NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF SCIENCE AND TECHNOLOGY

COURSE CODE: MTH 102

COURSE TITLE: ELEMENTARY MATHEMATICS II

NOUN

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COURSE GUIDE

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Course Guide

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Course Title: ELEMENTARY MATHEMATICS II

Introduction

MTH 102 - Elemetary Mathematics II is designed to teach you how differential and integral calculus could be used in solving problems in the contemporary business, technological and scientific world. Therefore, the course is structured to expose you to the skills required in other to attain a level of proficiency in Science, technology and Engineering Professions.

What you will learn in this Course

You will be taught the basis of mathematics required in solving scientific problems.

Course Aim

There are ten study units in the course and each unit has its objectives. You should read the objectives of each unit and bear them in mind as you go through the unit. In addition to the objectives of each unit, the overall aims of this course include:

(i) To introduce you to the words and concepts in Elementary mathematics

(ii) To familiarize you with the peculiar characteristics in Elementary mathematics.

(iii) To expose you to the need for and the demands of mathematics in the Science world.

(iv) To prepare you for the contemporary Science world.

Course Objectives

The objectives of this course are:

* To inculcate appropriate mathematical skills required in Science and Engineering.

* Educate learners on how to use mathematical Techniques in solving real life problems.

* Educate the learners on how to integrate mathematical models in Sciences and Engineering.

Working through this Course

{ You have to work through all the study units in the course. There are two modules and ten study units in all.

Course Materials

Major components of the course are:

1. Course Guide

2. Study Units

3. Textbooks

4. CDs

5. Assignments File

6. Presentation Schedule

Study Units

The breakdown of the three modules and eight study units are as follows:

MODULE ONE:

UNIT 1: FUNCTION AND GRAPHS

UNIT 2: LIMITS

UNIT 3: IDEA OF CONTINUITY

MODULE TWO:

UNIT 1: THE DERIVATIVE AS LIMIT OF RATE OF CHANGE

UNIT 2: DIFFERENTIATION TECHNIQUES

UNIT 3:: INTEGRATION

MODULE THREE:

UNIT 1: DEFINITE INTEGRALS

(Application to areas under curve and Volumes of solids)

UNIT 2: VOLUME OF SOLIDS OF REVOLUTION BY DEFINATE INTEGRAL

Recommended Texts

* { Seymour, L.S. (1964). Outline Series: Theory and Problems of Set Theory and related topics, pp. 1 – 133.

* Sunday, O.I. (1998). Introduction to Real Analysis (Real-valued functions of a real variable, Vol. 1)

* Pure Mathematics for Advanced Level By B.D Bunday H Mulholland1970.

* Introduction to Mathematical Economics By Edward T. Dowling.

* Mathematics and Quantitative Methods for Business and Economics. By Stephen P. Shao. 1976.

* Mathematics for Commerce & Economics By Qazi Zameeruddin & V.K. Khanne 1995.

* Engineering Mathematics By K. A Stroad.

* Business Mathematics and Information Technology. ACCA STUDY MANUAL By. Foulks Lynch.

* Introduction to Mathematical Economics SCHAUM�shS Out lines

Assignment File

{ In this file, you will find all the details of the work you must submit to your tutor for marking. The marks you obtain from these assignments will count towards the final mark you obtain for this course. Further information on assignments will be found in the Assignment File itself and later in this *Course Guide* in the section on assessment.

Presentation Schedule

The Presentation Schedule included in your course materials gives you the important dates for the completion of tutor-marked assignments and attending tutorials. Remember, you are required to submit all your assignments by the due date. You should guard against falling behind in your work.

Assessment

Your assessment will be based on tutor-marked assignments (TMAs) and a final examination which you will write at the end of the course.

}

Exercises TMAS

{ Every unit contains at least one or two assignments. You are advised to work through all the assignments and submit them for assessment. Your tutor will assess the assignments and select four which will constitute the 30% of your final grade. The tutor-marked assignments may be presented to you in a separate file. Just know that for every unit there are some tutor-marked assignments for you. It is important you do them and submit for assessment. }

Final Examination and Grading

{ At the end of the course, you will write a final examination which will constitute 70% of your final grade. In the examination which shall last for two hours, you will be requested to answer three questions out of at least five questions.

Course marking Scheme

This table shows how the actual course marking as it is broken down.

Assessment	Marks
Assignments	Four assignments, Best three marks of the four count at 30% of course marks
Final Examination	70% of overall course marks
Total	100% of course marks

How to Get the Most from This Course

In distance learning, the study units replace the university lecture. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace, and at a time and place that suits you best. Think of it as reading the lecture instead of listening to the lecturer. In the same way a lecturer might give you some reading to do, the study units tell you when to read, and which are your text materials or set books. You are provided exercises to do at appropriate points, just as a lecturer might give you an in-class exercise. Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit, and how a particular unit is integrated with the other units and the course as a whole. Next to this is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit. These learning objectives are meant to guide your study. The moment a unit is finished, you must go back and check whether you have achieved the objectives. If this is made a habit, then you will significantly improve your chances of passing the course. The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a Reading section. The following is a practical strategy for working through the course. If you run into any trouble, telephone your tutor. Remember that your tutor's job is to help you. When you need assistance, do not hesitate to call and ask your tutor to provide it.

In addition do the following:

1. Read this Course Guide thoroughly, it is your first assignment.

2. Organise a Study Schedule. Design a Course Overview " to guide you through the Course". Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the Semester is available from the study centre. You need to gather all the information into one place, such as your diary or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates and schedule of work for each unit.

3. Once you have created your own study schedule, do everything to stay faithful to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please, let your tutor know before it is too late for help.

4. Turn to Unit 1, and read the introduction and the objectives for the unit.

5. Assemble the study materials. You will need your set books and the unit you are studying at any point in time.

6. Work through the unit. As you work through the unit, you will know what sources to consult for further information.

7. Keep in touch with your study centre. Up-to-date course information will be continuously available there.

8. Well before the relevant due dates (about 4 weeks before due dates), keep in mind that you will learn a lot by doing the assignment carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the examination. Submit all assignments not later than the due date.

9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study materials or consult your tutor.

10. When you are confident that you have achieved a unit's objectives, you can start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.

11. When you have submitted an assignment to your tutor for marking, do not wait for its return before starting on the next unit. Keep to your schedule. When the Assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also the written comments on the ordinary assignments.

12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in the Course Guide).

Tutors and Tutorials

The dates, times and locations of these tutorials will be made available to you, together with the name, telephone number and the address of your tutor. Each assignment will be marked by your tutor. Pay close attention to the comments

your tutor might make on your assignments as these will help in your progress. Make sure that assignments reach your tutor on or before the due date.

Your tutorials are important therefore try not to skip any. It is an opportunity to meet your tutor and your fellow students. It is also an opportunity to get the help of your tutor and discuss any difficulties encountered on your reading. **Summary**

This course would train you on the concept of multimedia, production and utilization of it.

Wish you the best of luck as you read through this course

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MODULE ONE:

UNIT 1: FUNCTION AND GRAPHS

Content:

1.1 Introduction.

In everyday life, many quantities depend on one or more changing variable. For example:

a) Speed of a moving car or object depend on distance travelled and time taken

b) The voltage of electrical devices depends on current and resistance.

c) The volume of given mass of gas depends on the pressure at room temperature (I.e. temperature remain constant)

A function is a phenomenon that relates how one variable or quantity depends on the other variables or quantities.

For instance, in ohm's law $V \propto I$, mathematically $V=IR$ where R is constant of proportionality.

If I increase, so does the voltage V. If I decrease, so does the voltage. Hence, from this, we can say voltage is a function of current.

1.2 Objectives

In this module, you will cover the following topics:

Function(Definition of function)

Function Notation

Graphs of Function

Combinations of Functions

Inverse Function.

1.3 Function.

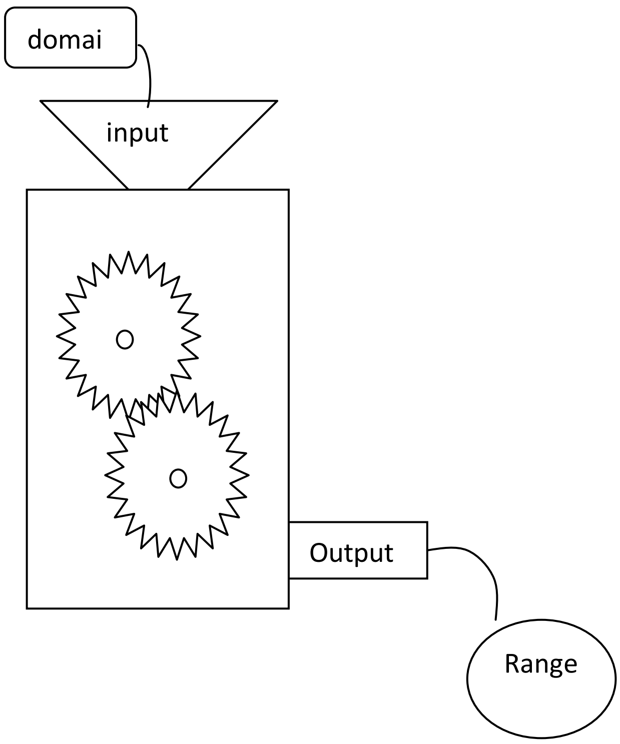
Definition of a function

A **function** is a relationship between two variables such that, to each value of the independent variable there is exactly one corresponding value of the dependent variable.

The **domain** of the function is the set of all values of the independent variable for which the function is defined. The **range** of the function is the set of all values taken on by the dependent variable.

OR

A **Function** is a correspondence from a first set, called the **domain**, to a second set, called the **range**, such that each element in the domain corresponds to exactly one element in the range.



In fig.1, notice that you can think of a function as a machine that inputs values of the independent variable and inputs values of the dependent variable.

Although function can be described by various means such as table, graphs and diagrams, they are most often specified by formulas or equation.

For instance, the equation $y = 4x^2 + 3$, describes y as a function of x. For this function, x is the independent variable and y is the dependent variable.

Example 1

Deciding whether relation are functions.

Which of the equation below define y as a function of x?

a) $x + y = 1$ b) $x^2 + y^2 = 1$

c) $x^2 + y = 1$ d) $x + y^2 = 1$

Solution

To decide whether an equation defines a function, it is helpful to isolate the dependent variable on the left side.

For instance, to decide whether the equation $x + y = 1$ defines y as a function of x, write the equation in the form.

$$y = v - x$$

From this form, you can see that for any value of x, there is exactly one value of y. So, y is a function of x.

Original EquationRewritten EquationTest: Is y a function of x ?

a. $x + y = 1$ $y = 1 - x$ Yes, each value of determines exactly one value of y

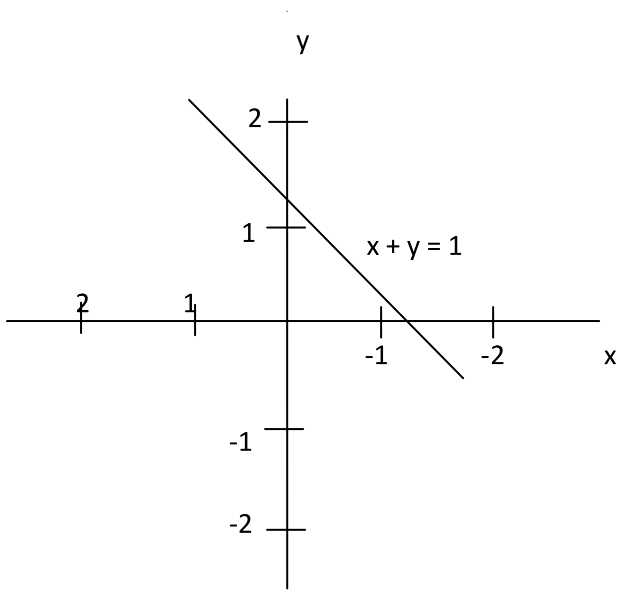
b. $x^2 + y^2 = 1$ $y = \pm\sqrt{1-x^2}$ No, some values of x determine two values of y.

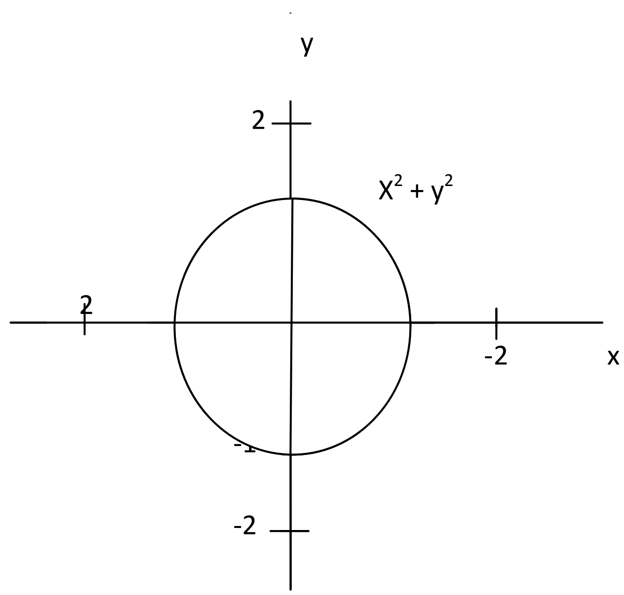
c. $x^2 + y = 1 y = 1 - x^2$ Yes, each value of x determine exactly one value of y.

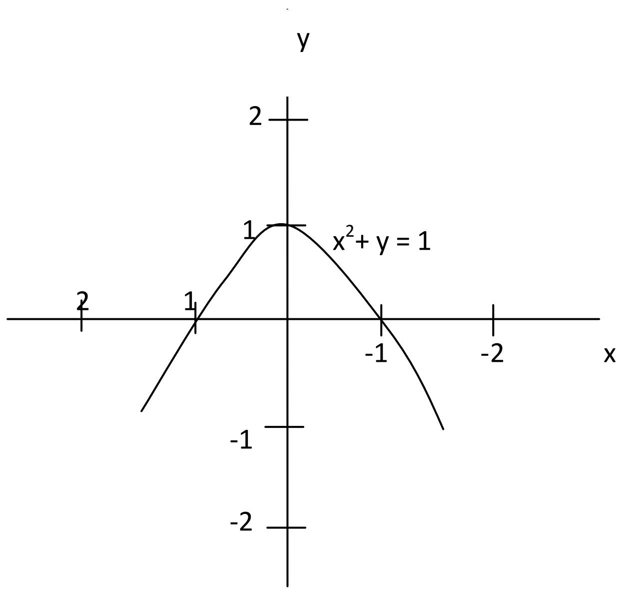
d. $x + y^2 = 1$ $y = \pm\sqrt{1-x}$ No, some value of x determine two values of y.

Note that the equations that assign two values ($\pm$) to the dependent variable for a given value of the independent variable do not define functions of x. For instance, in part (b), when x=0, the equation $y = \pm\sqrt{1-x^2}$indicates that $y = +1$ or $y = -1$.

Figure 1.12 shows the graphs of the four equations.







