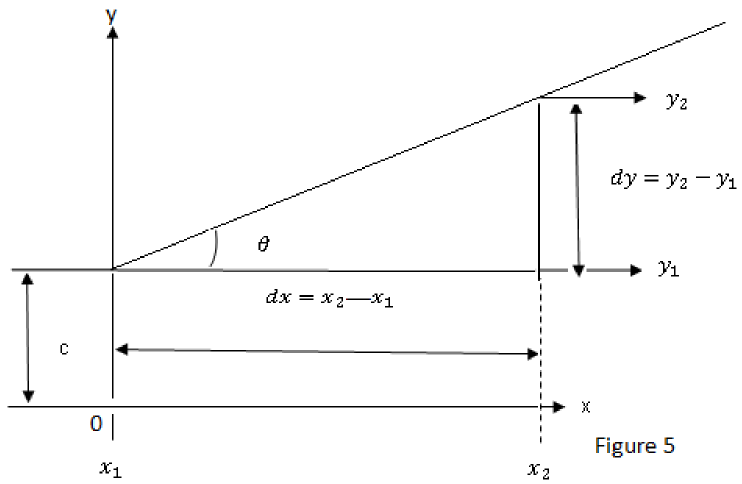


Figure 5

Therefore, the gradient $(m) = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

8.3.4.2 Angle of Slope

The angle of slope as in figure 5 is represented by θ. **It** is determined by:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{1}$$

Recall from the figure 5, that the diagram can be well represented below and can be used to determine angle of the slope.

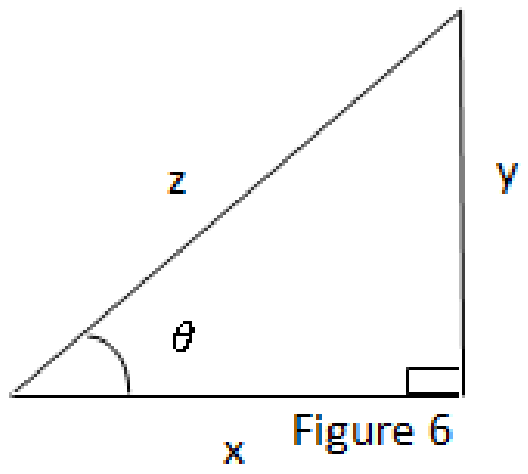


Figure 6

From the diagram above, you have that:

$$\sin\theta = \frac{y}{z} = \frac{y_2 - y_1}{z} \text{ (since } y = dy) \tag{1}$$

$$\text{Also } \cos\theta = \frac{x}{z} = \frac{x_2 - x_1}{z} \text{ (since } x = dx) \tag{2}$$

Substituting equation 2 and 3 into (1), you obtain that

$\tan\theta = \frac{y_2 - y_1}{z} \text{ x } \frac{z}{x_2 - x_1}$

$\Longrightarrow \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}$

$\Longrightarrow \theta = \tan^{-1} \frac{y_2 - y_1}{x_2 - x_1} = \tan^{-1} \frac{dy}{dx} = \tan^{-1}(m)$

Hence the Angle of a slope is defined as the inverse of tangent of the gradient (m) of a straight line.

Remark 8.3.3 *If* $0^o < \theta < 90^o$, $\tan\theta$ is positive and the line has a positive gradient.

If $\theta = 90^o$, the line has no gradient but perpendicular to the x-axis.

If $90^o < \theta < 180^o$, $\tan\theta$ is negative and the line has a negative gradient.

Example 8.3.4 *Calculate:*

1. The gradient of the line joining A(4,3) and B(9,7).
2. The angle of the slope

Solution

1. Using the formula
$m = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$
From the above, you have that $x_1 = 4,\ x_2 = 9,\ y_1 = 3,\ y_2 = 7$
$\Longrightarrow\ m = \frac{7-3\ 4}{9-4\ 5}$

2. $\theta = \tan^{-1}(m) = \tan^{-1}\left(\frac{4}{5}\right)$
$\Longrightarrow\ \theta = 38.66^o$

8.3.4.3 Equation of a Straight Line

A. Gradient-intercept form.
Using the diagram in figure 5, the equation of a straight line in a gradient intercept form is: $y = mx + c$. Where m = gradient= $\frac{dy}{dx}$, and
$c =$ intercept on real y-axis.

B. Gradient and one point form.

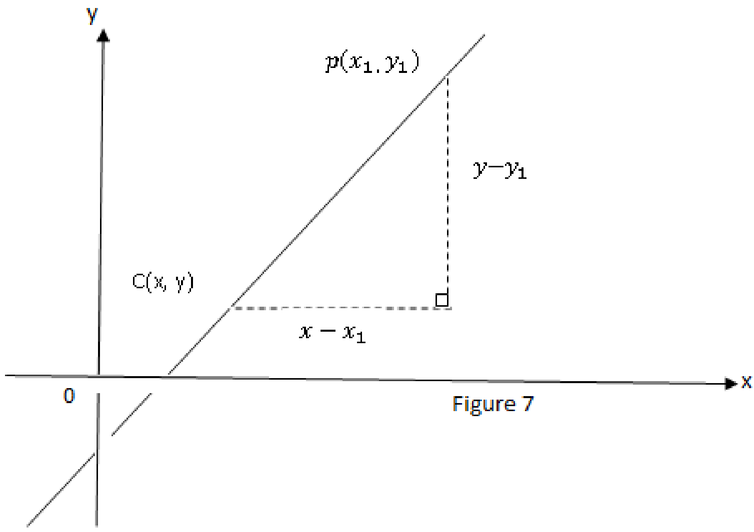


Figure 7

Using the diagram above, the equation of a straight line of gradient and one point form when a straight line passes through a given point p (x_1, y_1) is :

$y - y_1 = m(x - x_1)$

.

C. Gradient and two Points Form

If (x_1, y_1) and (x_2, y_2) are two points given on a line *AB* as shown in figure 8, which is used to get the equation of a straight line on a gradient and two point.

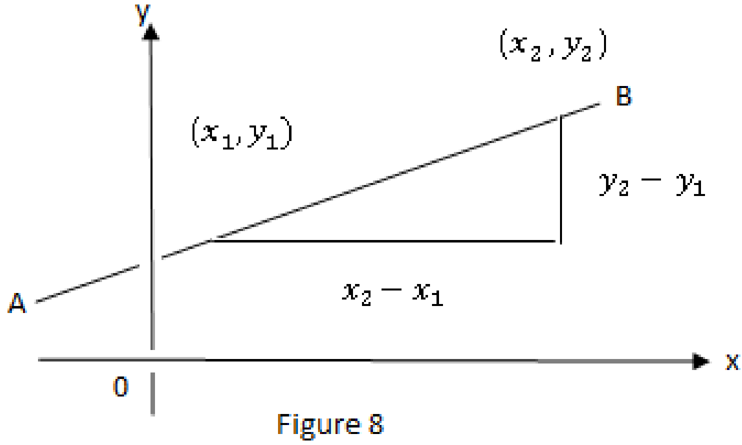


Figure 8

The gradient (m) of the line AB above is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1}$$

From one point form: $y - y_1 = m(x - x_1)$ (2)

Substituting equation (1) into equation (2), you have

$$y - y_1 = \frac{(y_2 - y_1)(x - x_1)}{x_2 - x_1} \tag{3}$$

$\Longrightarrow \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$

Therefore, the equation of a line through the two points is: $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$.

Example 8.3.5 *A* straight line has a gradient of 5/3 and it passes through the point (1,3). Find its equation and its intercept on the y axis.

Solution

Since the line passes through a gradient and one point, you will make use of gradient and one point form equation. That is

$y - y_1 = m(x - x_1)$

Given that $(x_1, y_1) = (1, 4)$ and $m = \frac{5}{3}$, you have that

$y - 4 = \frac{5}{3}(x - 4) \implies 3(y - 4) = \frac{5}{3}3(x - 4)$

$\implies 3y = 5x - 8$

$\implies \frac{3y}{3} = \frac{5x}{3} - \frac{8}{3}$

Hence the equation of the straight line at the point $(x_1, y_1) = (1, 4)$ and the gradient $(m) = \frac{5}{3}$ is:

$\implies y = \frac{5}{3}x - \frac{8}{3}$.

To obtain the intercept, you have to use the equation: $y = \frac{5}{3}x - \frac{8}{3}$

Comparing the above with the equation of straight line: $y = mx + c$, you have that

$c = -\frac{8}{3}$.

8.3.5 Angle Between two Lines

The angle between two lines in a plane is defined to be:

i. 0, if the lines are parallel

ii. the smaller angle having as sides the half-lines starting from the intersection point of the lines and lying on those two lines, if the lines are not parallel.