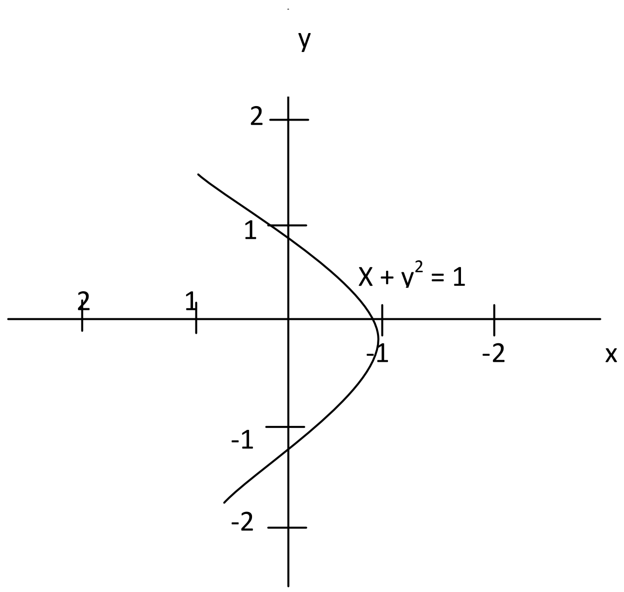


Fig 1.12

Checkpoint 1:

Which of the equations below define y as a function of x?

a. $x - y = 1$ b. $x^2 + y^2 = 4$ c. $y^2 + x = 2$ d. $x^2 - y = 0$ e. $2x + y = 6$ f. $x^2 + y^2 = 1$

Example 2: Determine whether an Equation Represents a Function.

Determine whether each equation defines as a function as a function of x:

a. $x^2 + y$ b. $x^2 + y^2 = 4$

Solution

Solve each equation for y in terms of x. If two or more values of y can be obtained for a given x, the equation is not a function.

a. $x^2 + y = 4$ **This is the given equation.**

$x^2 + y - x^2 = 4 - x^2$ **Solve for y by subtracting x^2 from both sides.**

$y = 4 - x^2$ **Simplify.**

From this last equation we can see that for each value of x, there is one and only one value of y. For example, if x=1, then $y = 4 - 1^2 = 3$. The equation defines y as a function of x.

b. $x^2 + y^2 = 4$ **This is the given equation.**

$x^2 + y^2 - x^2 = 4 - x^2$ **Isolate y^2 by subtracting x^2 from both sides.**

$y^2 = 4 - x^2$ **Simplify.**

$y = \pm\sqrt{4 - x^2}$ **Apply the square root properly: if u^2=d then $\boldsymbol{u} = \pm\sqrt{\boldsymbol{d}}$**

Then $\pm$ in this last equation shows that for certain values of x (all values between -2 and 2), there are two values of y. For example, if x=1, then $y = \pm\sqrt{4 - 1^2}$ = $\pm\sqrt{3}$. For this reason, the equation does not define y as a function of x.

1.4 Function Notation.

When using an equation to define a function, you generally isolate the dependent variable on the left. For instance, writing the equation $x + 3y = 1$ as

$$y = \frac{1-x}{3}$$

indicates that y is the dependent variable. In function notation, this equation has the form

$$f(x) = \frac{1-x}{3}$$

The independent variable (input) is x and the name of the function is ''f''. The symbol f(x) is read as ''f of x'' or ''f at x'' and it denotes the value of the independent variable (output) or **the value of the function at the number x.**

For instance, the value of f when x=3 is

$$f(3) = \frac{1-(3)}{3} = \frac{-2}{3} = \frac{-2}{3}$$

The value f(3) is called a **function value** and lies in the range of f. This means that the point [3, f(3)] lies on the graph of f. One of the advantages of function notation is that it allows you to be less wordy. For instance, instead of asking ''what is the value of y when x=3?'' you can ask ''what is f(3)?

N.B:

Study Tip: the notation f(x) does not mean ''f times x''. The notation describes the value of the function at x.

Example 3: Evaluating a function

Find the value of the function

$f(x) = 2x^2 - 4x + 1$

When x= −1, 0, and 2. Is f one to one?

<u>Solution</u>

When x= −1, the value of is

$f(x) = 2x^2 - 4x + 1$ **This is the given function.**

$f(-1) = 2(-1)^2 - 4(-1) + 1$ **Replace each occurrence of x with −1.**

$= 2(1) - 4(-1) + 1$ **Evaluate the exponential expression.** $(-1)^2 = 1$

$= 2 + 4 + 1$ **Perform the multiplications.**

$= 7 \therefore f(-1) = 7$

When x =0, the value of f is

$f(0) = 2(0)^2 - 4(0) + 1$

$f(0) = 0 - 0 + 1 = 1$

When x=2, the value of f is

$f(2) = 2(2)^2 - 4(2) + 1$

$= 2(4) - 4(2) + 1$

$= 8 - 8 + 1 = 1$

Diagram

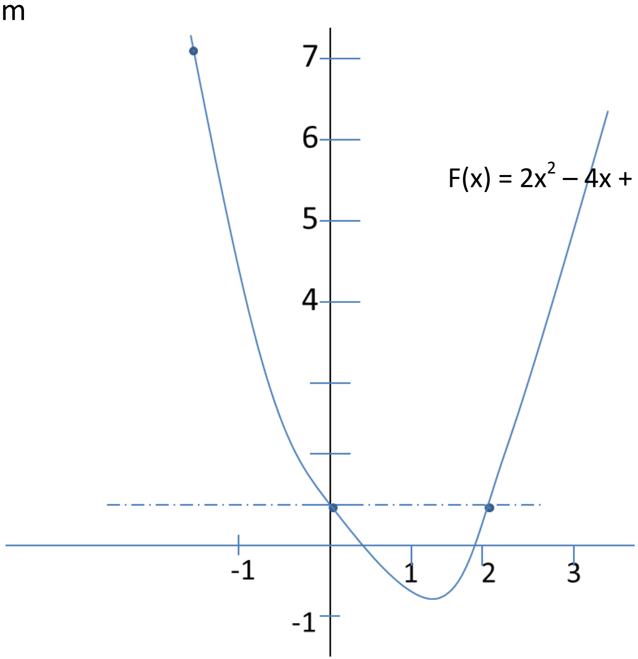


Fig 1.21

Because two different values of x yield the same value of f(x), the function is not one to one, as shown in fig 1.21.

Study tip:

You can use the horizontal line to test or determine whether the function in Example 3 is one to one. Because the line $y = 1$ intersects the graph of the function twice, the function is not one to one.

Checkpoint

Find the value of $f(x) = x^2 - 5x + 1$ when x is 0, 1 and 4?

Answer: $f(0) = 1$, $f(1) = -3$, $f(4) = -3$.

Example 4

If $f(x) = x^2 + 5x + 5$, evaluate each of the following

a. $f(x+2)$ b. $f(-x)$

Solution

a.

$f(x+2) = (x+2)^2 + 5(x+2) + 5$ **We find f(x+2) by substituting x +2 for x in the equation.**

$f(x+2) = (x+2)^2 + 5(x+2) + 5$ **Square x+2 using $(A+B)^2 = A^2 + 2AB + B^2$.**

$= x^2 + 4x + 4 + 5x + 10 + 5$ **Distribute 5 throughout the Parentheses.**

$= x^2 + 4x + 5x + 4 + 10 + 5$ **Combine like terms**

$= x^2 + 9x + 19$

b.

We find $f(-x)$ by substituting $-x$ for x in the equation $f(-x) = (-x)^2 + 5(-x) + 5$

$= x^2 - 5x + 5$

Example 5

Let $f(x) = x^2 - 4x + 7$, find

a. $f(x+\Delta x)$ b. $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Solution

a. To evaluate $f(x+\Delta x)$, substitute $x+\Delta x$ as x as shown in each set of parentheses as follows.

$$f(x+\Delta x) = (x+\Delta x)^2 - 4(x+\Delta x) + 7$$
$$= x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7$$

b. Using the result of part (a), you can write the following.

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \frac{[(x+\Delta x)^2 - 4(x+\Delta x) + 7 - (x^2 - 4x + 7)]}{\Delta x}$$

$$= \frac{[x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 7 - x^2 + 4x - 7]}{\Delta x}$$

$$= \frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x}$$

$$= 2x + \Delta x - 4 \quad \text{as } \Delta x \neq 0$$

Checkpoint:

1. If $f(x) = x^2 - 2x + 3$, and find
   a. $f(x + \Delta x)$ b. $\frac{f(x + \Delta x) - f(x)}{f(x)}$
2. If $f(x) = x^2 - 2x + 7$, evaluate each of the following

a. $f(-5)$ b. $f(x + 4)$ c. $f(-x)$

1.5 Graphs of Functions

The graph of a function is the graph of its ordered pairs. For example, the graph of f(x) = 3x is the set of points (x, y) in the rectangular coordinates system satisfying y = 3x.

When the graph of a function is sketched, the standard convention is to let the horizontal axis represents the independent variable. When this convention is used, the test described in Example 1 has **vertical line test**.

This test states that if every vertical line intersects the graph of an equation at most once, then the equation defines y as a function of x. For instance, in figure 1.12, the graph in parts (a) and (c) pass the vertical line test but those in parts (b) and (d) do not

The domain of a function may be described explicitly or it may be implied by an equation used to define the function. For example, the function given by $y = 1/(x^2 - 4)$ has an implied domain that consists of real x except $x = \pm 2$. These two values $(x^2 - 4)$ excluded from the domain because division by zero is **undefined.**

Another type of implied domain is that used to avoid even roots of negative number, as indicated in example below.

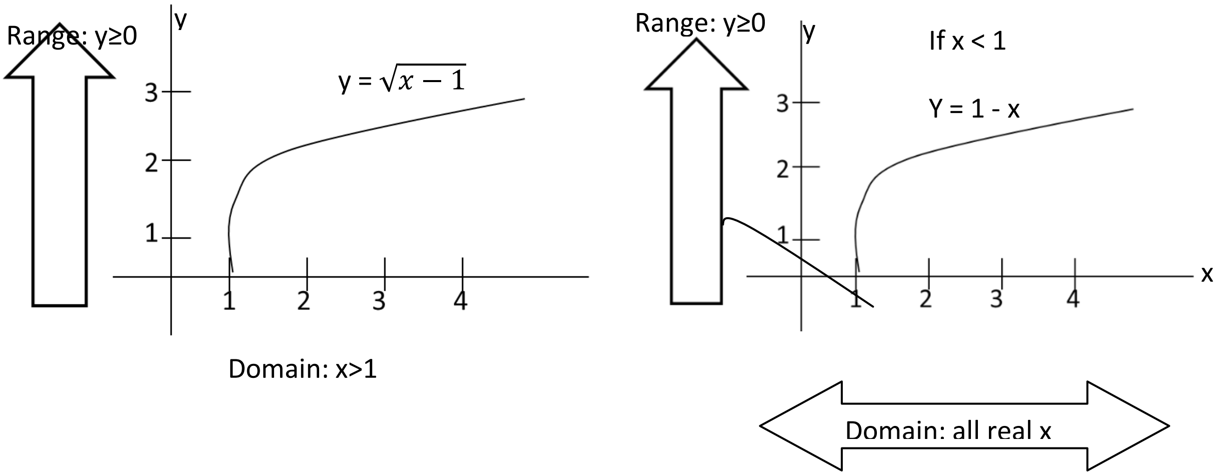
Example 6:Finding the Domain and Range of a function

Find the domain and range of each function

a. $y = \sqrt{x-1}$ b. $y = \begin{Bmatrix} 1-x, & x < 1 \\ \sqrt{x-1}, & x \geq 1 \end{Bmatrix}$

Solution

a. Because $\sqrt{x-1}$ is not defined for $X - 1 < 0$ (that is for $x < 1$), it follows that the domain of the function is the interval $x \geq 1$ or $[1, \infty]$. To find the range, observe $\sqrt{x-1}$ is never negative. Moreover, as x takes on the various values in the domain, y takes on all non negative values. So, the range is the interval $y \geq 0$or $[0, \infty]$. The graph of the function, shown in figure 1.22 (a) confirms these results.



Because this function is defined for x $<$ and for x $\geq$ I, the domain is the entire set of real numbers. This function is called **piecewise – definedfunction** because it is defined by two or more equations over a specified domain.

When x $\geq$ 1, the function behaves as in parts (a). For x <I, the values of I$-$x is positive, and therefore the range of the function is y$\geq$ 0 or [0, ∞], as shown in figure 1.22 b.

A function is **one to one** if to each value of the dependent variable in the range there corresponds exactly one value of the independent variable. For instance, the function in Example 6 a is one to one, where as the function in example 6b is not one- to -one

Geometrically, a function is one- to- one if every horizontal line intersects the graph of the function at most once. This geometrical interpretation is the **horizontal line test** for one to one functions. So, a graph that represents a one- to –one test must satisfy both the vertical line test and the horizontal line test.

Study Tips:

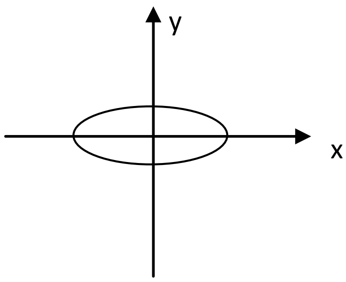
The vertical line test for function:

If any vertical line intersects a graph in more than one point, the graph does not define y as a function of x.

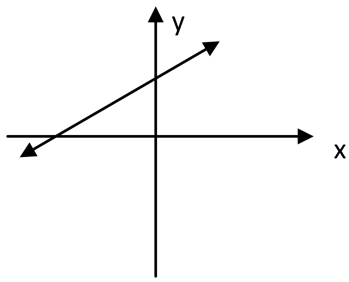
Example: Using the vertical line test

Use the vertical line test to identify graphs in which y is a function of x

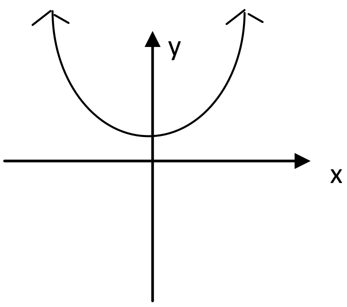
a.



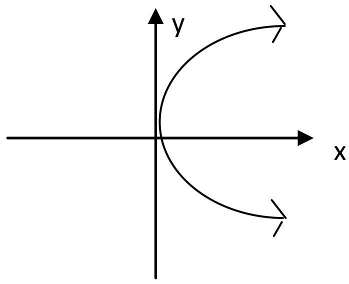
B.



c.

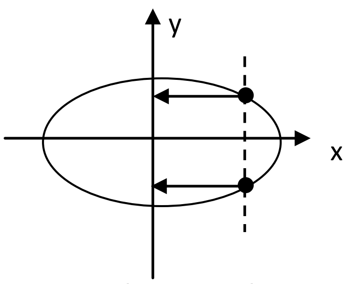


d.



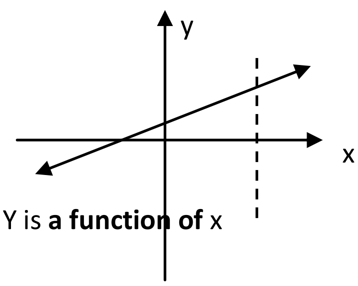
Solution

a.