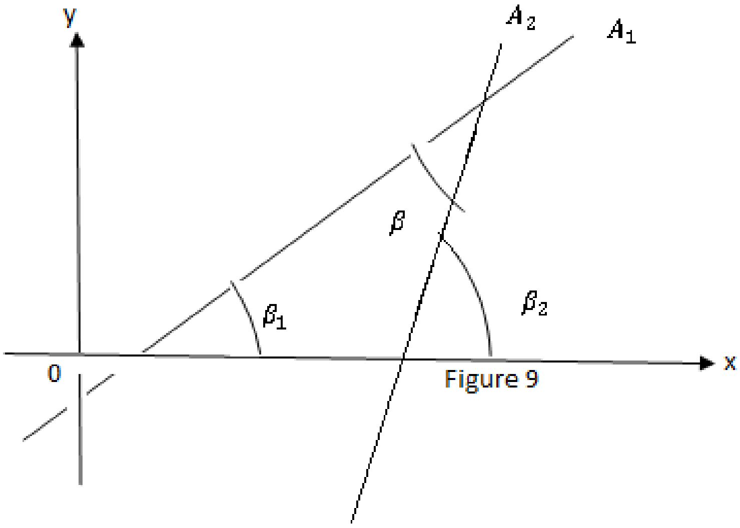


Figure 9

If line A_1 has a gradient m_1 and line A_2 has gradient m_2 making an angles of β_1 and β_2 respectively on the x- axis as shown in figure 9. β is the acute angle between the lines.

Recalling that $\tan\theta = m$ (gradient), this implies that $m_1 = \tan\theta$ and $m_2 = \tan\theta$. The exterior angle of the triangle, $\beta_2 = \beta_1 + \beta$

So $\beta = \beta_2 - \beta_1$

Therefore $\tan\beta = \tan(\beta_2 - \beta_1)$ But $\tan\beta = \frac{\tan\beta_2 - \tan\beta_1}{1+\tan\beta_2 \tan\beta_1}$

$\Longrightarrow \tan(\beta_2 - \beta_1) = \frac{\tan\beta_2 - \tan\beta_1}{1+\tan\beta_2 \tan\beta_1}$

Replacing back the values of β_2 and β_1, you have

$\tan\beta = \frac{m_2 - m_1}{1+m_2 m_1}$

Note: When the result is negative, an obtuse angle $180 - \beta$ is obtained. Hence the acute angle β between lines of gradient m_1, m_2 is given by: $\tan\beta = \frac{m_2 - m_1}{1+m_2 m_1}$ Where $m_1 m_2 \neq 1$.

8.3.5.1 Parallel Lines

When $\tan\beta$ is zero, the lines are parallel and then $m_1 = m_2$.

8.3.5.2 Perpendicular Lines

When the lines are perpendicular, the angle β is 90^o. Hence $\tan 90 = \frac{m_2 - m_1}{1+m_2 m_2}$ which has no finite value. So $1 + m_2 m_1 = 0$.
Therefore, if two lines of gradients m_1, m_2 are perpendicular, then $m_1 m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

Example 8.3.6 *Find* the angle between the two lines $-3x+4y = 8$ and $-2x - 8y - 14 = 0$

Solution

Finding the slope (gradient) of each equation by comparing them to $y = mx + c$, you have
$3x + 4y - 3x = -3x + 8$
$\Longrightarrow 4y = -3x + 8$
$\Longrightarrow y = -\frac{3}{4}x + 2$
$\Longrightarrow m_1 = -\frac{3}{4}$
Similarly, you follow the same argument to obtain m_2
$-2x - 8y - 14 = 0$
$\Longrightarrow 8y = -2x - 14$
$\Longrightarrow y = \frac{-2x}{8} - \frac{14}{8}$
$\Longrightarrow m_2 = \frac{-1}{4}$ Now, using $\tan\beta = \frac{m_2 - m_1}{1+m_2 m_1}$ to find angle between two lines.
Substituting the values of m_1 and m_2 to the equation, you have;
$\tan\beta = \frac{\frac{-3}{4} - (\frac{-1}{4})}{1+(-\frac{3}{4} \text{ X } \frac{-1}{4})}$

$\tan\beta = \frac{\frac{-3}{4}+\frac{1}{4}}{1+(\frac{3}{16})}$
$\tan\beta = \frac{\frac{-1}{2}}{1+(\frac{3}{16})}$
$\tan\beta = -\frac{8}{19}$
$\beta = \tan^{-1}\left(\tan\beta = -\frac{8}{19}\right)$
$\beta = -14.63^{o}$

Example 8.3.7 *Find* the equation of the line through the point $(-1, 2)$ which is
a) parallel to $y = 2x + 1$ b) perpendicular to $y = 2x - 1$
Solution
a) The gradient $(m) = 3$ and $(x, y) = (-1, 2)$
Now to obtain the equation, you make use of the formula: $y - y_1 = m(x - x_1)$
$y - 2 = 3(x - (-1))$
$y - 2 = 3(x + 1)$
$\Longrightarrow y = 3x + 5$

b) The gradient of the perpendicular line is $-\frac{1}{2}$.
Using $y - 2 = -\frac{1}{2}(x - (-1))$
$\Longrightarrow y - 2 = -\frac{1}{2}(+1)$
$\Longrightarrow y - 2 = -\frac{1}{2}x - \frac{1}{2}$
$\Longrightarrow y = -\frac{1}{2}x + \frac{3}{2}$

8.4 Conclusion

In this unit, you have learnt the meaning of straight line and the equation of straight line. You have also learnt the concept of angle of slope and angle between a straight line. Finally, you were also introduced to the distance between two points and midpoint of two points.

8.5 Summary

Having gone through this unit, you have learnt the following:

i. Gradient $(m) = \frac{dy}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$

ii. Angle of slope $\theta = \tan^{-1}\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$

iii. Distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

iv Midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$

v. Gradient and y-intercept form is $y = mx + c$

vi. Gradient and one point form is $y - y_1 = m(x - x_1)$

vii. Two point form is $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

viii. Acute angle β between lines of gradients m_1, m_2 is given by $\tan\beta = \frac{m_2 - m_1}{1 + m_2 m_1}$

ix For parallel lines, $m_1 = m_2$. For perpendicular line, $m_1 m_2 = -1$

8.6 Exercises

1. Find the distance between the points of the following:
   (a) (1, -4) and (7,6)
   (b) (-1,-3) and (5,4)
2. What are the coordinate of the midpoint of the line joining:
   (a) (3, 5) and (6, 7)
   (b) (-3,-9) and (6, 8)
3. Find the gradients of the line joining the points:
   (a) (2,5) and (5,9)
   (b) (1,3) and (5,3)
4. Find the equations of the line which passes through the following pairs of points:
   (a) (-1,-2) and (3, 0)
   (b) (3, -2) and (-5,-2)
5. State the gradient and the intercept of the following lines
   (a) $3x - 2y - 8 = 0$
   (b) $\frac{x}{4} + \frac{y}{3} = 1$
6. A straight line has a gradient of $\frac{-5}{2}$ and passes through the point $(2, 5)$. Find its equation and its intercept on y-axis.

7. Find the equation of a line AB which passes through (3, -3) and (-4,-7). Hence find the equation of the straight line CD parallel to AB, which passes through the point (4, 9).

8. Find the equation of the line which passes through (-6,7) and is parallel to the line $y = 3x + 7$.

9. Find the equation of the line which is parallel to the line $2y + 3 = 3$ and passes through the midpoint of (-2,3) and (4, 5).

10. Find the acute angle between the lines:

    (a) $2y = 3x - 8$ and $5y = x + 7$

    (b) $2x + y = 4$ and $y - 3x + 7 = 0$

11. A line AB passes through the point P (3, -2) with gradient $-\frac{1}{2}$. Find its equation and also the equation and also the equation of the line CD through P perpendicular to AB.

8.7 References

Blitzer. *Algebra and Trigonometry Custom*. 4th Edition
Godman and J.F Talbert. *Additional Mathematics*
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 9

Coordinate Geometry (Circle)

9.1 Introduction

A circle is the locus of all points equidistant from a central point.

9.2 Objectives

In this unit, you shall learn the following:

i. Equation of a circle

ii. Parametric eqaution of a circle

iii. Points outside and inside a circle

iv. Touching circles

v. Tangents to a circle

9.3 Main Content

9.3.1 Equation of a circle

9.3.1.1 Center at the Origin and a Radius r

Let us consider a simplest case of a circle with centre at the origin and radius r as shown in figure 1. If the circle has a centre (0, 0) and radius r.

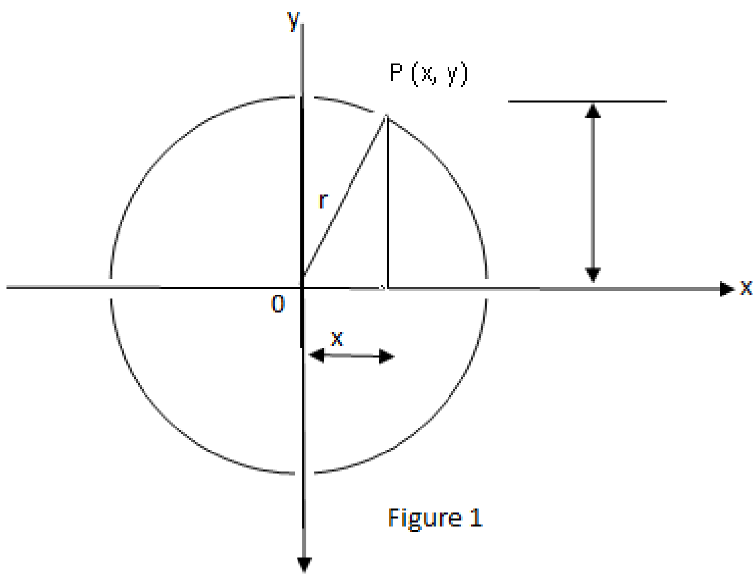


Figure 1

Using Pythagoras theorem, the equation of the circle from the figure above, becomes;
$x^2 + y^2 = r^2$

9.3.1.2 Center and one point given

Equation of a circle by moving the centre to a new point (h, k) which has a radius r as shown in figure 2.

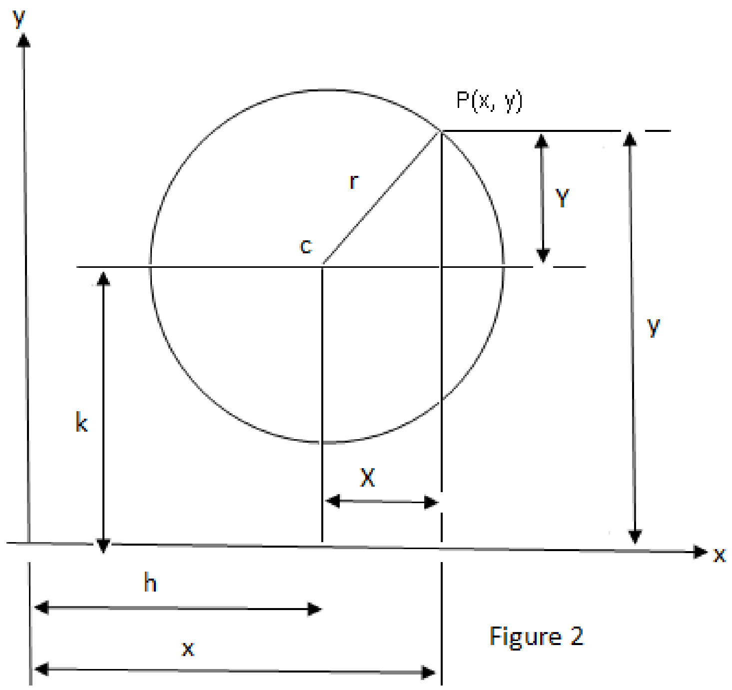


Figure 2

The length (radius) of the hypotenuse of the right triangle is determine by the Pythagoras formula:

$$X^2 + Y^2 = r^2 \quad (1)$$

Where $Y = y - k$ and $X = x - h$. Substituting these values into equation (1), you have
$(x - h)^2 + (y - k)^2 = r^2$
Note: (h ,k) is the centre of the circle and r is the radius.

Example 9.3.1 *Find* the center and radius of the circle $x^2 + y^2 + 8x + 6y = 0$. Sketch the circle.

Solution

The aim is to get the equation into the form: $(x-h)^2+(y-k)^2 = r^2$.

$x^2+y^2+8x+6y=0$

$$\implies x^2+8x+y^2+6y=0 \qquad (1)$$

Applying completing the square method to equation, you have that:

$x^2+8x+16+y^2+6y+9=16+9$

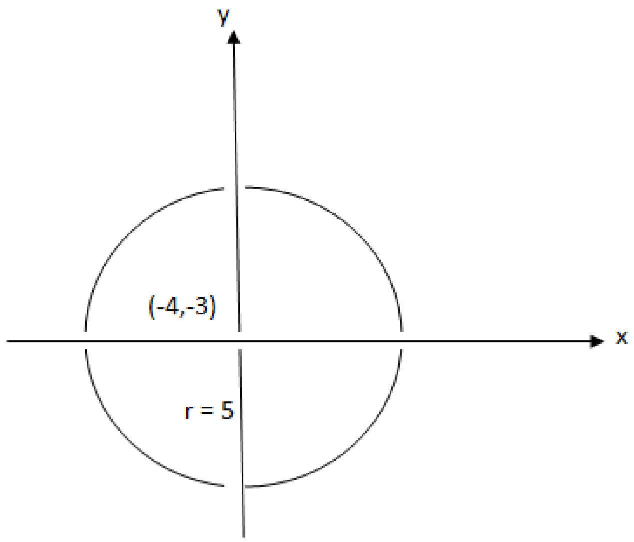
$\implies x^2+8x+16+y^2+6y+9=25$

$$\implies (x+4)^2+(y+3)^2=5^2 \qquad (2)$$

Hence, this is now in the form: $(x-h)^2+(y-k)^2=r^2$ that is required and you can determine the centre and the radius by comparing it with equation (2). So the centre of the circle is: $h=-4$ and $k=-3$

Thus the radius is $r=5$

Note: that the circle passes through (0, 0). This is logical, since $(-4)^2+(-3)^2=(5)^2$.



9.3.1.3 General equation of a circle

The equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where the centre is the point (-g,-f) and the radius $r = \sqrt{g^2 + f^2 - c}$

Note:

For a second degree equation to represent a circle:

a the coefficients of x^2 and y^2 must be identical

b there is no product term in xy.

Example 9.3.2 *The* equation of a circle is $x^2+y^2+2x-6y-15 = 0$. What are the center and the radius?

Solution

Comparing $x^2+y^2+2x-6x-15 = 0$ with $x^2+y^2+2gx+2fy+c = 0$

Taking the coefficient:
Coefficient of x: $2g = 2 \implies g = 1$
Coefficient of y: $2\boldsymbol{f} = -6 \implies \boldsymbol{f} = -3$
Coefficient of c: $c = -15$
To get the radius, you apply the formula, $r = \sqrt{g^2 + \boldsymbol{f}^2 - c}$
So that $\boldsymbol{r} = \sqrt{1^2 + (-3)^2 - (-15)^2} = \sqrt{25} = 5$
Hence, the center $(-g, -\boldsymbol{f}) = (-1, 3)$ and the radius $\boldsymbol{r} = 5$.

Example 9.3.3 *Find* the equation of the circle with center $(-2, 3)$ and radius 6.

Solution
Using the equation, $(x - h)^2 + (y - k)^2 = \boldsymbol{r}^2$.
Where $h = -2$, $k = 3$, $\boldsymbol{r} = 6$
You now have that,
$[x - (-2)]^2 + [y - 3]^2 = 6^2$
$\implies x^2 + 4x + 4 + y^2 - 6y + 9 = 36$
$\implies x^2 + y^2 + 4x - 6y - 23 = 0$

Example 9.3.4 *Find* the equation of the circle with center (-3,4) which passes through the point (2,5)

Solution
You first have to find the radius r, using the distance between two points formula since radius r is the distance from (-3,4) to (2, 5).
$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[2 - (-3)]^2 + (5 - 4)^2} = \sqrt{5^2 + 1^2}$
$\implies r^2 = 26$
Hence, using the equation, $(x - h)^2 + (y - k)^2 = \boldsymbol{r}^2$, where $\boldsymbol{r}^2 = 26$, $h = -3$, $k = 4$
$[x - (-3)]^2 + (y - 4)^2 = 26$

$\Longrightarrow\ x^2 + 6x + 9 + y^2 - 8y + 16 = 26$
$\Longrightarrow\ x^2 + y^2 + 6x - 8y - 1 = 0$

Example 9.3.5 *Find* the points of intersection of the circle $x^2 + y^2 - x - 3y = 0$ with the line $y = x - 1$.