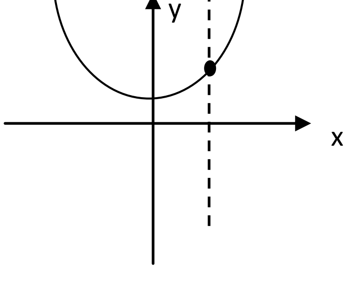
y is **not a function of** x. Two values of y corresponds to an x - value

B.

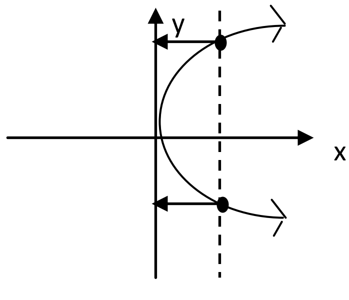


Y is **a function of** x

C.



d.



Y is **a function of** x

y is **not a function of** x. two values of y correspond to an x-value.

Example7

Find the value of $f(x) = x^2 - 5x + 1$ when x is 0, 1, and 4. Is f one to one?

Solution

When x = 0, the value of f is

$f(0) = 0^2 - 5(0) + 1 = 1$

When x = 1, the value of f is

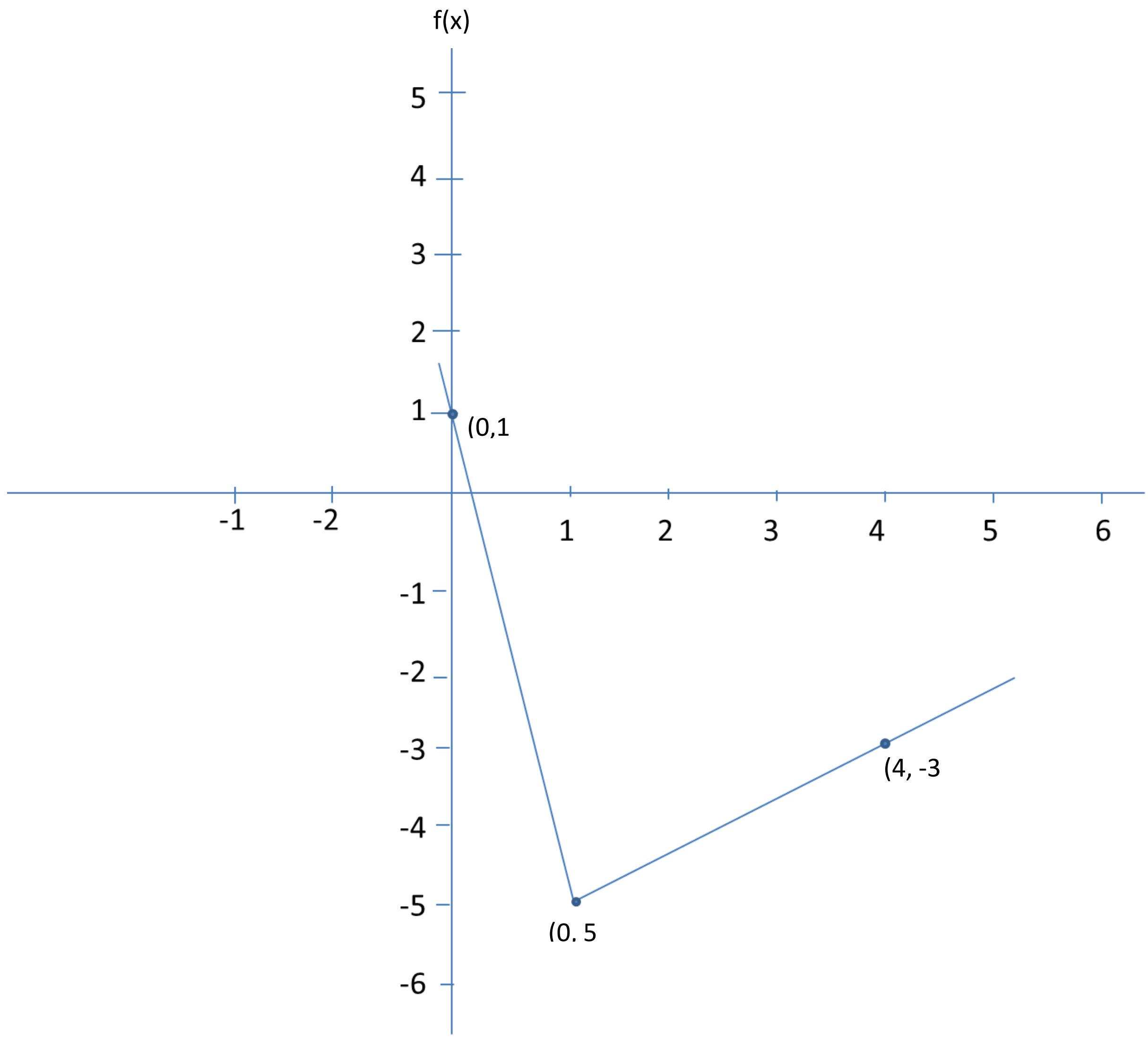
$f(1) = 1^2 - 5(1) + 1$

$= 1 - 5 + 1 = -5$

When x = 4, the value of f(x) is

$= 4^2 - 5(4) + 1$

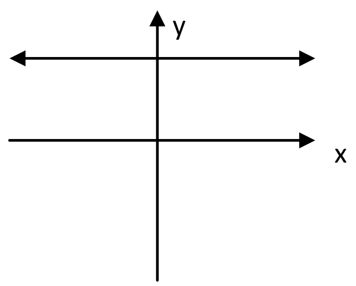
$=16 - 20 + 1 = -3$



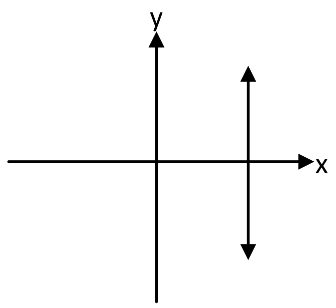
Checkpoint

In the following exercise, use the vertical line test to identify graphs in which y is a function of x.

i.



ii.



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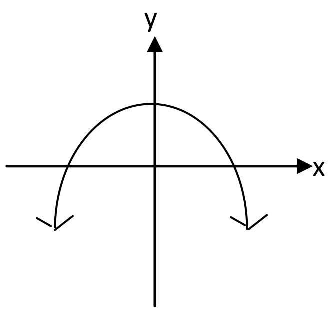
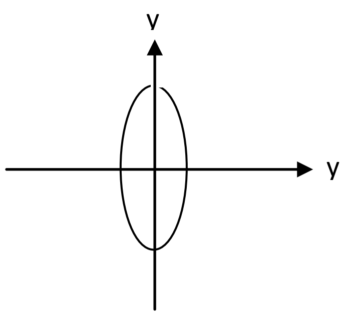
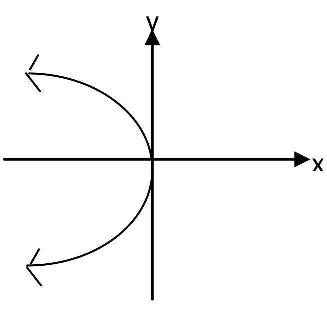


iv.



v.



1.4 combinations of functions (composite functions)

Definition

The function given by (f.g) (x) = f (g(x)) is the composite of f with g. The domain of (f.g) is the set of all x in the domain of g such that g(x) is the domain of f.

Two functions can be combined in various ways to create new functions.

For instance, if f (x) = 2x – 3 and g(x) = x^2 + 1, you can form the following functions.

f(x) + g(x) = (2x – 3) + (x^2 + 1) = x^2 + 2x – 2 **Sum**.

f(x) – g(x) = (2x - 3) – (x^2 + 1) = - x^2 + 2x – 4 **Difference.**

f(x) g(x) = (2x – 3) (x^2 + 1) = $2x^3$ – $3x^2$ + 2x – 3**Product.**

f(x)/g(x) = $\frac{2x-3}{x^2}$**Quotient/ division**

Example

Forming composite functions

Let f(x) = 2x – 4 and g(x) =x^2+ 3, and find

a. f (g(x)) b. g(f(x))

Solution

a. The composite of f with g is given by

g (f(x)) = 2(g(x)) – 4 **Evaluate f at g(x).**

= 2(x^2 + 3) – 4 **Substitute x^2+ 3 for g(x).**

= $2x^2$ + 6 – 4 **Simplify**.

= $2x^2$ – 2

= 2(x^2 - 1)

b. The composite of g with f is given by

g (f(x)) = $(f(x))^2$ + 1**Evaluate g at f(x).**

= $(2x – 4)^2$ + 1**Substitute 2x – 4 for f(x).**

= $4x^2$ − 16x + 16 + 1**Simplify.**

= $4x^2$ − 16x + 17

Checkpoint

Let f(x) = 2x + 1 and g(x) = x^2+ 2, and find

a) f (g(x)). b) g (f(x)).

1.7 INVERSE FUNCTION

Definition of the inverse of a function.

Let f and g be two functions such that:

f (g(x)) = x for each x in the domain of g

and g(f(x)) = x for each x in the domain of f

Under these conditions, the function g is the **inverse** of the function f. The function is donated byf^{-1}, which is read as “f- inverse”. So,

$$f\left(f^{-1}(x)\right) = x \text{and} f^{-1}\left(f(x)\right) = x$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain off^{-1}.

Example 8

Finding inverse functions

Several functions and their inverse are shown below. In each case, note that the inverse function “undoes” the original function. For instance, to undo multiplication by 2, you should divide by 2.

a) f(x) = 2x $f^{-1}(x) = \frac{1x}{2}$

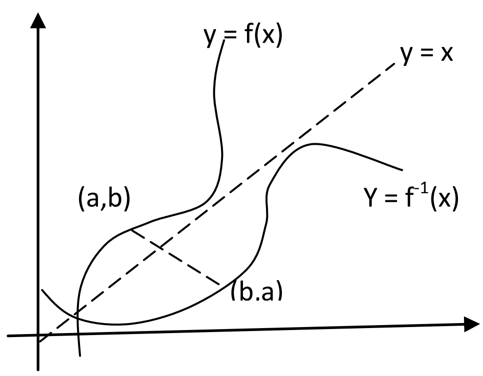
b) f(x) = 1/5x f^{-1} (x) = 5x

c) f(x) = x + 8 f^{-1} (x) = x – 8

d) f(x) = 3x + 7 f^{-1} (x) = 1/3 (x + 7)

e) f(x) = x^3 f^{-1} (x) = $\sqrt[3]{x}$

f) f(x) = 1/x f^{-1}(x) = $\frac{1}{x}$



The graphs of f and f^{-1} are mirror images of each other (with respect to the line y = x), as in figure 1.23

Checkpoint

Informally find the inverse function of each function

a. f(x) = 1/5x b. f (x) = 3x + 2

Example 9: Finding the inverse function

Find the inverse function of f(x) = $\sqrt{2x-3}$.

Solution

Begin by substituting f(x) with y. Then interchange x and y and solve for y.

$f(x) = \sqrt{2x-3}$ Write original function.

$y = \sqrt{2x-3}$ Replace f(x) with y.

$x = \sqrt{2y-3}$ Interchange x and y.

$x^2 = 2y - 3$ Squaring each side.

$x^2 + 3 = 2y$ Add 3 to each side.

$\frac{x2+3}{2}$ =y Divide each side by 2.

So, the inverse function has the form

$$f^{-1}(\_\_) = \frac{(\_)^2 + 3}{2}$$

Using x as the independent variable, you can write $f^{-1}(x) = \frac{x^2+3}{2}$ as x ≥ 0.

In the figure 1.24, note that the domain of f^{-1} coincides with the range of f.

Diagram

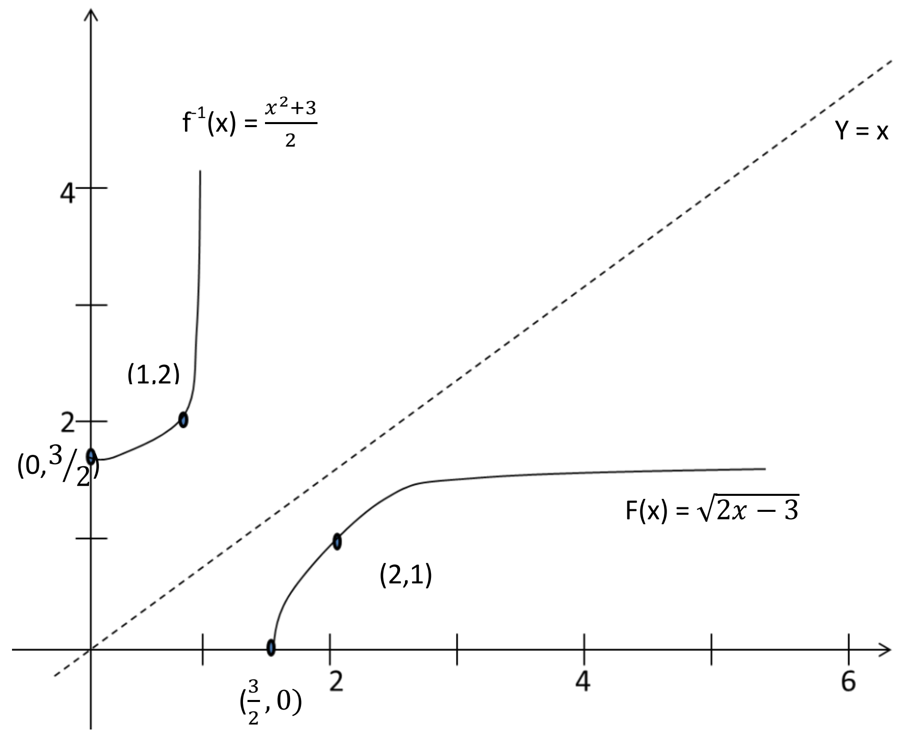


Figure 1.24

After you have found the inverse of a function, you should check your result. You can check your results graphically by observing that graphs of f and $f^{-1}$are reflections of each other in the line y =x. You can check your results algebraically by evaluating$f(f^{-1}(x))$and $f^{-1}(f(x))$ – both should be equal to x.

Check that $f(f^{-1}(x)) = x$

Check that $f^{-1}\big(f(x)\big) = x$

$f(f^{-1}(x)) = f(\frac{x^2+3}{2})$

$f^{-1}(f(x)) = f^{-1}(\sqrt{2x-3}\,)$

$$f^{-1}\big(f(x)\big) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \frac{(\sqrt{2x-3})^2 + 3}{2}$$

$$= \sqrt{x^2} = \frac{2x}{2}$$

$= x, \quad x \geq 0.$

=x, $x \geq \frac{3}{2}$

Checkpoint

Find the inverse of function of $f(x) = x^2 + 2 \; for \; x \geq 0.$

NB: Not every function has an inverse function. In the fact, for a function to have an inverse function, it must be one- to- one.

Example 10

A function that has no inverse function.

Show that the function f(x) = $x^2 - 1$ has no inverse function. (Assume that the domain of f is the set of all real numbers)

Solution

Begin by sketching the graph of f, as shown in figure 1.25 Note that

$f(x) = x^2 - 1$

$f(2) = 2^2 - 1 = 3$

And

$f(-2) = (-2)^2 - 1 = 3$

So, f does not pass the horizontal line test, which implies that if is not one- to- one and therefore has no inverse function. The same conclusion can be obtained by trying to find the inverse of f algebraically.