Solution

Solving the equations simultaneously, you have;

$$x^2 + y^2 - x - 3y = 0 \tag{1}$$

$$y = x - 1 \tag{2}$$

Substituting equation (2) into equation (1), you will have;
$x^2 + (x - 1)^2 - x - 3(x - 1) = 0$
$\Longrightarrow\ x^2 + x^2 - 2x + 1 - x - 3x + 3 = 0$
$\Longrightarrow\ 2x^2 - 6x + 4 = 0$
$\Longrightarrow\ x^2 - 3x + 2 = 0$
$\Longrightarrow\ (x - 1)(x - 2)$
$x = 1$ or $x = 2$
To get the corresponding y values, you make use of equation(2) and substituting the values of x.
$y = 1 - 1$ or $y = 2 - 1$
$\Longrightarrow\ y = 0$ or $y = 1$
So that the points of intersection are at $(1, 0)$ and $(2, 1)$.

9.3.2 Parametric Equation of a Circle

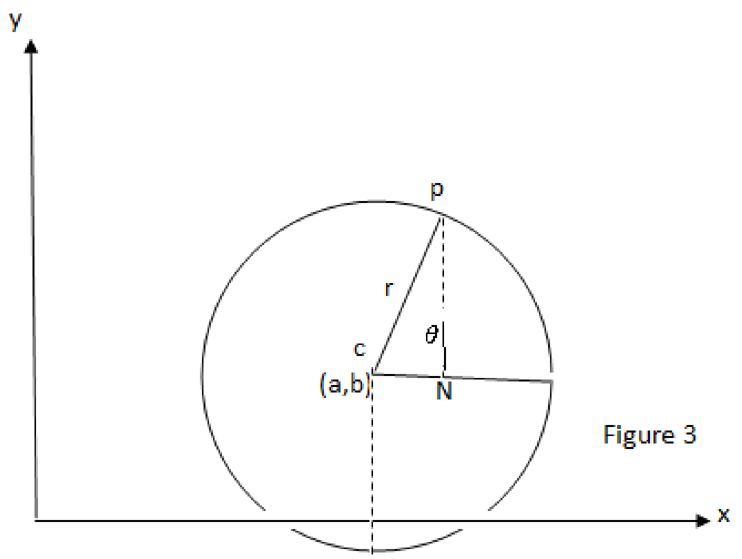


Figure 3

In Figure 3, the circle with centre C(a, b) and radius r. If P is any point on the circle and the radius CP makes an angle $\theta(0 \leq \theta \leq 360^o)$ with CN which is parallel to the axis.
$CN = r\cos\theta$ and $PN = r\sin\theta$. So the coordinate (x, y) of P will be given by the two equations:

$$x = a + r\cos\theta, \; y = b + r\sin\theta \qquad (1)$$

These is called the parametric equations of the circle as they give the coordinates of every point on the circle in terms of one variable or parameter θ (where r is always greater than zero).
From these equations, you can deduce the ordinary x-y equation by eliminating between them.

From equation (1), you have
$\cos\theta = \frac{x-a}{r}, \quad \sin\theta = \frac{y-b}{r}$.
Recall that $\sin^2\theta + \cos^2\theta = 1$
Hence, $\frac{(x-a)^2}{r^2} + \frac{(y-b)^2}{r^2} = 1$
$\Longrightarrow (x - a)^2 +)(y - b)^2 = r^2$. This is still the usual form of the equation of a circle.

Example 9.3.6 a. State the parametric equations of a circle with centre (2, -1) and radius 3.

b. State the centre and radius of a circle given by $x4 = 2c\cos\theta,\ y+ 3 = 2\sin\theta$

Solution

a. Using the formula of parametric equations of the circle, $x = a + r\cos\theta,\ y = b + r\sin\theta$.
Then $x = 2 + 3\cos\theta$
$y = -1 + 3\sin\theta$
b. Rewriting the parametric equation:
$x = 4 + 2\cos\theta,\ y = -3 + 2\sin\theta$. The center is $(4, -3)$ and the radius is 2.

9.3.3 Points Outside and Inside a Circle

Let the function $f(x, y) = (xa)^2 + (y - b)^2 - r^2$. $f(x, y) =$ 0= defines a circle, radius r and centre (a, b). When the point (h, k) lies inside this circle as shown in the figure 4.

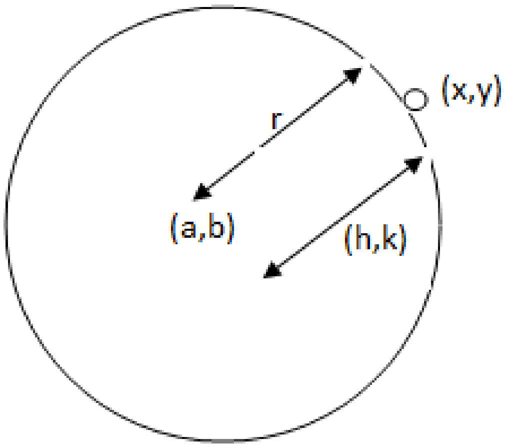


Figure 4

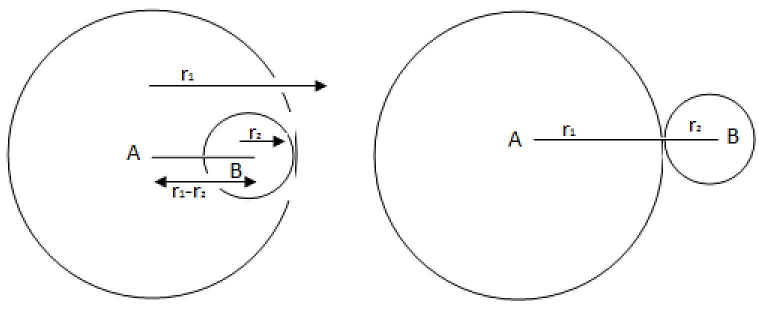
Then, $(h - a)^2 + (k - b)^2 < r^2$ i.e. $(h - a)^2 + (k - b)^2 - r^2 < 0$ or $f(h, k) < 0$.
So, if (h, k) lies inside the circle given by $f(x, y) = 0$, then $f(h, k) < 0$. Similarly if (h, k) lies outside the circle, $f(h, k) > 0$.

Example 9.3.7 *Are* the points A(1,-1) B(5, 2) inside or outside the circle $x^2 + y^2 - 3x + 4y = 12$?

Solution
Take $f(x, y) = x^2 + y^2 - 3x + 4y - 12$
$f(1, -1) = 1 + 1 - 3 - 4 - 12 = -17 < 0$. So A is inside the circle.
$f(5, 2) = 25 + 4 - 15 + 8 - 12 = 10 > 0$. So B is outside the circle.

9.3.4 Touching Circles

Two circles with centres A, B and radii r_1, r_2 can touch each other internally or externally, as shown below.



Example 9.3.8 *Find* how the circles $x^2 + y^2 + 2x + 6y = 39$ and $x^2 + y^2 - 4x - 2y + 1 = 0$ touch each other. Find the coordinates of their point of contact.

Solution

Comparing the two equations with $x^2 + y^2 + 2gx + 2fy + c = 0$ to find their respective centre and their respective radius.

First, you find the centre and radius at point A:

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

$$x^2 + y^2 + 2x + 6y - 39 = 0 \tag{2}$$

Comparing the coefficient of each other:

Coefficient of x: $2g = 2.$

$\Longrightarrow g = 1$

Coefficient of y: $2f = 6.$

$\Longrightarrow f = 3$

Coefficient of constant term: $c = -39.$

Hence, $(-g, -f) = (-1, -3).$ and

$r = \sqrt{g^2 + f^2 - c} = \sqrt{1^2 + 3^2 - (-39)} = \sqrt{49} = 7$

Center $A(-1, -3)$ and its radius $r = 7$.

Similarly, to find the center and radius at point B

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{1}$$

and

$$x^2 + y^2 - 4x - 2y + 1 = 0 \tag{2}$$

Comparing the coefficients;
Coefficient of x: $2g = -4$
$\Longrightarrow g = -2$
$f = -1$
Coefficient of constant term: $c = 1$
Therefore, center $(-g, -f) = (2, 1)$
$r = \sqrt{g^2 + f^2 - c} = \sqrt{(-2)^2 + (-1)^2 - 1} = \sqrt{4} = 2$
To get the difference of the two radii, you must use of distance between two points A(-1,-3) and B(2,1).
$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[2 - (-1)]^2 + [1 - (-3)]^2} = \sqrt{3^2 + 4^2} = 5$
To find the coordinates (x_p, y_p) of the point of contact p, you have that;
$\frac{x_p - (-1)}{x_p - 2} = \frac{7}{2}$ and $\frac{y_p - (-3)}{y_p - 1} = \frac{7}{2}$
This implies that $x_p = 3\frac{1}{5}$, $y_p = 2\frac{3}{5}$. The point of contact is $\left(3\frac{1}{5}, 2\frac{3}{5}\right)$.

9.3.5 Tangents to a Circle

A tangent of a circle is the point of contact where the perpendicular line drawn from radius meets with the line that runs across the circumference of the circle (i.e. tangential line).