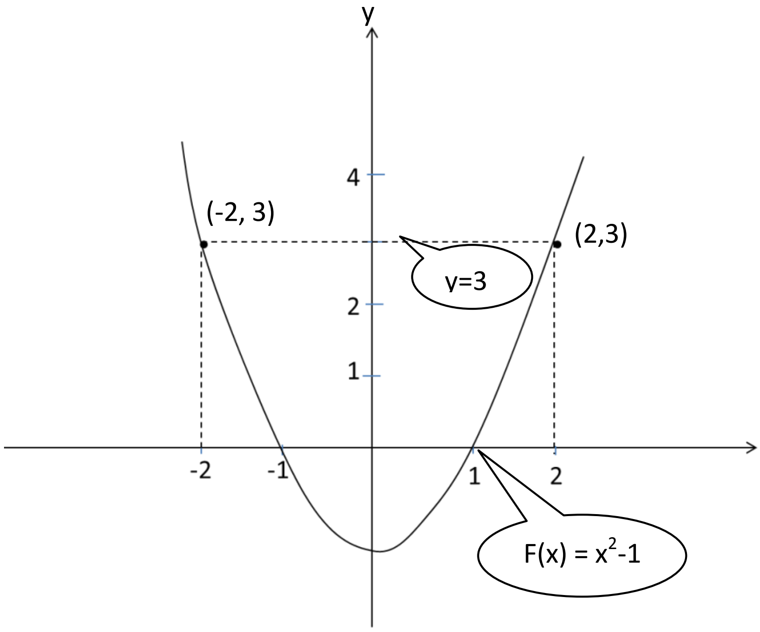
$f(x) = x^2 - 1$	**Write original function.**
$y = x^2 - 1$	**Replace f(x) with y**
$x = y^2 - 1$	**Interchange x and y**
$x + 1 = y^2$	**Add 1 to each side**
$\pm\sqrt{x+1} = y$	**Take square root of each side**

The last equation does not define y as a function of x, and so f has no inverse function

y

$F(x) = x^2 - 1$	write original function.
$Y = x^2 - 1$	Replace f(x) with y
$X = y^2 - 1$	Interchange x and y
$X + 1 = y^2$	Add1 to each side
$+\sqrt{x + 1 = y}$	Take square root of each side

The last equation does not define y as a function of x, and so f has no inverse function



Checkpoint:

Show that the function $f(x) = x^2 + 4$ has no inverse function.

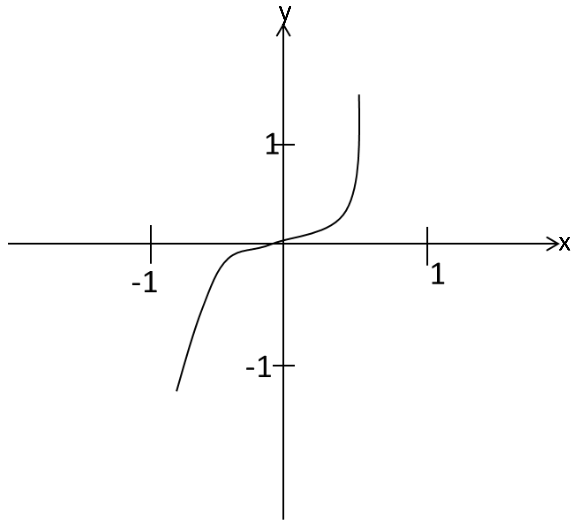
1.6 Exercises

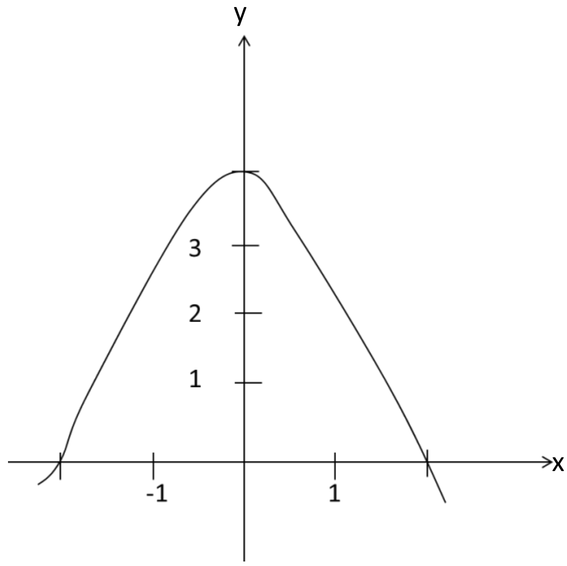
A. In the following exercises 1 - 4, decide whether the equation define y as a function of x.

1. $x^2 + y^2 = 4$ 2. $\frac{1}{2}x - 6y = -3$ 3. $x^2 + y = 4$ 4. $y^2 = x^2 - 1$

B. In the following exercises 5 – 6, find the domain and range of the function. Use interval notation to write your result.

5. $f(x) = x^3$ 6. $f(x) = 4 - x^2$





C. In exercises 7 – 8 evaluate the function at the specified the values of the independent variable. Simplify the result.

7. f(x) = 2x – 3

(a) f(0) (b). f(- 3) (c). f (x – 1) (d). f $(1+\Delta x)$

8. g(x) = $\frac{1}{x}$

(a) g(2) (b). g(¼) (c). g(x $+\Delta x)$ (d). g(x $+\Delta x) - g(x)$

D. In exercises 9 – 10, evaluate the difference quotient and simplify the result

9. f(x) = $x^3 - x$ 10. g(x) $= \sqrt{x+3}$

$$\frac{f(x-\Delta x) - f(x)}{\Delta x} \quad \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

11. f(x) $= \frac{1}{x-2}$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

E.Find:

(a) f(x) + g(x) (b) f(x) .g(x) (c) $f(x)/g\,(x)$ (d) f(g(x)) and (e) g(f(x)) if defined.

12. f(x) $= 2x - 5$,

g(x) $= 5$

13. f(x) $= x^2 + 1,$

g(x)$= x - 1$

F.

14. Given that f(x) = $\sqrt{x}$ and g (x) = x^2 - 1, find the composite functions:

a. f(g(1)) b. g(f(1)) c. g(f(0)) d. f(g(-4)) e. f(g(x)) f. g(f(x))

G. In exercises 15 – 16, show that f and g are inverse functions by showing that f(g(x)) = x and g (f(x)) = x. Then sketch the graphs of f and g on the same coordinate axes .

15. f(x) $= 5x + 1$ g(x) $= \frac{x-1}{5}$

16. f(x)$= 9 - x^2$, $x \geq 0$ g(x) $= \sqrt{9 - x}$, $x \leq 9$

17. Find the inverse function of f. Then sketch the graph of f and f^{-1} on the same coordinate axis.

f(x) $= \sqrt{9 - x^2}$, $0 \leq x \leq 3$.

H. In the exercises 18 and 19, use the vertical line test to determine whether y is a function of x.

18. $x^2 + y^2 = 9$ 19. $x^2 = xy - 1$

1.8 Conclusion and Summary

I. A function is a correspondence from a first set, called the domain to a second set, called the range such that each element in the domain corresponds to exactly one element in the range.

II. A function is one -to -one if each value of the dependent variable in the range there corresponds exactly one value of the independent variable

1.9 References

Calculus an Applied Approach Larson Edwards Sixth Edition

Blitzer Algebra and Trigonometry Custom 4th Edition

Engineering Mathematics by K.A Stroud.

MODULE 2: LIMITS.

Contents:

2.1 Introduction:

In everyday language, one always refers to limit one's endurance, speed limit of a car, a wrestler's weight limit or stretching a spring to its limit. These phrases all suggest that a limit is a bound, which on some instances may not be reached but on other instance may be reached or exceeded.

Hooke's law is a perfect illustration of limit which states provided an elastic limit of a spring is not exceeded; the extension (e) is directly proportional to the tension or force acting on it. That is a spring has a limit of extension when a load is suspended on it. If it exceeds the boundary, it will reach a point of plasticity or break without returning to its initial position.

Limit is a concept/Fundamental to Calculus.

2.2 Objectives:

In this Module, you will cover the following topics:

2.3 Limits (Definition)
2.4 Properties of Limits
2.5 The Limit of a Polynomial Function
2.6 Techniques for Evaluating Limits

2.1 Definition of Limit of a Function:

If $f(x)$ becomes arbitrarily close to a single number **L** as $\boldsymbol{x}$ approaches **c** from either side, $\lim_{x\to c} f(x) = L$, then $\mathrm{Lim}\, f(x)$ which is read as **“the limit of $\boldsymbol{f(x)}$ as $\boldsymbol{x}$ approaches c is L”**

2.2 Properties of Limit:

Many times, the limit of $f(x)$ as x approaches c is simply $f(c)$ .Whenever the limit of as $f(x)$ approaches c is $\lim_{x\to c} f(x) = f(c)$.The limit can be evaluated by direct substitution.

Properties of Limits:

Let b and c be real numbers, and let n be a positive integer.

i. $\lim_{x \to c} b$ = b
ii. $\lim_{x \to c} x$ = c
iii. $\lim_{x \to c} x^n = x^c$
iv. $\lim_{x \to c} x^n = c^n$
In the property IV, if n is even, then c must be positive.

By combining the properties of limits with the rules for operating with limits shown below, you can find limits for a wide variety of algebraic function.

Operations with limits:

Let b and c be real numbers let n be a positive integer, and let f and g be functions with the following limits.

$\lim_{x \to c} f(x)$ = L and $\lim_{x \to c} g(x)$ = k

I. Scalar multiple: $\lim_{x \to c}[b\, f(x)]$ = b L .
II. Sum or Difference: $\lim_{x \to c}[f(x) \pm g(x)]$= L $\pm$ k.
III. Product: $\lim_{x \to c}[f(x)\, .\, g(x)]$ = L . k.
IV. Quotient: $\lim_{x \to c} f(x)/g(x)$ = L/k provided that k $\neq$ 0.
V. Power: $\lim_{x \to c}[f(x)]^n = L^n$
VI. Radical: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
In property VI, if n is even, then L must be positive.

Example 1: Find the limit:

$\lim_{x \to 1}(x^2+1)$

Solution.

Using direct substitution by substituting 1 for x

$\lim_{x \to 1}(x^2+1) = 1^2 + 1 = 2$

Example 2:

Find the limit: $\lim_{x \to 1} f(x)$.

a. $f(x) = \frac{x^2-1}{x-1}$ b. $f(x) = \frac{|x-1|}{|x-1|}$ c. $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$

Solution

a. $\lim_{x \to 1} f(x) = \frac{x^2-1}{x-1} = \frac{x^2-1^2}{x-1}$

$= \frac{(x+1)(x-1)}{(x-1)}$ **Factorizing the numerator by the difference of two square [$a^2 - b^2 = (a+b)(a-b)$].**

$\lim_{x \to 1}(x+1) = 1 + 1 = 2$ **Substituting 1 for x.**

Therefore, $\lim_{x \to 1} f(x) = \frac{x^2-1}{x-1} = 2$

b. $\lim_{x \to 1} \frac{|x-1|}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} = 0$ **Substituting 1 for x.**

Therefore, $\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

c. $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$ = 1

2.3 The limit of a polynomial function

If p is a polynomial function and c is any real number, then $\lim_{x \to c} p(x) = p(c)$

Example 3

Finding the limit of a polynomial function

Find the limit:$\lim_{x\to 2}(x^2 + \ 2x - 3)$

Solution:

$\lim_{x\to 2} x^2 + 2x - 3 = \ \lim_{x\to 2} x^2$+ $\lim_{x\to 2} 2x - \lim_{x\to 2} 3$**Applying property II.**

= $2^2 \ + 2(2) - 3$**Use direct substitution.**

4 + 4 – 3 = 5 **Simplify.**

= 5

Note: Example 3 show or state that the limit of polynomial can be evaluated by direct substitution.

Check point:

1. Find the limit: $\lim_{x\to 2} f(x)$

   a. f(x) = $\frac{x^2-4}{x-2}$ b. f(x) = $\frac{|x-2|}{x-2}$ c. $f(x) = \begin{cases} x^2, & x \neq 0 \\ 0, & x = 2 \end{cases}$

2. Find the limit : $\lim_{x\to 2} 2x^2 - x + 4$

2.4 **Techniques for Evaluating Limits.**

There are several techniques for calculating limits and these are based on the following important theorem. Basically, the theorem states that **"if two functions agree at all but a single point c, then they have identical limit behavior at x = c.**

- ❖ **The Replacement Theorem/Technique**

Let c be a real number and f(x) = g(x) for all x ≠ c. if the limit of g(x) exists as x→c, then the limit of f(x) also exists and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$$

To apply the Replacement Theorem, you can use a result from algebra which states that for a polynomial function p, p(c) = 0 if and only if (x-c) is a factor of p(x).

Example 4

Finding the limit of a function

Find the limit: $\lim_{x\to 1}\frac{x^3-1}{x-1}$

Solution

Note that the numerator and the denominator are zero when x=1

$\lim_{x\to 1}\frac{x^3-1}{x-1}$ .For the numerator $\lim_{x\to 1} x^3 - 1 = 1^3 - 1=0$ &the denominator $\lim_{x\to 1} x - 1=1-1=0$.

This implies or means that x - 1 is a factor of both and you can divide out this like factor using division of polynomial.

$$\frac{x^3-1}{x-1} = (x^2 + x + 1)(x - 1)$$

$\therefore \lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$

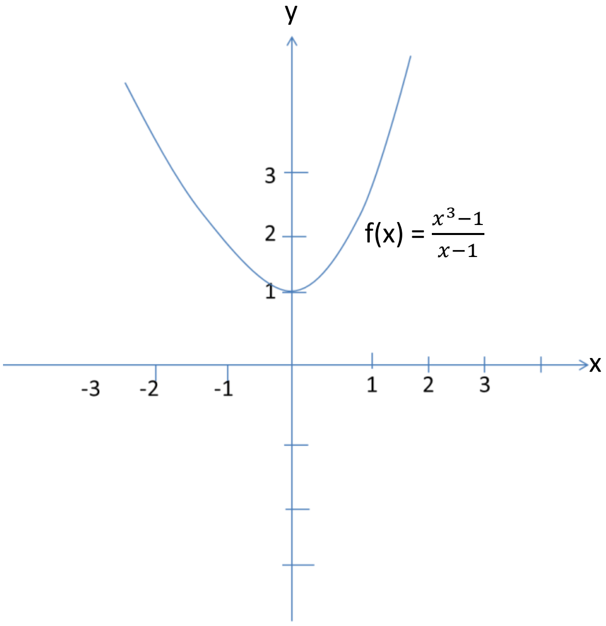
$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$ **factor numerator.**

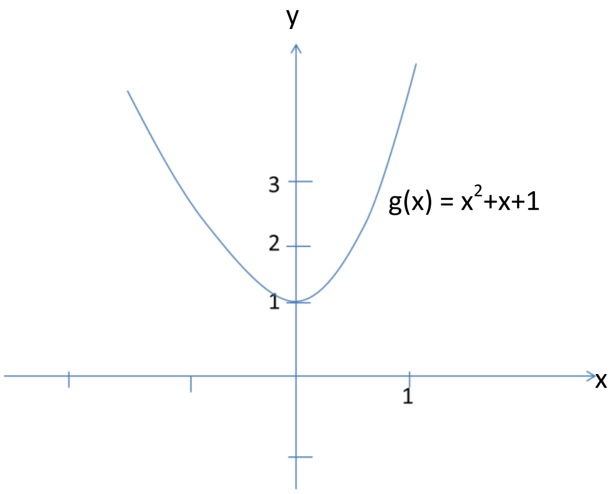
$= \frac{(x-1)(x^2+x+1)}{x-1}$ **Divide out factor.**

$= x^2 + x + 1$, $x \neq 1$ **Simplify.**

So, the rational function $(x^3 - 1)/(x - 1)$ and the polynomial function $x^2 + x + 1$ agree for all value of x other than x = 1, and you can apply the Replacement theorem.

$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2 + x + 1) = 1^2 + 1 + 1 = 3$





In fig2.1 illustrates this result graphically. Note that the two graphs are identical except that the graph of g contains the point (1, 3), whereas this point is missing on the graph of f. (in fig 2.1, the missing point is denoted by an open dot.)

Checkpoint.

Find the limit: $\lim_{x\to 2} \frac{x^3-8}{x-2}$

Dividing Out Technique.

Example 5

Find the limit: $\lim_{x\to -3} \frac{x^2+x+6}{x+3}$

Solution.

Using direct substitution will fails because both the numerator and thedenominator are zero when x = -3.

$$\lim_{x\to -3} \frac{x^2+x+6}{x+3}.$$

Check:

For the numerator : $\lim_{x\to -3}(x^2+x+6)$ = -3^2+ (-3)- 6 = 9 – 3 – 6 =0.

Similarly for the denominator: $\lim_{x\to -3}$(x + 3) = - 3+3= 0.

Since the limits of both numerator and denominator are zero, you know that they have a common factor of x + 3 by factorizing the numerator. So, for all x ≠ 3, you can divide out this factor to obtain the following:

$\lim_{x\to -3} \frac{x^2+x-6}{x+3} = \lim_{x\to -3} \frac{(x-2)(x+3)}{x+3}$ **Factor numerator by factorization.**

$= \lim_{x\to -3} \frac{(x-2)(x+3)}{x+3}$ **Divide out like factor.**

$= \lim_{x\to -3}(x-2)$ **Simplify.**

= - 3 – 2 **Substituting – 3 to be x.**

= - 5

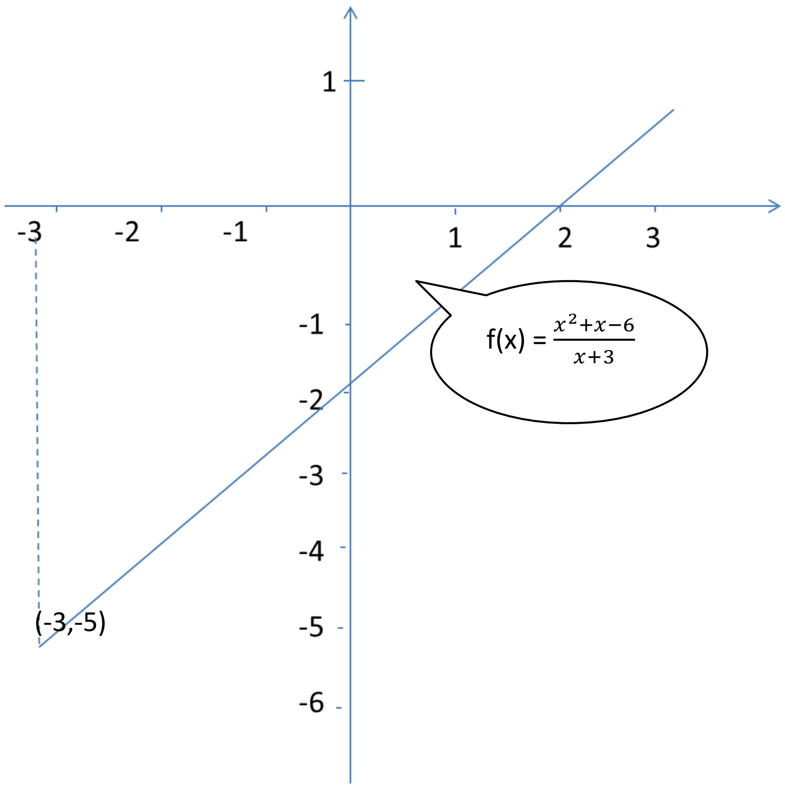


Fig 2.2: f is undefined when x = -3

This result is shown graphically above in fig 2.2. Note that the graph of f coincides with the graph of g(x) = x – 2, except that the graph of f has a hole at (-3, -5).

Checkpoint

Find the limit: $\lim_{x \to 3} \frac{x^2 + x - 12}{x - 3}$

Rationalizing The Numerator Technique.

Example 6

Find the limit: $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$.

Solution

Direct substitution fails because both the numerator and the denominator are zero when x = 0. In this case, you can rewrite the fraction by rationalizing the numerator by taking the conjugate of the numerator and using it to both the numerator and denominator.

Taking the conjugate of the numerator of the numerator $\sqrt{x+1} - 1$ will be$\sqrt{x+1} + 1$.

Conjugate of $\sqrt{x+1} - 1$ is$\sqrt{x+1} + 1$.

$$\therefore \frac{\sqrt{x+1} - 1}{x} = \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right)$$

$$= \frac{x + 1 + \sqrt{x+1} - \sqrt{x+1} - 1}{x(\sqrt{x+1} + 1)}$$

$= \frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$, x≠ 0

Now, using the replacement theorem, you can evaluate the limit as follows:

$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = ½$

Checkpoint

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$.

One Sided Limit

One way in which a limit fails to exist is when a function approachesa different value from the left of c than it approaches from right of c. This type of behaviour can be described more concisely with the concept of a **one-sided limit**.

$\lim_{x\to c^-} f(x) = L$ Limit from the left.

$\lim_{x\to c^+} f(x) = L$ Limit from the right.

The first of these two limits is read as "the limit of f(x) as x approaches c from the left is L". The second is read as "limitf(x) as x approaches c from the right is L".

Example 7

Find the limit as x→0 from the left and the limit as x→0 from the right for the function:

f(x) = $\frac{|2x|}{x}$

Solution

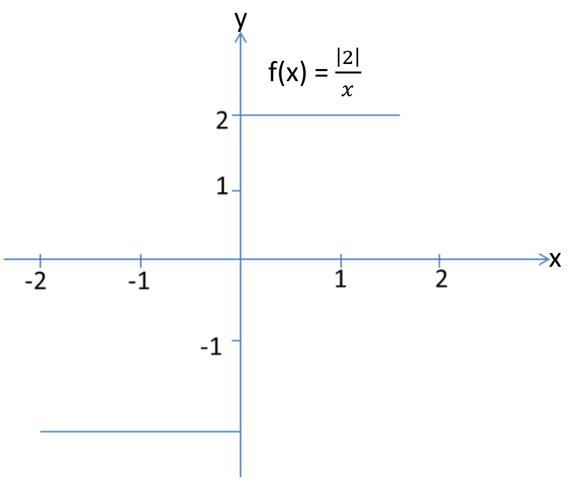


Fig 2.3

From the graph of f, shown in fig 2.3, you can see that f(x)= -2 for all x<0. Therefore, the limit from the left is:

$\lim_{x \to 0^-} \frac{|2x|}{x}$ = - 2 **Limit from the left.**

Because f(x) = 2 for all x > 0, the limit from the right is:

$\lim_{x \to 0^+} \frac{|2x|}{x}$ = 2 **Limit from the right.**

Unbounded Behaviour.

A Limit can fail to exist when f(x) increases or decrease without bound as x approaches c.The equal sign in the statement $\lim_{x \to c^+} \infty$does not mean that the limit exists. On the contrary, it tells you how the limit fails to exist by denoting the unbounded behaviour of f(x) as x approaches c.

Example 8

Find the limit (if possible) : $\lim_{x \to 2} \frac{3}{x-2}$

Solution

$\lim_{x \to 2^-} \frac{3}{x-2} = \infty$

and $\lim_{x \to 2^+} \frac{3}{x-2} = \infty$

Because f is a unbounded as x approaches 2, the limit does not exist.

Checkpoint:

Find the limit (if possible): $\lim_{x \to -2} \frac{5}{x+2}$

Solution.

$\lim_{x \to -2} \frac{5}{x+2} = \frac{5}{-2+2} = \frac{5}{0} = \infty$

2.7 Exercises

A. In exercise I and II, find the limit of (a) f(x) + g(x), (b) f(x).g(x) and (c) $\frac{f(x)}{g(x)}$as x approaches c.

I. $\lim_{x\to c} f(x) = 3$ II. $\lim_{x\to c} f(x) = \frac{3}{2}$

$$\lim_{x\to c} g(x) = 9 \lim_{x\to c} g(x) = \frac{1}{2}$$

B. In exercise III – XVI, find the limit:

III.$\lim_{x\to 2} x^4$ IV. $\text{Lim}_{x\to -2} x^3$ V. $\lim_{x\to -3}(3x+2)$ VI. $\text{Lim}_{x\to 1}(1-x^2)$

VII. $\lim_{x\to 2}(-x^2+x-2)$IX. $\lim_{x\to 3}\sqrt{x+1}$ X. $\lim_{x\to 4}\sqrt[3]{x+4}$

XI. $\lim_{x\to -3}\frac{2}{x+2}$ XII. $\lim_{x\to -2}\frac{3x-1}{2-x}$ XIII. $\lim_{x\to -1}\frac{4x-5}{3-x}$ XIV. $\lim_{x\to 7}\frac{5x}{x+2}$

XV. $\lim_{x\to 3}\frac{\sqrt{x+1}}{x-4}$ XVI. $\lim_{x\to -2}\frac{x^2-1}{2x}$

In the following exercise XVII – XXX, find the limit (if it exists):

XVII. $\lim_{x\to -1}\frac{x^2-1}{x+1}$ XVIII. $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$ XIX. $\lim_{x\to 2}\frac{x-2}{x^2-4x+4}$ XX. $\lim_{x\to 2}\frac{2-x}{x^2-4}$

XXI. $\lim_{t\to 5}\frac{t-5}{t^2-25}$ XXII. $\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$ XXIII. $\lim_{x\to -2}\frac{x^3-1}{x-1}$ XXIV. $\lim_{x\to 2}\frac{x^3+8}{x+2}$

XXV. $\lim_{x\to 0}\frac{|x|}{x}$ XXVI. $\lim_{x\to 2}\frac{|x-2|}{x-2}$

XXVII. $\lim_{x\to 3} f(x)$, where f(x) = $f(x) = f(x) = \begin{cases} \frac{1}{3}x, & x \le 3 \\ -2x+5, & x > 3 \end{cases}$

XXVIII. $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x}$ XXIX. $\lim_{\Delta x\to 0}\frac{\sqrt{x+2+\Delta x}-\sqrt{x+2}}{\Delta x}$

XXX.$\lim_{\Delta t}\frac{(t+\Delta t)^2-5(t+\Delta t)-(t^2-5t)}{\Delta t}$

2.8 Summary.

I. If f(x) becomes arbitrary close to a single number L as x approach c from either side, then $\lim_{x\to c} f(x)$ = L which is as the limit of f(x) as x approaches c is L.

II. If p is a polynomial function and c is any real number, then $\lim_{x\to c} p(x) = p(c)$

2.9References:

I. Engineering Mathematics by K. A Stroud.

II. Blitzer Algebra and Trigonometry Custom 4th Edition.

III. Calculus An Applied Approach Larson Edwards Sixth Edition.

MODULE ONE

UNIT 2: LIMITS.

Contents:

2.1 Introduction:

In everyday language, one always refers to limit one's endurance, speed limit of a car, a wrestler's weight limit or stretching a spring to its limit. These phrases all suggest that a limit is a bound, which on some instances may not be reached but on other instance may be reached or exceeded.

Hooke's law is a perfect illustration of limit which states provided an elastic limit of a spring is not exceeded; the extension (e) is directly proportional to the tension or force acting on it. That is a spring has a limit of extension when a load is suspended on it. If it exceeds the boundary, it will reach a point of plasticity or break without returning to its initial position.

Limit is a concept/Fundamental to Calculus.

2.2 Objectives:

In this Module, you will cover the following topics:

2.3 Limits (Definition)
2.4 Properties of Limits
2.5 The Limit of a Polynomial Function
2.6 Techniques for Evaluating Limits

2.1 Definition of Limit of a Function:

If $f(x)$ becomes arbitrarily close to a single number **L** as $\boldsymbol{x}$ approaches **c** from either side, $\lim_{x\to c} f(x) = L$, then $\text{Lim}\, f(x)$ which is read as **"the limit of $\boldsymbol{f(x)}$ as $\boldsymbol{x}$ approaches c is L"**

2.2 Properties of Limit:

Many times, the limit of $f(x)$ as x approaches c is simply $f(c)$ .Whenever the limit of as $f(x)$ approaches c is $\lim_{x\to c} f(x) = f(c)$.The limit can be evaluated by direct substitution.

Properties of Limits:

Let b and c be real numbers, and let n be a positive integer.

v. $\lim_{x \to c} b$ = b

vi. $\lim_{x \to c} x$ = c

vii. $\lim_{x \to c} x^{n}$ = x^{c}

viii. $\lim_{x \to c} x^{n}$ = c^{n}

In the property IV, if n is even, then c must be positive.

By combining the properties of limits with the rules for operating with limits shown below, you can find limits for a wide variety of algebraic function.

Operations with limits:

Let b and c be real numbers let n be a positive integer, and let f and g be functions with the following limits.

$\lim_{x \to c} f(x)$ = L and $\lim_{x \to c} g(x)$ = k

V. Scalar multiple: $\lim_{x \to c}[b\, f(x)]$ = b L .

VI. Sum or Difference: $\lim_{x \to c}[f(x) \pm g(x)]$= L $\pm$ k.

VII. Product: $\lim_{x \to c}[f(x)\, .\, g(x)]$ = L . k.

VIII. Quotient: $\lim_{x \to c} f(x)/g(x)$ = L/k provided that k $\neq$ 0.

V. Power: $\lim_{x \to c}[f(x)]^{n}$ = L^{n}

VI. Radical: $\lim_{x \to c} \sqrt[n]{f(x)}$ = $\sqrt[n]{L}$

In property VI, if n is even, then L must be positive.

Example 1: Find the limit:

$\lim_{x \to 1}(x^2+1)$

Solution.