Example 9.3.9 *Find* the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y = 15$ at the point (-1, 2).

Solution

$x^2 + y^2 - 2x + 4y = 15$

Rearranging this equation of the circle and comparing with the general equation of the circle, you will have

$$x^2 + y^2 - 2x + 4y - 15 = 0 \tag{1}$$

$$x^2 + y^2 + 2gx + 2fy + c = 0 \tag{2}$$

Coefficient of x: $2g = -2 \implies g = -1$

Coefficient of y: $2f = 4 \implies f = 2$

Therefore the centre $(-g, -f) = (1, -2)$. The gradient of the radius to the point $(-1, 2)$ is -2. Hence, the gradient of the tangent is $\frac{1}{2}$. Its equation is therefore:

$y - 2 = 1/2(x + 1)$

i.e $2y = x + 5$.

9.4 Conclusion

In this unit, you have learnt the concept of circle. You were also introduced to the equation and parametric equations of a circle.

9.5 Summary

Having gone through this unit, you have learnt that:

i. The equation of a circle with center 0, radius r is: $x^2 + y^2 = \boldsymbol{r}^2$.

ii. Circle, centre (a, b), radius r has equation : $(x - h)^2 + (y - k)^2 = \boldsymbol{r}^2$.

iii. General equation of circle: $x^2 + y^2 + 2gx + 2\boldsymbol{f}y + c = 0$.

9.6 Exercises

1. Given a circle with centre (-3,-4), which passes through the point (2,-1). Find the equation.
2. If the circle centre (1, -3) passes through the origin, find its equation.
3. Find the centre and radius of each of the following circles:
   (a) $x^2 + y^2 - 2x - 6y = 15$.
   (b) $x^2 + y^2 - 6x + 14y = 49 = 0$.
4. (a) State the parametric equations of a circle with centre (-1,-3) and radius 5.
   (b) State the centre and radius of the circle given by $x = 2 + 4\cos\theta$, $y = -5 + 4\sin\theta$
5. . Show that the circle $x^2+y^2-2x-4y+4 = 0$ lies completely inside the circle $x^2 + y^2 = 15$.
6. Find the equation of the tangents to the circles:
   (a) $x^2 + y^2 + 4x - 10y - 12 = 0$ at (3, 1).

(b) $x^2 + y^2 - 6x - 4y - 16 = 0$ at (-2,0).

9.7 References

Blitzer. *Algebra and Trigonometry Custom*. 4th Edition
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K.A Stroud. *Engineering Mathematics*. 5th Edition
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UNIT 10

Parabola

10.1 Introduction

A parabola has many applications in real life situations where:

a. radiation needs to concentrated at one point(e.g. radio telescope, pay TV dishes, solar radiation collectors) or

b. radiation needs to be transmitted from a single point into wide parallel beam(e.g headlight reflectors).

10.2 Objectives

In this unit, you shall learn the following:

i. definition of a parabola

ii. the formula for a parabola vertical axis

iii. parabola with horizontal axis

iv. shifting the vertex of a parabola from the origin.

10.3 Main Content

10.3.1 Definition of Parabola

A parabola is defined as the locus of a point which moves so that it is always the same distance from the fixed point (called the focus) and a given line (called the directrix).

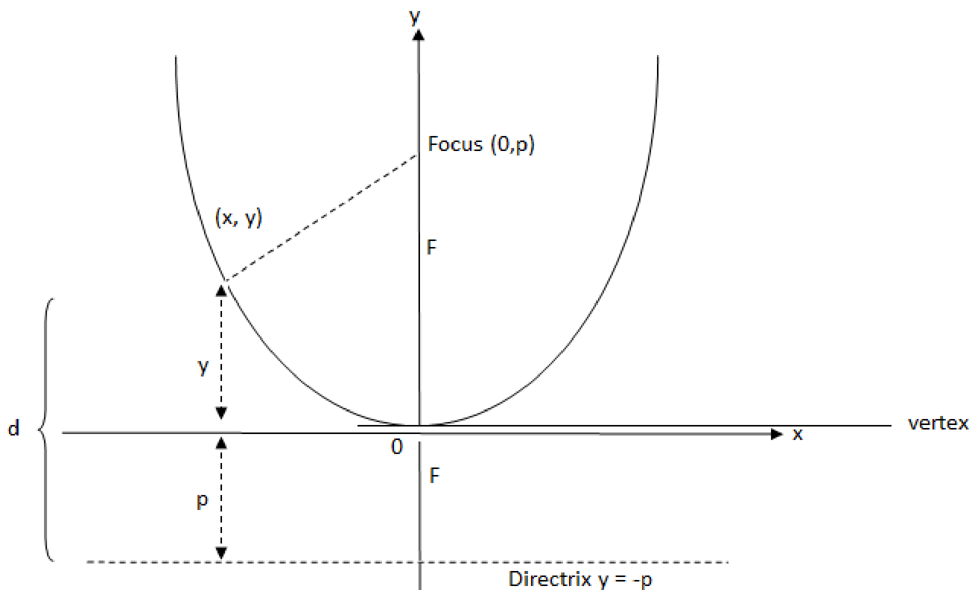


Figure 1

In the graph above:

i. The focus of the parabola is at $(0, p)$

ii. The line $y = -p$ is the directrix.

iii. The focal distance is $|p|$ (Distance from the origin to the focus and from the origin to the directrix. You take absolute value because distance is positive.)

iv. The point (x, y) represent any point on the curve.

v. The distance d from any point (x, y) to the focus (0, p) is the distance from (x, y) to the directrix.

vi. The axis of symmetry (goes through the focus, at right angles to the directrix).

vii. The vertex (where the parabola makes its sharpest turn) is halfway between the focus and directrix.

viii. Reflector is any ray parallel to the axis of symmetry which gets reflected off the surface straight to the focus. This explains why that dot is called the focus, because that is where all the rays get focused.

10.3.2 The formula for a parabola vertical axis.

From the diagram, in figure 1, If the distance $d = y + p$. Using the distance formula on the general points (0, p) and (x, y), and equating it to our value $d = y + p$, you have:

$\sqrt{(x - 0)^2 - (y - p)^2} = y + p$

Squaring both sides gives;

$(x - 0)^2 - (y - p)^2 = (y +$

$p)^2$

Simplifying gives you the formula for a parabola:
$x^2 - (y^2 - 2py + p^2) = y^+2py + p^2$
Collecting like terms of the equation above.
$x^2 - y^2 + 2py - p^2 = y^2 + 2py + p^2.$
$x^2 = y^2 - y^2 + 2py + 2py + p^2 - p^2.$
$x^2 = 4py.$
Making y subject of the formula: $y = \frac{x^2}{4p}$
Where p is the **focal distance** of the parabola.

Example 10.3.1 *What* is the equation of the directrix of the parabola whose equation is: $x^2 - 30y = 0.$

Solution

$x^2 - 30y = 0.$
Rearranging this and making x^2 subject of the formula, you will obtain:

$$x^2 = 30y \tag{1}$$

Equating equation (1) with the general formula of parabola, you will have;
$30y = 4py \implies p = 7.5$
The directrix is on the opposite side of the origin to the focus (0, 7.5).
So, the directrix of $x^2 - 30y = 0$ is $y = -7.5.$

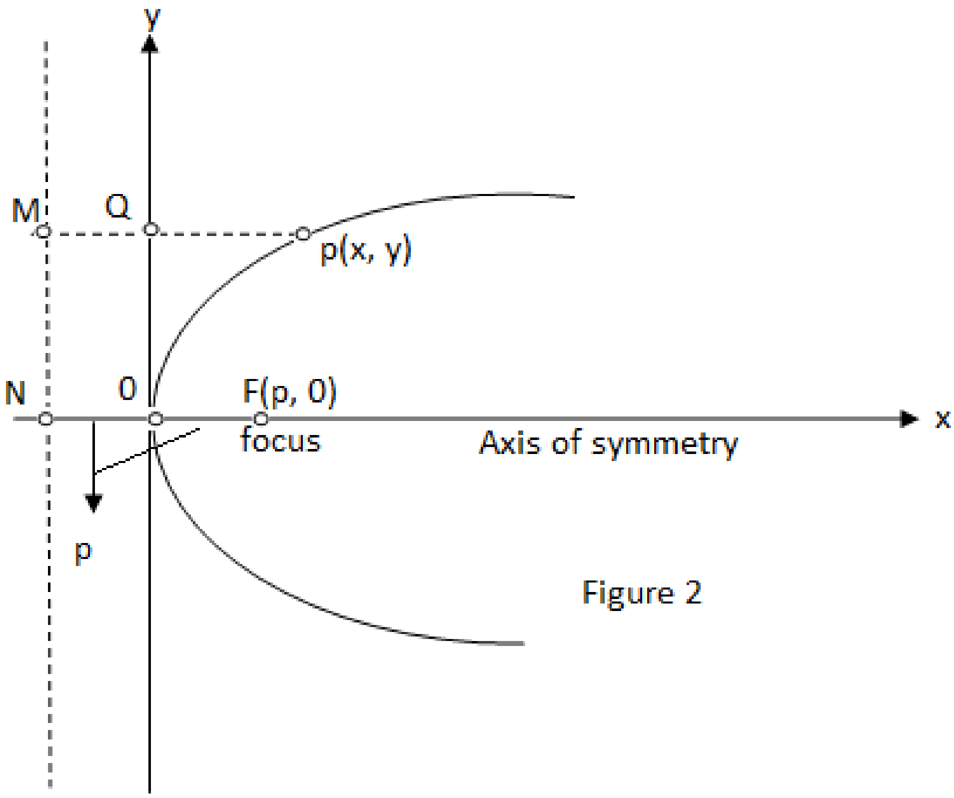
Example 10.3.2 *Where* is the focus of the parabola whose equation is $y^2 + 32x = 0$