Using direct substitution by substituting 1 for x

$\lim_{x \to 1}(x^2+1)$ = $1^2 + 1 = 2$

Example 2:

Find the limit: $\lim_{x \to 1} f(x)$.

a. f(x) = $\frac{x^2-1}{x-1}$ b. f(x) = $\frac{|x-1|}{|x-1|}$ c. $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases}$

Solution

a. $\lim_{x \to 1} f(x) = \frac{x^2-1}{x-1} = \frac{x^2-1^2}{x-1}$

$= \frac{(x+1)(x-1)}{(x-1)}$ **Factorizing the numerator by the difference of two square [$\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$].**

$\lim_{x \to 1}(x+1)$ = 1 + 1 = 2 **Substituting 1 for x.**

Therefore, $\lim_{x \to 1} f(x) = \frac{x^2-1}{x-1} = 2$

b. $\lim_{x\to 1} \frac{|x-1|}{x-1} = \frac{1-1}{1-1} = \frac{0}{0} = 0$ **Substituting 1 for x.**

Therefore, $\lim_{x\to 1} \frac{|x-1|}{x-1}$ does not exist.

c. $f(x) = \begin{cases} x, & x \neq 1 \\ 0, & x = 1 \end{cases} = 1$

2.3 The limit of a polynomial function

If p is a polynomial function and c is any real number, then $\lim_{x\to c} p(x) = p(c)$

Example 3

Finding the limit of a polynomial function

Find the limit:$\lim_{x\to 2}(x^2 + 2x - 3)$

Solution:

$\lim_{x\to 2} x^2 + 2x - 3 = \lim_{x\to 2} x^2 + \lim_{x\to 2} 2x - \lim_{x\to 2} 3$ **Applying property II.**

$= 2^2 + 2(2) - 3$ **Use direct substitution.**

$4 + 4 - 3 = 5$ **Simplify.**

$= 5$

Note: Example 3 show or state that the limit of polynomial can be evaluated by direct substitution.

Check point:

3. Find the limit: $\lim_{x\to 2} f(x)$

   b. $f(x) = \frac{x^2-4}{x-2}$ b. $f(x) = \frac{|x-2|}{x-2}$ c. $f(x) = \begin{cases} x^2, & x \neq 0 \\ 0, & x = 2 \end{cases}$

4. Find the limit : $\lim_{x\to 2} 2x^2 - x + 4$

2.4 **Techniques for Evaluating Limits.**

There are several techniques for calculating limits and these are based on the following important theorem. Basically, the theorem states that **"if two functions agree at all but a single point c, then they have identical limit behavior at x = c.**

- **The Replacement Theorem/Technique**

Let c be a real number and f(x) = g(x) for all x ≠ c. if the limit of g(x) exists as x→c, then the limit of f(x) also exists and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$$

To apply the Replacement Theorem, you can use a result from algebra which states that for a polynomial function p, p(c) = 0 if and only if (x-c) is a factor of p(x).

Example 4

Finding the limit of a function

Find the limit: $\lim_{x\to 1}\frac{x^3-1}{x-1}$

Solution

Note that the numerator and the denominator are zero when x=1

$\lim_{x\to 1}\frac{x^3-1}{x-1}$ .For the numerator $\lim_{x\to 1} x^3-1 = 1^3-1$=0 &the denominator $\lim_{x\to 1} x-1 = 1-1 = 0$.

This implies or means that x - 1 is a factor of both and you can divide out this like factor using division of polynomial.

$$\frac{x^3-1}{x-1} = (x^2 + x + 1)(x - 1)$$

$\therefore \lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}$

$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$ **factor numerator.**

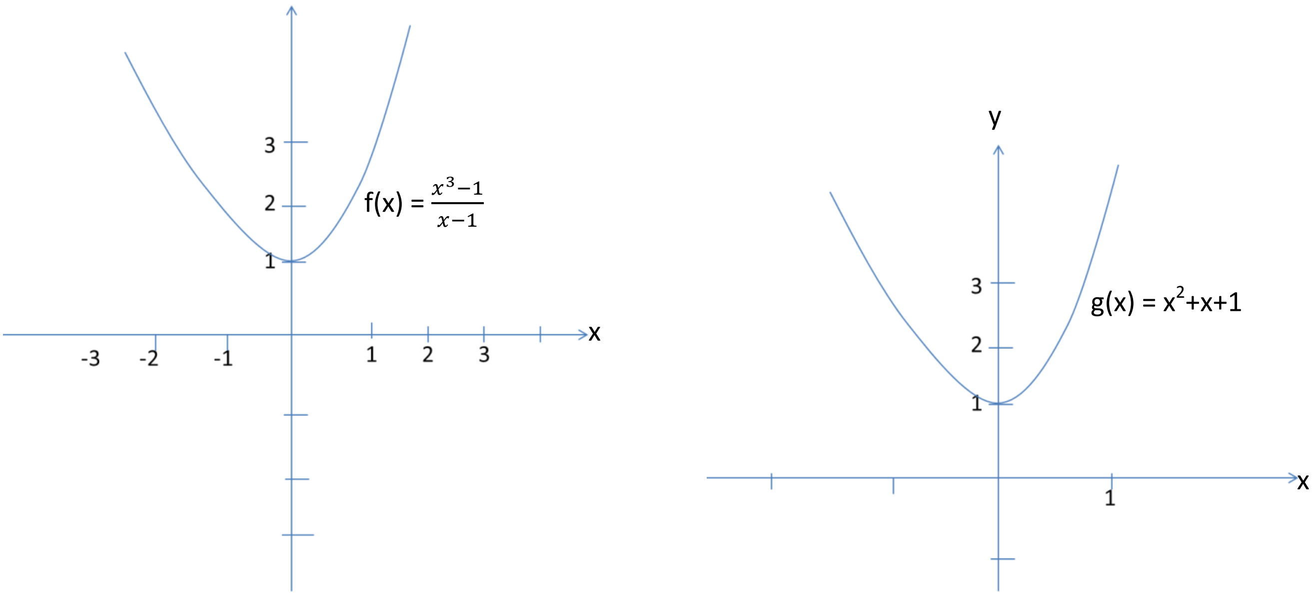
$=\frac{(x-1)(x^2+x+1)}{x-1}$ **Divide out factor.**

$= x^2+x+1$, $x \neq 1$ **Simplify.**

So, the rational function $(x^3 - 1)/(x - 1)$ and the polynomial function $x^2 + x + 1$ agree for all value of x other than x = 1, and you can apply the Replacement theorem.

$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}(x^2+x+1) = 1^2 + 1+1 = 3$

y



In fig2.1 illustrates this result graphically. Note that the two graphs are identical except that the graph of g contains the point (1, 3), whereas this point is missing on the graph of f. (in fig 2.1, the missing point is denoted by an open dot.)

Checkpoint.

Find the limit: $\lim_{x\to 2}\frac{x^3-8}{x-2}$

Dividing Out Technique.

Example 5

Find the limit: $\lim_{x\to -3}\frac{x^2+x+6}{x+3}$

Solution.

Using direct substitution will fails because both the numerator and thedenominator are zero when x = -3.

$$\lim_{x\to-3}\frac{x^2+x+6}{x+3}.$$

Check:

For the numerator : $\lim_{x\to -3}(x^2+x+6)$ = -3^2+ (-3)- 6 = 9 – 3 – 6 =0.

Similarly for the denominator: $\lim_{x\to-3}$(x + 3) = - 3+3= 0.

Since the limits of both numerator and denominator are zero, you know that they have a common factor of x + 3 by factorizing the numerator. So, for all x ≠ 3, you can divide out this factor to obtain the following:

$\lim_{x\to -3} \frac{x^2+x-6}{x+3} = \lim_{x\to -3} \frac{(x-2)(x+3)}{x+3}$ **Factor numerator by factorization.**

$= \lim_{x\to -3} \frac{(x-2)(x+3)}{x+3}$ **Divide out like factor.**

$=\lim_{x\to -3}(x-2)$ **Simplify.**

$= -3 - 2$ **Substituting – 3 to be x.**

= - 5

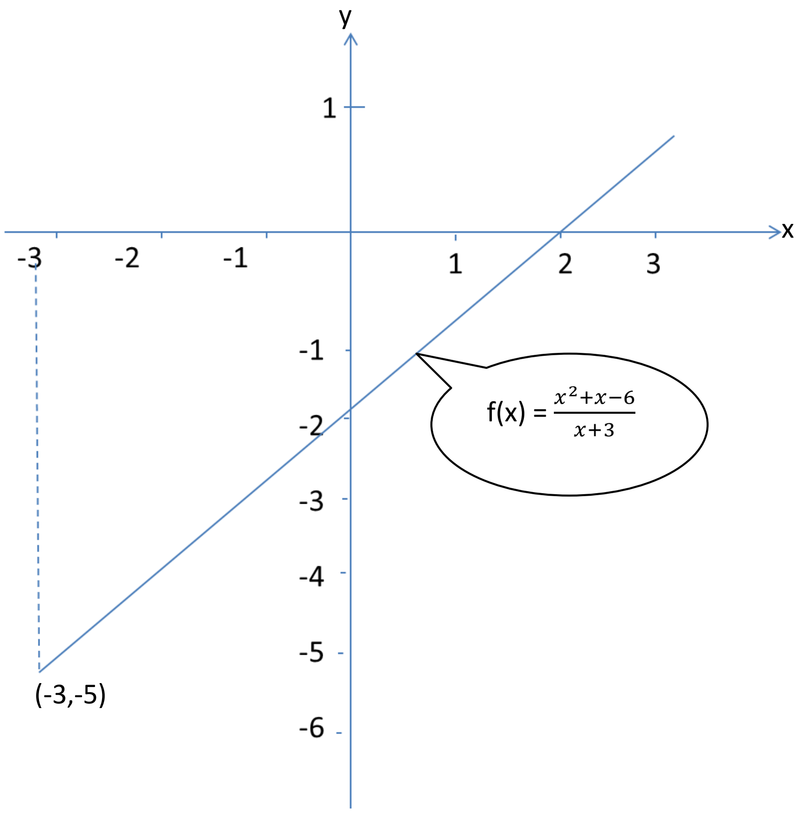


Fig 2.2: f is undefined when x = -3

This result is shown graphically above in fig 2.2. Note that the graph of f coincides with the graph of g(x) = x – 2, except that the graph of f has a hole at (-3, -5).

Checkpoint

Find the limit: $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

Rationalizing The Numerator Technique.

Example 6

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution

Direct substitution fails because both the numerator and the denominator are zero when x = 0. In this case, you can rewrite the fraction by rationalizing the numerator by taking the conjugate of the numerator and using it to both the numerator and denominator.

Taking the conjugate of the numerator of the numerator $\sqrt{x+1}-1$ will be $\sqrt{x+1}+1$.

Conjugate of $\sqrt{x+1}-1$ is $\sqrt{x+1}+1$.

$\therefore \frac{\sqrt{x+1}-1}{x} = \left(\frac{\sqrt{x+1}-1}{x}\right)\left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right)$

$= \frac{x+1+\sqrt{x+1}-\sqrt{x+1}-1}{x(\sqrt{x+1}+1)}$

$= \frac{x+1-1}{x(\sqrt{x+1}+1)} = \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$, $x \neq 0$

Now, using the replacement theorem, you can evaluate the limit as follows:

$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = ½$

Checkpoint

Find the limit: $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$.

One Sided Limit

One way in which a limit fails to exist is when a function approachesa different value from the left of c than it approaches from right of c. This type of behaviour can be described more concisely with the concept of a **one-sided limit**.

$\lim_{x\to c^-} f(x) = L$ Limit from the left.

$\lim_{x\to c^+} f(x) = L$ Limit from the right.

The first of these two limits is read as “the limit of f(x) as x approaches c from the left is L”. The second is read as “limitf(x) as x approaches c from the right is L”.

Example 7

Find the limit as x→0 from the left and the limit as x→0 from the right for the function:

$f(x) = \frac{|2x|}{x}$

Solution

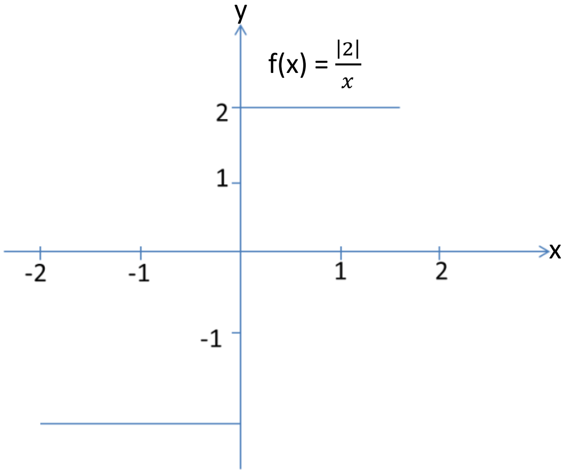


Fig 2.3

From the graph of f, shown in fig 2.3, you can see that f(x)= -2 for all x<0. Therefore, the limit from the left is:

$\lim_{x \to 0^-} \frac{|2x|}{x}$ = - 2 **Limit from the left.**

Because f(x) = 2 for all x > 0, the limit from the right is:

$\lim_{x \to 0^+} \frac{|2x|}{x}$ = 2 **Limit from the right.**

Unbounded Behaviour.

A Limit can fail to exist when f(x) increases or decrease without bound as x approaches c.The equal sign in the statement $\lim_{x \to c^+} \infty$does not mean that the limit exists. On the contrary, it tells you how the limit fails to exist by denoting the unbounded behaviour of f(x) as x approaches c.

Example 8

Find the limit (if possible) : $\lim_{x \to 2} \frac{3}{x-2}$

Solution

$\lim_{x \to 2^-} \frac{3}{x-2} = \infty$

and $\lim_{x \to 2^+} \frac{3}{x-2} = \infty$

Because f is a unbounded as x approaches 2, the limit does not exist.

Checkpoint:

Find the limit (if possible): $\lim_{x\to -2} \frac{5}{x+2}$

Solution.

$\lim_{x\to -2} \frac{5}{x+2} = \frac{5}{-2+2} = \frac{5}{0} = \infty$

2.7 Exercises

A. In exercise I and II, find the limit of (a) f(x) + g(x), (b) f(x).g(x) and (c) $\frac{f(x)}{g(x)}$as x approaches c.

I. $\lim_{x\to c} f(x) = 3$ II. $\lim_{x\to c} f(x) = \frac{3}{2}$

$$\lim_{x\to c} g(x) = 9 \lim_{x\to c} g(x) = \frac{1}{2}$$

B. In exercise III – XVI, find the limit:

III.$\lim_{x\to 2} x^4$ IV. $\text{Lim}_{x\to -2} x^3$ V. $\lim_{x\to -3}(3x+2)$ VI. $\text{Lim}_{x\to 1}(1-x^2)$

VII. $\lim_{x\to 2}(-x^2+x-2)$IX. $\lim_{x\to 3} \sqrt{x+1}$ X. $\lim_{x\to 4} \sqrt[3]{x+4}$

XI. $\lim_{x\to -3} \frac{2}{x+2}$ XII. $\lim_{x\to -2} \frac{3x-1}{2-x}$ XIII. $\lim_{x\to -1} \frac{4x-5}{3-x}$ XIV. $\lim_{x\to 7} \frac{5x}{x+2}$

XV. $\lim_{x\to 3} \frac{\sqrt{x+1}}{x-4}$ XVI. $\lim_{x\to -2} \frac{x^2-1}{2x}$

In the following exercise XVII – XXX, find the limit (if it exists):

XVII. $\lim_{x\to -1} \frac{x^2-1}{x+1}$ XVIII. $\lim_{x\to -1} \frac{2x^2-x-3}{x+1}$ XIX. $\lim_{x\to 2} \frac{x-2}{x^2-4x+4}$ XX. $\lim_{x\to 2} \frac{2-x}{x^2-4}$

XXI. $\lim_{t\to 5} \frac{t-5}{t^2-25}$ XXII. $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$ XXIII. $\lim_{x\to -2} \frac{x^3-1}{x-1}$ XXIV. $\lim_{x\to 2} \frac{x^3+8}{x+2}$

XXV. $\lim_{x\to 0} \frac{|x|}{x}$ XXVI. $\lim_{x\to 2} \frac{|x-2|}{x-2}$

XXVII. $\lim_{x\to 3} f(x)$, where f(x) = $f(x) = f(x) = \begin{cases} \frac{1}{3}x, & x \le 3 \\ -2x+5, & x > 3 \end{cases}$

XXVIII. $\lim_{\Delta x\to 0} \frac{2(x+\Delta x)-2x}{\Delta x}$ XXIX. $\lim_{\Delta x\to 0} \frac{\sqrt{x+2+\Delta x}-\sqrt{x+2}}{\Delta x}$

XXX.$\lim_{\Delta t} \frac{(t+\Delta t)^2-5(t+\Delta t)-(t^2-5t)}{\Delta t}$

2.8 Summary.

I. If f(x) becomes arbitrary close to a single number L as x approach c from either side, then $\lim_{x\to c} f(x)$ = L which is as the limit of f(x) as x approaches c is L.

II. If p is a polynomial function and c is any real number, then $\lim_{x\to c} p(x) = p(c)$

2.9References:

I. Engineering Mathematics by K. A Stroud.

II. Blitzer Algebra and Trigonometry Custom 4th Edition.

III. Calculus An Applied Approach Larson Edwards Sixth Edition.

MODULE ONE

UNIT 3: IDEA OF CONTINUITY

3.1 Introduction:

In mathematics, continuity means rigorous formulation of the intuitive concept of function that varies with no abrupt breaks or jumps.

Continuity of a function is expressed some times by saying if the X values are closed together, then the y value of the function will also be close.

3.2 Objective

In this module, you will cover the following topics

3.3 Definition of Continuity

3.4 Continuity of Polynomial and Rational functions

3.5 Continuity on a closed interval

3.3 Idea of Continuity

3.3.1 Definition of Continuity

Let **c** be a number in the interval (a, b), and let **f** be a function whose domain contains the interval (a, b). The function **f** is **continuous at the point c**if the following conditions are true:

i. F(c) is defined

ii. $\lim_{x \to c} f(x)$ exists

iii. $\lim_{x \to c} f(x)$ = f(c)

If f is continuous at every point in the interval (a, b), then it is **continuous on an open interval (a, b).**

N:B

- Roughly, we can say that a function is said to be continuous on an interval if its graph on the interval can be traced using a pencil and paper without lifting the pencil from the paper (i.e. the graph of f is unbroken at c, and there are no holes, jumps, or gaps or to say that a function is continuous at x=c when there is no interruption in the graph of f at c.
- Specifically, when direct substitution can be used to evaluate the limit of a function at c, then the function is continuous at c.The two types of functions that have this property are polynomial functions and rational functions.

3.4 Continuity of polynomial and Rational Functions.

i. A polynomial function is continuous at every real number.

iiA rational function is continuous at every number in its domain.

Determining continuity of a polynomial function

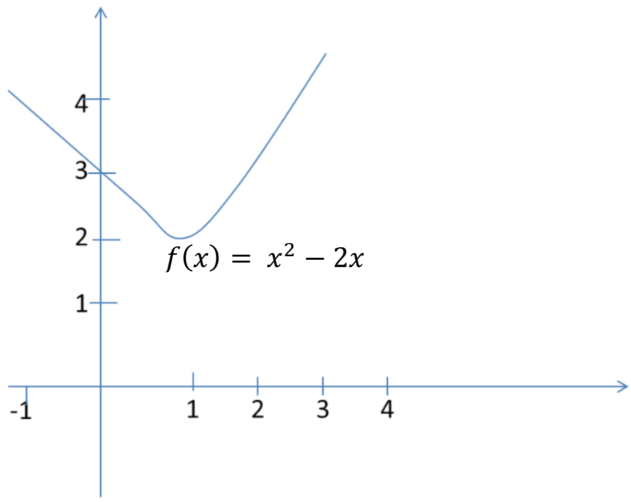
Example 1

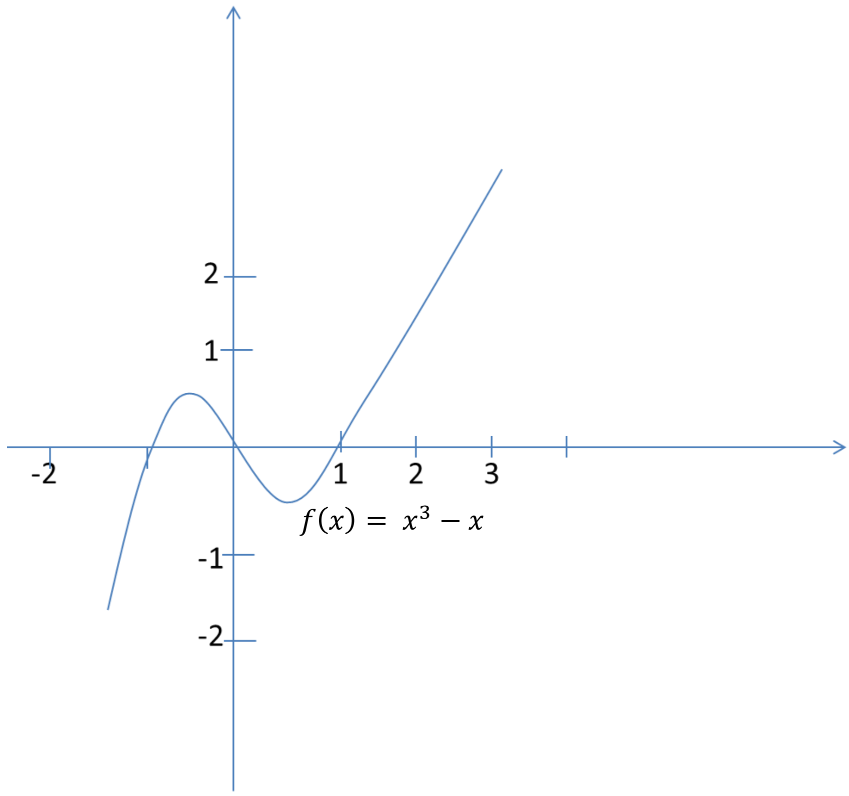
Discuss the continuity of each function

a. $f(x) = x^2 - 2x + 3$ b. $f(x) = x^3 - x$

Solution

Each of these functions is a polynomial function. So, each is continuous on the entire real line.





Both functions are continuous on (-∞,∞).

Determining continuity of each function

Example 2

Discuss the continuity of each function

(a) f (x) = $\frac{1}{x}$(b) f(x) = $\frac{x^2 -1}{x-1}$ (c) f(x) = $\frac{1}{x^2+1}$

Solution

Each of these functions is a rational function and is therefore continuous at every number in its domain.

(a) The domain of f(x) = $\frac{1}{x}$consist of all real numbers except x = o. So, this function is continuous on the intervals (-∞, 0) and (0,∞).

(b) The domain of f(x) = $(x^2 - 1)/ (x - 1)$ consists of all real numbers except x = 1. So, this function is continuous on the intervals (-∞, 1) and (1,∞).

(c) The domain of f(x) = $\frac{1}{x^2 + 1}$consists of all real numbers. So, this function is continuous on the entire real line.

Class exercise

Discuss the continuity of each function:

(a) f(x) = $\frac{1}{x- 1}$ (**Answer**: Continuous (-∞, 1) and (1, ∞))

(b) f(x) = $\frac{x^2 -4}{x-2}$ (**Answer**: Continuous (-∞, 2) and (2, -∞))

(c) f(x) = $\frac{1}{x^2 + 2}$ (**Answer**: Continuous on the entire real line)

Consider an open interval I that contains a real number **c**. If a function **f** is defined on I (except possibly at **c**), and **f** is not continuous at **c**, then f is said to have a **discontinuity** at **c**. Discontinuities fall into two categories: **removable** and **non-removable.**

A discontinuity is called removable if **f** can be made continuous by appropriately defining (or redefining) **f(c)**. For instance, the function in Example 2b has a removable discontinuity at (1, 2). To remove the discontinuity, all you need to do is redefine the function so that F (1) = 2.

A discontinuity at x = c is non removable if the function cannot be made continuous at x=c by defining or redefining the function at x=c. For instance, the function in Example 2a has a non removable discontinuity at x=0.

3.5 Continuity on a Closed Interval

Definition

Let **f** be defined on a closed interval [a, b]. If **f** is continuous on the open interval (a, b) and $\lim_{x \to a^+} f(x)$ = f(a) and $\lim_{x \to b^-} f(x)$ = f(b)

Then f is **continuous on the closed interval [a, b].** Moreover, f is **continuous from the right** at a and**continuous from the left** at b.

Similar definitions can be made to cover continuity on intervals of the form (a, b] and [a,b), or on infinite intervals. For example the function

f(x) = $\sqrt{x}$ is continuous on the infinite interval [0, ∞).

Examining Continuity at an End point

Example 3

Discuss the continuity of f(x) = $\sqrt{3-x}$

Solution

Notice that the domain of f is the set (-∞, 3].

Moreover, f is continuous from the left at x = 3 because:

$\lim_{x\to 3} f(x)$ = $\lim_{x\to 3^-} \sqrt{3-x}$

= 0

= f(3)

For all x<3, the function f satisfies the three conditions foe continuity. So, you can conclude that f is continuous on the interval (-∞, 3].

Working Tip

When working with radical functions of the form f(x) =$\sqrt{g(x)}$, remember that the domain of f coincides with the solution of g (x) ≥ 0.

Examining Continuity on a closed interval.

Example 4:

Discuss the continuity of $g(x) = \begin{cases} 5-x, & -1 \le x \le 2 \\ x^2-1, & 2 < x \le 3 \end{cases}$

Solution

The polynomial functions 5 – x and X^2 – 1 are continuous on the intervals [- 1, 2) and (2, 3], respectively. So, to conclude that g is continuous on the entire interval [-1, 3], you need only check the behavior of g when x =2. You can do this by taking the one – sided limit when x=2.

$\lim_{x\to 2^-} g(x)$ = $\lim_{x\to 2^-}(5-x)$ = 3 limit from the left

and

$\lim_{x\to 2^{\mp}} g(x)$ = $\lim_{x\to 2^{\mp}}(x^2-1)$ = 3 limit from the right

Because these two limits are equal

$\lim_{x \to 2} g(x)$ = g(2) = 3.

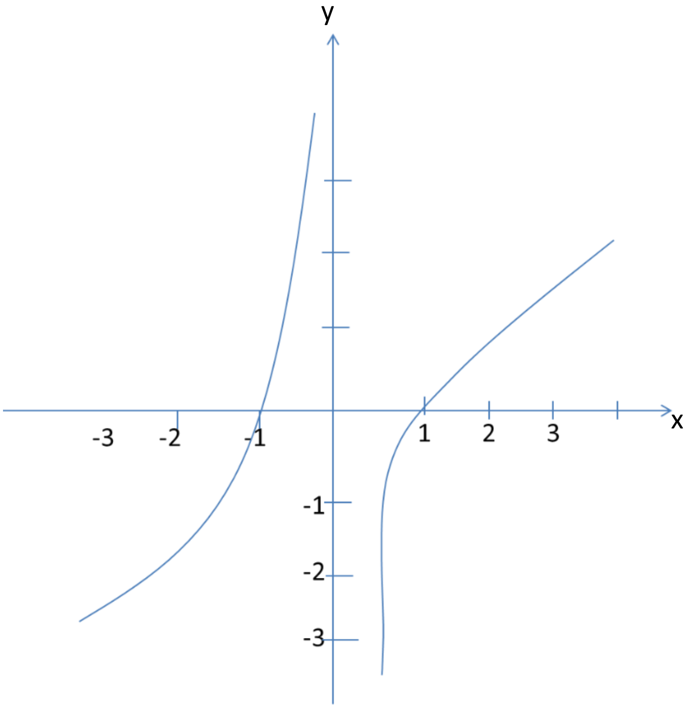
So, g is continuous at x = 2 and consequently, it is continuous on the entire interval [-1, 3]

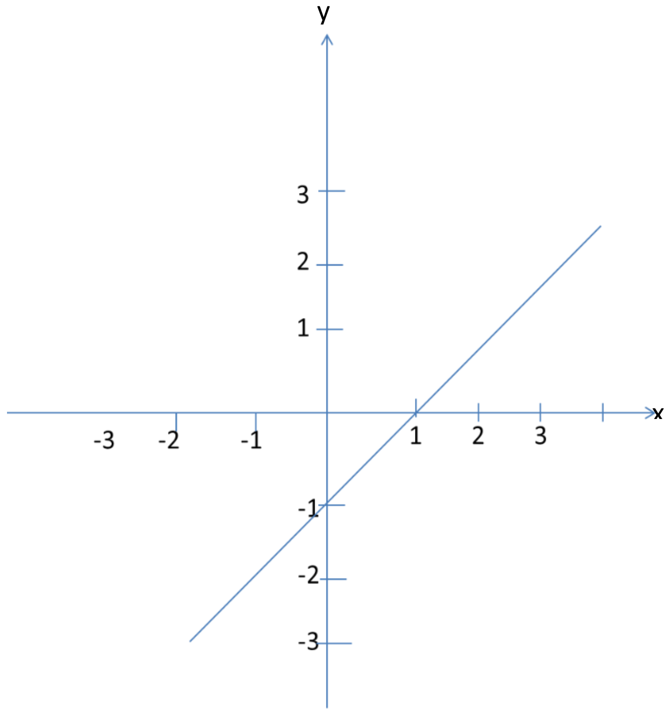
3.6 Exercises

A. In exercise i to iii, determine whether the function is continuous on the entire real line. Explain your reasoning.

i. $f(x) = 5x^3 - x^2 + 2$

ii. $f(x) = \frac{1}{x^2 - 4}$





B. In Exercises iv to xvi, describe the interval(s) on which the function is continuous.

iii. $f(x) = \frac{1}{9 - x^2}$

iv. $f(x) = \frac{x^2 - 1}{x}$

v. $f(x) = \frac{x^2 - 1}{x+1}$

vi. $f(x) = x^2 - 2x + 1$

vii. $f(x) = \frac{x}{x - 1}$

viii. $f(x) = \frac{x}{x^2 + 1}$

ix. $f(x) = \frac{x-5}{x^2 - 9x + 20}$

x. = $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x > 1 \end{cases}$

xi.

$$f(x) = \begin{cases} 3 + x, & x \leq 2 \\ x^2 + 1, & x > 2 \end{cases}$$

xii. $f(x) = \frac{|x+1|}{x+1}$ xiii. $f(x) = [[x-1]]$

xvi. $h(x) = f(g(x)), f(x) = \frac{1}{\sqrt{x}}, g(x) = x-1, x > 1$

3.7 Summary

i. A function is said to be continuous if and only if it is continuous at every point of its domain.

ii. Continuity can be defined in terms of limits by saying that f(x) is continuous at x(0) of its domain if and only if, for values of x in its domain

$$\lim_{x \to x_0} f(x) = f(x_0)$$

3.8 References

i. Blitzer Algebra and Trigonometry Custom 4th Edition

ii. Calculus: An Applied Approach. Larson Edwards Sixth Edition

iii. Engineering Mathematics by K.A Stroud.