Solution

$y^2 + 32x = 0.$
Rearranging the terms and comparing with the general formula $y^2 = 4px.$, you will obtain that

$4px = -32x \Longrightarrow P = -8$
So the focus of $y^2 + 32x = 0$ is $F(-8, 0)$.

10.3.3 Parabola with horizontal axis.



In this case as shown in figure 2, the parabola place on the Cartesian coordinate (x- y graph) with:

a. Its vertex at the origin 0 and

b. Its axis of symmetry lying on the x axis, then the curves is defined by: $y^2 = 4px$.

Example 10.3.3 *Where* is the focus in the equation $y^2 = 5x$?

Solution

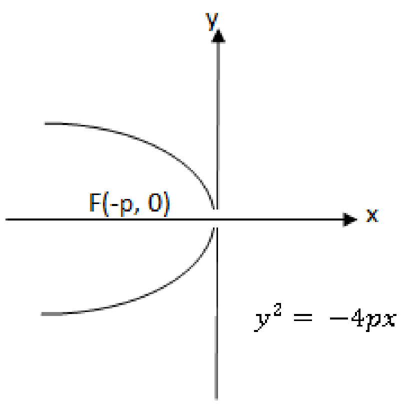
Comparing $y^2 = 5x$ with equation of a parabola on a horizontal axis, you have that:

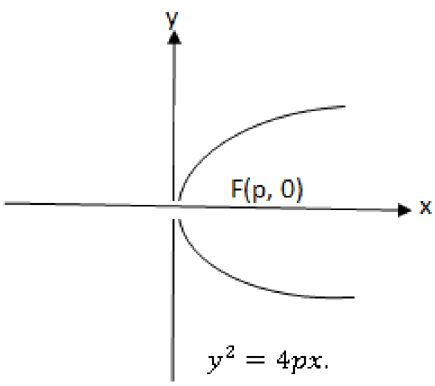
$$y^2 = 5x \qquad (1)$$

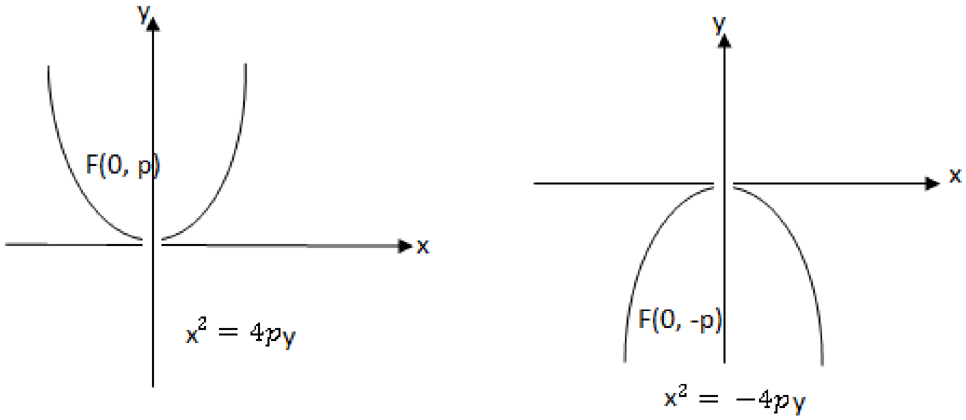
$$y^2 = 4px \qquad (2)$$

Equating equation (1) to equation(2), you will have:
$5x = 4px \implies p = \frac{5}{4}$
Substituting the value of p into equation (2), you will obtain that : $y^2 = 4\frac{5}{4}x \implies y^2 = 5x$
So that $F(p, 0) = \left(\frac{5}{4}, 0\right)$.

The equations of a parabola in different orientations are as follows:

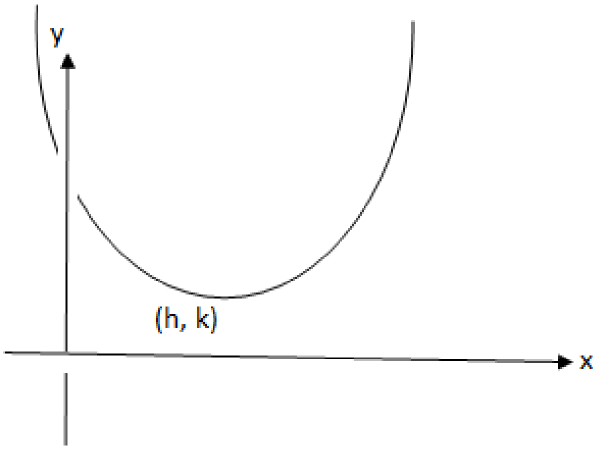




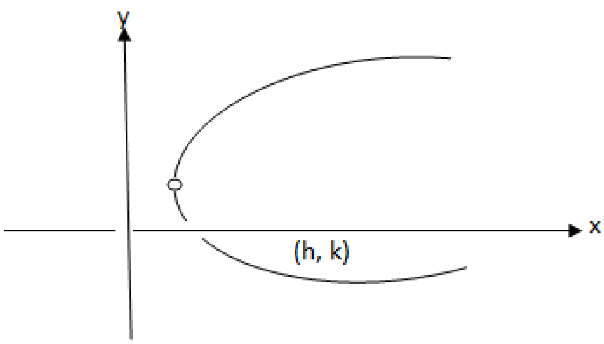


10.3.4 Shifting the vertex of a parabola from the origin.

This is similar concept to the case when we shifted the centre of a circle from the origin. To shift the vertex of a parabola from $(0, 0)$ to (h, k), each x in the equation becomes $(y - k)$.
So if the axis of a parabola is vertical and the vertex is at (h, k), you have:
$(x - h)^2 = 4p(y - k)$.



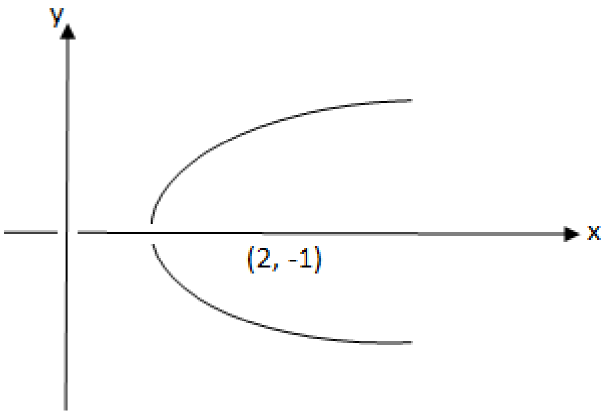
If the axis of a parabola is horizontal and the vertex is at (h, k), the equation becomes:
$(y - k)^2 = 4p(x - h)$.



Example 10.3.4 *Find* the equation of the parabola having vertex $(0, 0)$, axis along the x- axis and passing through $(2, -1)$.

Solution

The curve must have this orientation, since you know it has horizontal axis and passes through $(2, -1)$:



Using the general formula $y^2 = 4px$. you need to find the value of p since the curve goes through $(2, -1)$. Since $x = 2$ and $y = -1$, then substituting the value of x and y into the general formula. $(-1)^2 = 4(p)(2)$. $\Longrightarrow 1 = 8p \Longrightarrow p = \frac{1}{8}$

Now substituting the value of p into the general equation to get equation of the parabola.

$y^2 = 4\frac{1}{8}x$

This implies that, the equation of the parabola is $y^2 = \frac{x}{2}$

10.4 Conclusion

In this unit, you studied the concept of parabola. You were also introduced to the equation of a parabola with vertical and horizontal axis.

10.5 Summary

Having gone through this unit, you have learnt that:

i. A parabola is defined as the locus of a point which moves so that it is always the same distance from the fixed point (called the focus) and a given line (called the directrix).

ii. The equation of a parabola with horizontal axis is: $y^2 = 4px$

iii. The equation of a parabola with vertical axis is: $x^2 = 4py$

iv. If the axis of a parabola is vertical and the vertex is at (h, k), the the equation of the parabola is: $(x - h)^2 = 4p(y - k)$

v. If the axis of a parabola is horizontal and the vertex is at (h, k), the the equation of the parabola is: $(y - h)^2 = 4p(x - k)$

10.6 References

Blitzer. *Algebra and Trigonometry Custom*. 4th Edition
K.A Stroud. *Engineering Mathematics*. 5th Edition
Larson Edwards *Calculus: An Applied Approach*. Sixth Edition.

UNIT 11

Ellipse and Hyperbola

11.1 Introduction

Ellipse and hyperbola take a huge rule curves and curves fittings that devise reliable methods for establishing the relationship between two variables for minimizing errors.

11.2 Objectives

In this unit, you shall study the concept of ellipse and hyperbola, together with their properties.

11.3 Main Content

11.3.1 Ellipse

Definition 11.3.1 *An* Ellipse can be defined as the locus of all points that satisfies the equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$
Where:
$x, \ y$ are the coordinates of any point on the ellipse,
$a =$ radius on the x axis or the semi major axis , and
$b =$ radius on the y axis or the semi minor axis.

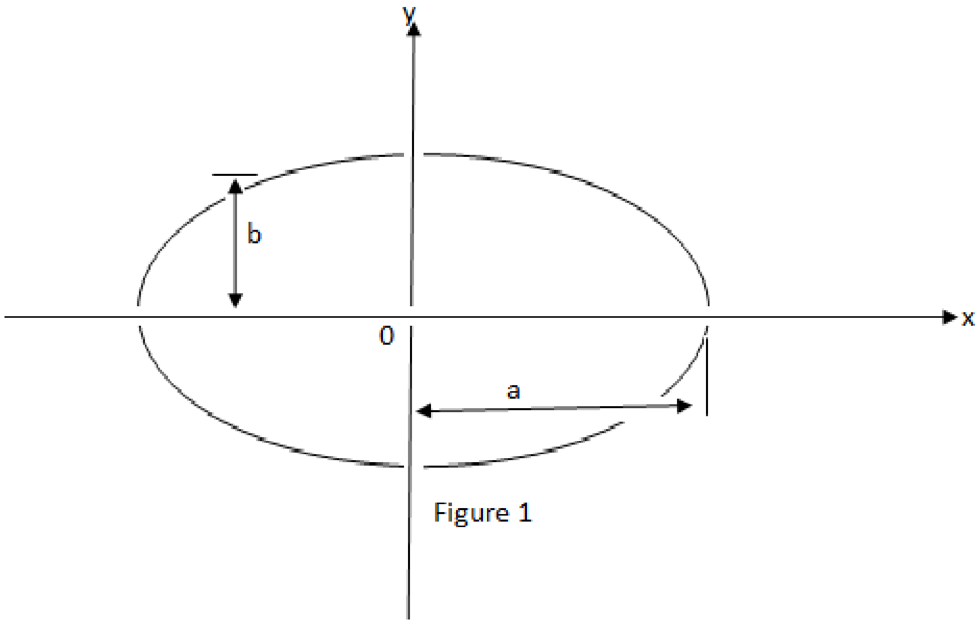


Figure 1

This equation is very similar to the one used to define a circle. The only difference between a circle and an ellipse, is that in an ellipse, there are two radius measures, one horizontally along the x- axis, the other vertically along the y axis as shown in figure 1.

Clearly, for a circle both of these have the same value. By convention, the y radius is usually called b and the x - axis is called *a*.

11.3.1.1 Ellipses Centred at the Origin.

If the ellipse is centred on the origin, (its centre at (0, 0)) the equation is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where: $a =$ radius on the x- axis, this implies that $y = 0$, $x = \pm a$ $b =$ radius on the y- axis, this implies $x = 0$, $y \pm b$. Note that the equations above are true only for ellipses that are aligned with the coordinate plane, that is, where the major and minor axes are parallel to the coordinate system.

11.3.1.2 Ellipses not Centred at the Origin

Just as with the circle equations, we subtract offsets from the x and y terms to translate (or move) the ellipse back to the origin. So the full form of the equation is: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
Where a is the radius along the x axis, b is the radius along the y axis, h, k are the x, y coordinates of the ellipses center.

11.3.1.3 Derivation of Ellipse formula.

You recall the basic equation of a circle: $x^2 + y^2 = r^2$
From the above equation, you have $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$
Replace the radius with the a separate radius for the x- axis and b separate radius for the y- axis.
$\frac{x^2}{a^2} + \frac{y^2}{r^2} = 1$.

With $a > b > 0$. The length of the major axis is 2a and the length of the minor axis is $2b$. The two foci (foci is the plural of focus) are at $(\sim \pm \sim c, 0)$ or at $(\sim +mn \sim c)$

Example 11.3.2 *Given* that ellipse has an equation of $9x^2 + 4y^2 = 36$.

a. Find the x and y intercepts of the graph of the equation.

b. Find the coordinates of the foci.

c. Find the length of the major and minor axis.

d. Sketch the graph of the equation.

Solution

a. Recall that the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Hence, rewrite the equation in the form of the ellipse equation: $9x^2 + 4y^2 = 36$.

$$\Longrightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1 \tag{2}$$

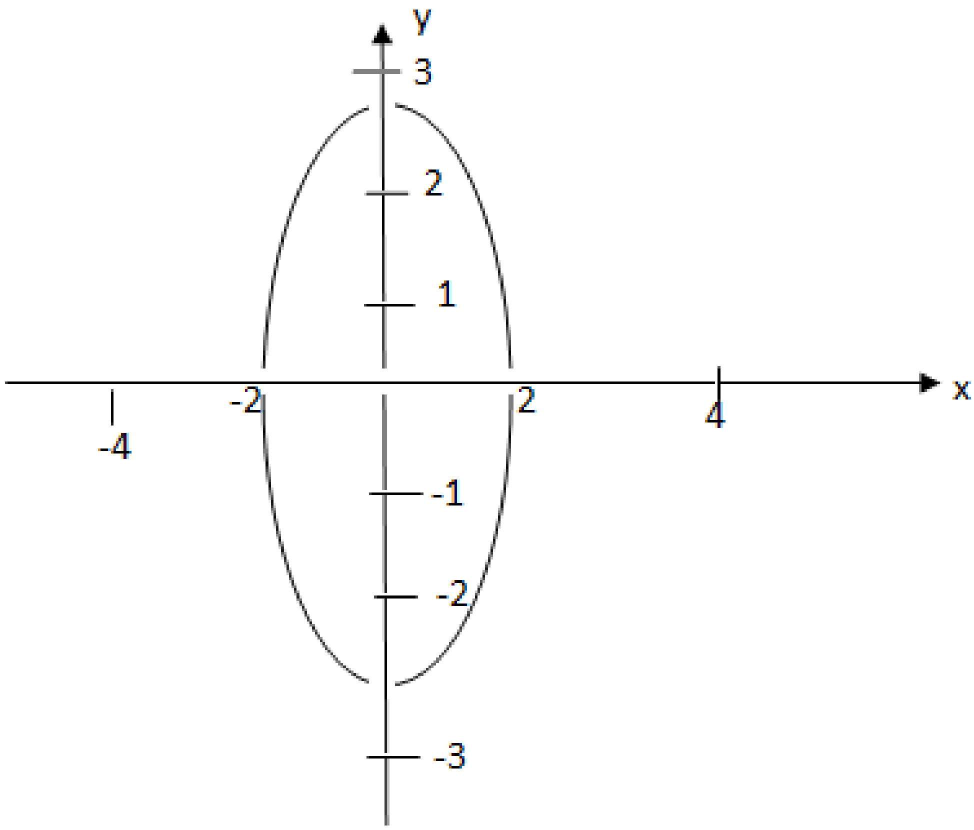
Comparing equation (1) and equation (2) together.
Then a =3, and b =2 (where $a > b$).
Set y=0 in equation (2) obtained and find the x- intercepts.
Then equation (2) becomes: $\frac{x^2}{2^2} + \frac{0^2}{3^2} = 1$
$\Longrightarrow \frac{x^2}{2^2} = 1$
$\Longrightarrow x^2 = 2^2$
$\Longrightarrow x =\sim +mn \sim 2$

Set $x = 0$ in equation (2) obtained and find the y- intercepts.
Then equation (2) now becomes: $\frac{0^2}{2^2} + \frac{y^2}{3^2} = 1$
$\Longrightarrow \frac{y^2}{3^2} = 1$
$\Longrightarrow y^2 = 3^2$
$\Longrightarrow y = \sim +mn \sim 3.$
Hence, x and y intercept is: $(x, y) = (2, 3)$.

b. To find the coordinate of the foci, you need to find c first.
$c^2 = a^2 - b^2$. Substituting a and b that was found in part (a) into the equation, you obtain:
$c^2 = a^2 - b^2 = 3^2 - 2^2 = 9 - 4 = 5.$
$\Longrightarrow c^2 = 5. \Longrightarrow c = \sim +mn \sim (5)^{\frac{1}{2}}$
The foci are $F_1 \; 0, (5)^{\frac{1}{2}}$ and $F_2 \; 0, -(5)^{\frac{1}{2}}$

c. The major axis length is given by $2a = 2 \text{ x } 3 = 6$
The minor axis length is given by $2b = 2 \text{ x } 4 = 8$

d. Locate the x and y intercepts, find the extra points if needed and sketch.



11.3.2 Hyperbola

Hyperbola is a symmetrical open curve formed by the intersection of a circular cone with a plane at a smaller angle with its axis than the side of the cone.
Mathematically, **it** is defined as a curve where the distance of any point from:

i. a fixed point (the focus).

ii. a fixed straight line (the directrix) are always in the same ratio.

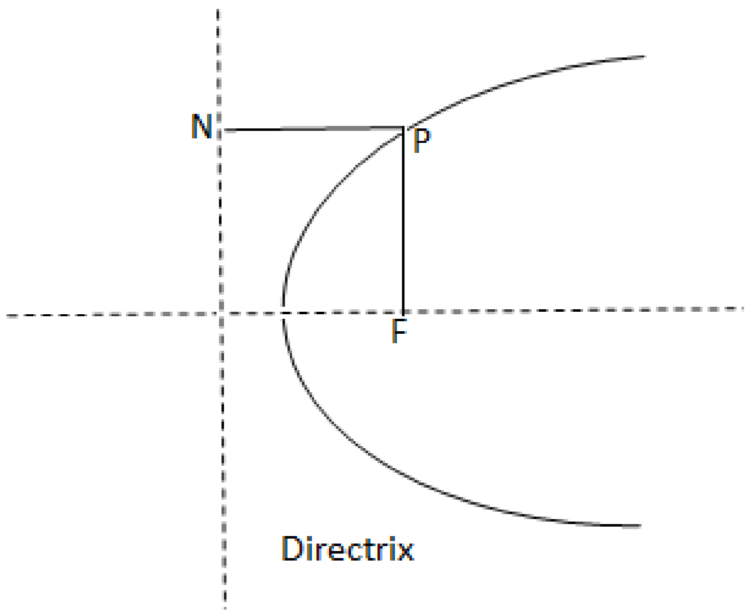
This ratio is called the eccentricity, and for the hyperbola **it** is always greater than 1. The hyperbola is an open curve (has no ends). Hyperbola is actually **t**wo separate curves in mirror image.

11.3.2.1 Equation of Hyperbola

By placing a hyperbola on an x- y graph (centred over the x- axis and y axis), the equation of the curve is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Also:
One vertex is at (a, 0), and the other is at $(-a, 0)$.
The asymptotes are the straight lines: $y = \frac{b}{a}x$
$y = -\frac{b}{a}x$.

11.3.2.2 Eccentricity

Eccentricity (e) shows how uncurvy (varying from being a circle) the hyperbola is.



In the diagram above, you have that

a. P is the point on the curve

b. F is the focus and

c. N is the point on the directrix so that PN is perpendicular to the directrix.

The ratio $\frac{PF}{PN}$ is the eccentricity of the hyperbola (for a hyperbola the eccentricity is always greater than 1). It can also be given by the formula: $e = \frac{\sqrt{a^2+b^2}}{a}$

Example 11.3.3 *Given* the following equation $9x^2 - 16y^2 = 144$

a. Find the x and y intercepts of the graph of the equation.

b. Find the coordinate of the foci.

c. Sketch the graph of the equation.

Solution

a. Rewriting the equation in the form of: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ you have
$9x^2 - 16y^2 = 144$

$$\Longrightarrow \frac{x^2}{4^2} - \frac{y^2}{3^2} = 1 \quad (1)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (2)$$

Comparing equation(1) and equation (2)
$a = 4$ and $b = 3$.
Set $y = 0$ in the equation obtained and find the x intercepts.
$\frac{x^2}{4^2} - \frac{0^2}{3^2} = 1 \Longrightarrow \frac{x^2}{4^2} = 1$
$\Longrightarrow x^2 = 4^2$
$\Longrightarrow x \sim +mn \sim 4$
Set x =0 in the equation obtained and find the y intercepts.
$\frac{0^2}{4^2} - \frac{y^2}{3^2} = 1 \Longrightarrow \frac{-y^2}{3^2} = 1$
$\Longrightarrow -y^2 = 3^2$.
Hence, y intercept does not exist, since the equation has no real solution.

b. You need to find c first.
$c^2 = a^2 + b^2$
$c^2 = 4^2 + 3^2 \Longrightarrow c^2 = 25$.
Solve for c, you have that
$c =\sim +mn \sim 5$.
The foci are $F_1(5, 0)$ and $F_2(-5, 0)$

c. Find extra points (if necessary)
Set $x = 6$ and find y.
$9(6)^2 - 16y^2 = 144 \implies -16y^2 = 144 - 324.$
$\implies y^2 = \frac{45}{4}$
$\implies y = \frac{3(5)^{\frac{1}{2}}}{2}$ and $y = -\frac{3(5)^{\frac{1}{2}}}{2}$
So the points $\left(6, \frac{3(5)^{\frac{1}{2}}}{2}\right)$ and $\left(6, -\frac{3(5)^{\frac{1}{2}}}{2}\right)$ are on the graph of the hyperbola.
Also because of the symmetry of the graph of the hyperbola, the points $\left(-6, \frac{3(5)^{\frac{1}{2}}}{2}\right)$ and $\left(-6, -\frac{3(5)^{\frac{1}{2}}}{2}\right)$ are also on the graph of the hyperbola.

11.4 Conclusion

In this unit, you have studied the concept of ellipse and hyperbola. You also learnt how to derive the formula for ellipse and the equation of hyperbola.

11.5 Summary

Having gone through this unit, you have learnt that:

i. An Ellipse can be defined as the locus of all points that satisfy the equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

ii. By placing a hyperbola on an x- y graph (centred over the x- axis and y axis), the equation of the curve is: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

11.6 Exercises

1. Given the following equation: $x^2 - y^2 = 9.$
   (a) Find the x and y intercepts, if possible, of the graph of the equation.
   (b) Find the coordinate of the foci.
   (c) Sketch the graph of the equation.
2. Given that ellipse has an equation of $9x^2 + 16y^2 = 144.$
   (a) Find the x and y intercepts of the graph of the equation.
   (b) Find the coordinates of the foci.
   (c) Find the length of the major and minor axis.
   (d) Sketch d graph of the equation.
3. What is the major axis and length for the ellipse? $\frac{1}{9}x^2 + \frac{9}{25}y^2 = \frac{1}{25}$
4. ellipse is given by the equation $8x^2 + 2y^2 = 32.$ Find:
   (a) the major axis and the minor axis of the ellipse and their lengths.
   (b) the vertices of the ellipse.
   (c) the foci of this ellipse.
5. Find the equation of the ellipse whose centre is the origin of the axes and has a focus at (0, 4) and a vertex at (0, -6).
6. Find the equation of the ellipse whose foci are at (0,-5) and (0, 5) and the length of its major axis is 14.

7. An ellipse has the x axis as the major axis with a length of 10 and the origin as the center. Find the equation of this ellipse if the point $(3, \frac{16}{5})$ lies on the graph.
8. An ellipse has the following equation: $0.2x^2 + 0.6y^2 = 0.2.$
   (a) find the equation of part of the graph of the given ellipse that is to the left of the axis.
   (b) Find the equation of part of the graph of the given ellipse that is below the x axis
9. ellipse is given by the equation: $\frac{(x-1)^2}{9} + \frac{(y+4)^2}{16} = 1$. Find
   (a) its center
   (b) its major and minor axes.
   (c) its vertices.
   (d) and the foci
10. Find the equation of the ellipse whose foci are at (-1,0) and (3, 0) and the length of its minor axis is 2.
11. An ellipse is defined by its parametric equations as follows: $x = 6\sin(t)$ and $y = 4\cos(\boldsymbol{t})$. Find the centre, the major and minor axes and their lengths of the ellipse.

11.7 References

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