MODULE TWO

UNIT 1: THE DERIVATIVE AS LIMIT OF RATE OF CHANGE

Contents

4.1 Introduction

The derivative of a function of a real variable measures the sensitivity to change of a quantity (a function value or dependent variable) which is determined by another quantity (the independent variable).

Derivatives are fundamental tools of calculus. The derivative of a function of a single variable at a chosen input value is the slope of the tangent line to the graph of the function at that point. This means that it describe the best linear approximation of the function near that input value. For this reason, the derivative is often described as the "instantaneous rate of change", the ratio of the instantaneous change in the dependent variable to that of the independent variable.

Differentiation is the action of computing a derivative.

4.2 Objective

In this module, you will cover the following topics:

4.3 Differentiation (First Principle)
4.4 Derivative for Power of x^n

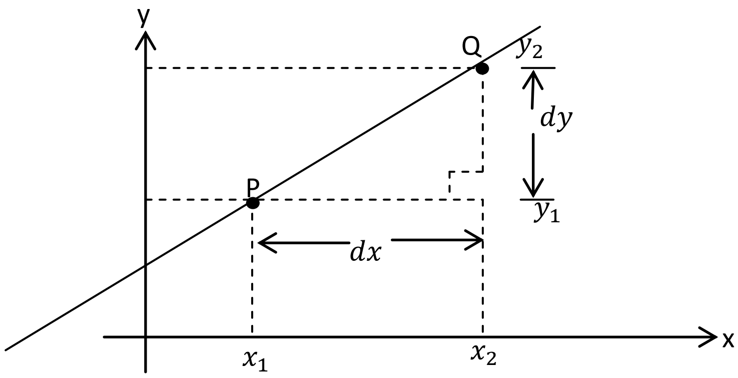
4.5 Differentiation of Polynomials

4.6 Standard derivative

4.3 Differentiation

Before introducing differentiation, it is always good to study gradient at point on a curve since it is the basis of differentiation. The process of finding a derivative is called differentiation.

**** The Gradient of a straight line graph**

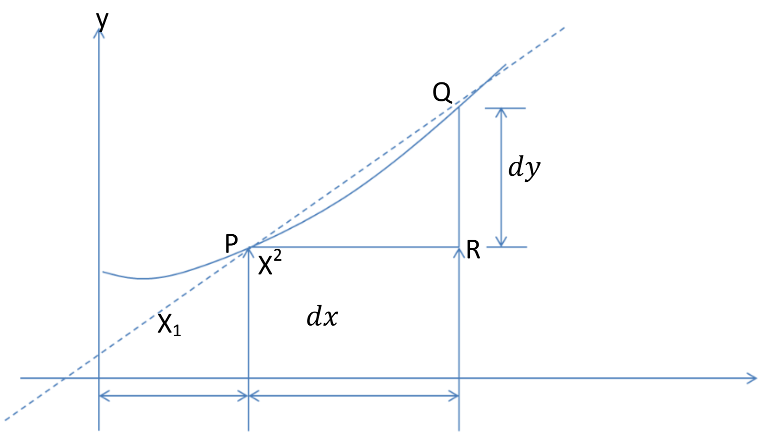


Gradient is defined as the ratio of the vertical distance the line rises or falls between two points P and Q to the horizontal distance between P and Q

m is the symbol used denoting gradient of a straight line graph

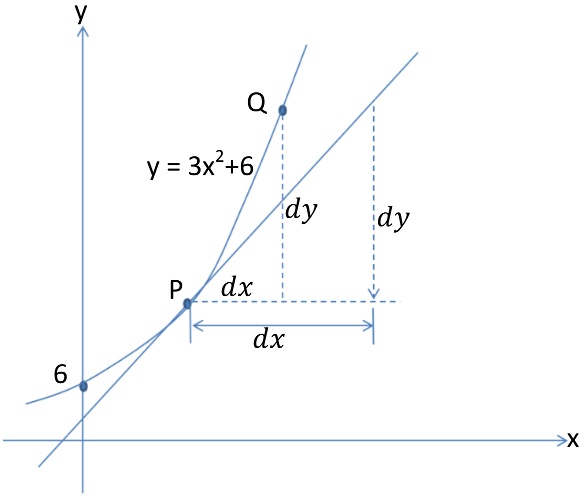
i.e. $\quad m = \frac{\delta y}{\delta x} = \frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1}$

**** The Gradient of a curve at a given point (Algebraic Determination)**



Let P be a fixed point(x, y) on the curve y and Q be a neighbouring. We will notice a slight change; that we frequently use δx and δy to denote the respective differences in the x and y values of the points P and Q on the curve. The δx and δy are called the differentials.

For example using fig 4.1 to illustrate



On the graph, P is a fixed point on the curve y=$3x^2 + 6$and Q a neighbouring point. There is a slight difference in x (i.e. x+δx) and in y (i.e. y+δy)

Using the first principle of differentiation to evaluate it.

Let y=$3x^2 + 6$(1) ————————

At Q: with the little increment

y+δy = 3(x+δx)^2 +6 ———————— (2)

Expanding the bracket in equation (2)

y+δy = 3(x^2+2x-δx+|δx|^2) +6

Subtracting y from both sides

y+δy-y = 3x^2+6xδx+3|δx|^2+6-y

Replacing the value of y in equation (1) in the term above

$y+\delta y- y = 3x^2+6x\delta x+3|\delta x|^2+6-(3x^2+6x)$

$\delta y = 3x^2+6x\delta x+3|\delta x|^2+6-3x^2-6x$

Collecting the common terms

$\delta y = 3x^2+6x\delta x\ 3x^2 - 3x^2+6x\delta x+3|\delta x|^2 \quad +6 - 6$

$\delta y = 6x\delta x+3|\delta x|^2$

Dividing both side by δx

$\frac{\delta y}{\delta x} = 6x\frac{\delta x}{\delta x} + \frac{3|\delta x|^2}{\delta x}$

$\frac{\delta y}{\delta x}= 6x+3|\delta x|$

Lim $\delta y \to 0$, $\delta x \to \quad 0$

$\frac{\delta y}{\delta x} = 6x + 3|0|$

$\frac{\delta y}{\delta x}=6x$

This is called the First Principle of Differentiation

Example 2

If y $=x^2 + 3x$, find $\frac{\delta y}{\delta x}$ using first principle

$y = x^2 + 3x$ ------------------------------ (1)

$y+\delta y = (x + \delta x)^2+3(x + \delta x)$---------------------- (2)

Expanding the bracket in equation (2)

$y+\delta y = x^2 + 2x\,dx + \delta x + |\delta x|^2 + 3x + 3\delta x$

Subtracting y from both sides

$$y + \delta y - y = x^2 + 2x\,dx + \delta x + |\delta x|^2 + 3x + 3\delta x - y$$

$$\delta y = x^2 + 2x\,dx + |\delta x|^2 + 3x + 3\delta x - y$$

Replacing the value of y in equation (1) in the term above

$$\delta y = x^2 + 2x\,dx + |\delta x|^2 + 3x + 3\delta x - (x^2 + 3x)$$

$$\delta y = x^2 + 2x\delta x + |\delta x|^2 + 3x + 3\delta x - x^2 - 3x$$

Colleting like terms

$$\delta y = x^2 - x^2 + 2x\delta x + |\delta x|^2 + 3x - 3x + 3\delta x$$

$$8y = 2x\delta x + |\delta x|^2 + 3\delta x$$

Dividing throughout by δx

$$\frac{\delta y}{\delta x} = 2\,x\frac{\delta x}{\delta x} + \frac{|\delta x|^2}{\delta x} + \frac{3\delta x}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2\,x + |\delta x| + 3$$

Lim $\delta y \rightarrow 0$, $\delta x \rightarrow 0$

$$\frac{\delta y}{\delta x} = 2x + 0 + 3$$

$$\frac{\delta y}{\delta x} = 2x + 3$$

4.4 Derivative of Power of X^n

If $y = x^n$ ------------------- (1)

We can established that if $y = x^n$

Using Binomial theorem to find the derivative of $y = x^n$

If $(a+b)^n = a^n + n\,a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots$

If y= x^n, y + δy = $(x + \delta x)^n$ using first principle

y + δy = $x^n + nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \ldots\ldots\ldots\ldots\ldots$ (2)

y= x^n

Subtracting y from both side of equation (2)

$\therefore \delta y = nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^3 + \cdots$

Dividing throughout by δx

$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(\delta x) + \frac{n(n-1)(n-2)}{3!}x^{n-3}(\delta x)^2 + \ldots\ldots\ldots\ldots\ldots\ldots$

If δx $\to$0, $\frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$ and all terms on the RHS, except the first$\to$ 0

If $dx \to 0, \frac{\delta y}{\delta x} = nx^{n-1}$+0+0+0+.....................

If y= x^n, $\frac{\delta y}{\delta x} = nx^{n-1}$

Generally, if y=ax^n then, $\frac{\delta y}{\delta x} = nax^{n-1}$where **a** is a constant

If y = K (where K is constant) then $\frac{\delta y}{\delta x}$= 0

4.5 Differentiation of Polynomial

When differentiating a polynomial, it has to be differentiated in turn of each term.

Example 1

If y= $x^3 + 5x^2 - 4x + 2$, differentiate with respect to x

Solution

y= $x^3 + 5x^2 - 4x + 2$

$\frac{\delta y}{\delta x}$= $3x^{3-1} + 2.5x^{2-1} - 1.4x^{1-1} + 0$ $\quad$ ($using$ y= x^n,$\frac{\delta y}{\delta x} = nx^{n-1}$)

= $3x^2 + 10x - 4$

Example 2

If y = $x^4 + 6x^3 - 4x^2 + 7x - 2, find\ \frac{\delta y}{\delta x}\ and\ the\ value\ of\ \frac{\delta y}{\delta x}\ at\ x = 2$

<u>Solution</u>

y = $x^4 + 6x^3 - 4x^2 + 7x - 2$

$\frac{\delta y}{\delta x}$= $4x^{4-1} + 3.6x^{3-1} - 2.4x^{2-1} + 1.7x^{1-1} - 0$

$\frac{\delta y}{\delta x}$ = $4x^3 + 18x^2 - 8x^1 + 7$

At x = 2, $\frac{\delta y}{\delta x}$= $4(2)^3 + 18(2)^2 - 8(2)^1 + 7$

= 32+72-16+7

= 95

4.6 Standard Derivative

Derivative of Trigonometric expression

This is established by using a number of trigonometric formulas:

I. Derivative of y = sin x

If y =sin x --------------------------- (1)

Using 1st Principle of Differentiation

y+δy = sin(x + δx) ………………………(2)

Subtract y from both side of the term above (equ. 2)

$\mathrm{y}+\delta\mathrm{y}-\mathrm{y}=\sin(\mathrm{x}+\mathrm{y}+\delta\mathrm{x})-\mathrm{y}$

$$\delta\mathrm{y}=\sin(\mathrm{x}+\delta\mathrm{x})-\mathrm{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Replacing both the value of y in equ. (3)

(3) becomes: $\delta\mathrm{y}=\sin(\mathrm{x}+\delta\mathrm{x})-\sin\mathrm{x}$

We now apply the trigonometrical formulae:

$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

Where A = x+ δx and B = x

$\mathrm{y}=2\cos(\frac{\mathrm{x}+\delta\mathrm{x}+x}{2})\sin(\frac{\mathrm{x}+\delta\mathrm{x}-x}{2})$

$\delta\mathrm{y}=2\cos(\frac{2\mathrm{x}+\delta\mathrm{x}}{2})\sin(\frac{\delta\mathrm{x}}{2})$

$\delta\mathrm{y}=2\cos(\frac{2x}{2}+\frac{\delta\mathrm{x}}{2})(\sin\frac{\delta\mathrm{x}}{2})$

$\delta\mathrm{y}=2\cos(x+\frac{\delta\mathrm{x}}{2})\sin(\frac{\delta\mathrm{x}}{2})$

Dividing both side by δx

$\frac{\delta y}{\delta x}=\frac{2\cos\left(x+\frac{\delta\mathrm{x}}{2}\right)+\sin(\frac{\delta\mathrm{x}}{2})}{\delta\mathrm{x}}$

$\frac{\delta y}{\delta x}=\frac{\cos\left(x+\frac{\delta\mathrm{x}}{2}\right)+\sin(\frac{\delta\mathrm{x}}{2})}{\frac{\delta\mathrm{x}}{2}}$

$\frac{\delta y}{\delta x}=\cos(x+\frac{\delta\mathrm{x}}{2}).\frac{\sin(\frac{\delta\mathrm{x}}{2})}{\frac{\delta\mathrm{x}}{2}}$

When $\delta\mathrm{y}\to 0,\frac{\delta y}{\delta x}\to\frac{\delta y}{\delta x}$

$\frac{\delta y}{\delta x}=\cos(x+\frac{0}{2})\frac{\sin(\frac{0}{2})}{\frac{0}{2}}$

$\frac{\delta y}{\delta x} = \cos x$

∴ $\boxed{\text{If } y = \sin x, \frac{\delta y}{\delta x} = \cos x}$

II. Derivative of y = $\cos x$

If y = $\cos x$ ……………………….. (1)

Using 1st Principle of Differentiation

y + δy = $\cos(x + \delta x)$……………………………… (2)

Subtracting y from both side of the equ. (2) above

y + δy − y = $\cos(x + \delta x) - y$

δy = $\cos(x + \delta x) - y$…………………………….(3)

Replacing back the value of y in equation (1) to equation (3)

(3) becomes: δy = $\cos(x + \delta x) - \cos x$

Now we can use the trigonometrical formula

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

Where A= $x + \delta x$ and B = x

$\delta y = -2\sin\left(\frac{x+\delta x+x}{2}\right).\sin\left(\frac{x+\delta x-x}{2}\right)$

$\delta y = -2\sin\left(\frac{2x+\delta x}{2}\right).\sin\left(\frac{\delta x}{2}\right)$

Dividing both side by δx

$\frac{\delta y}{\delta x} = \frac{-2\cos\left(\frac{2x}{2}+\frac{\delta x}{2}\right)\ \sin\left(\frac{\delta x}{2}\right)}{\delta x}$

$\frac{\delta y}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right) \quad \frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}$

As $\delta x \to 0, \frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}$

$\frac{\delta y}{\delta x}$ = - sin (x + $\frac{0}{2}$) sin ($\frac{0}{2}$)/$\frac{0}{2}$

$\frac{\delta y}{\delta x}$ = - sin x

∴If y= cos x, $\frac{\delta y}{\delta x} = -sinx$

III. Derivative of y = Tan x

If y = tan x

From trigonometric expression tan x = $\frac{Sin\ x}{\cos x}$

Since the expression is y = $\frac{u}{v}$, we will use the quotient rule

$$\frac{\delta y}{\delta x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$$

Let u = sin x, v = cos x

$\frac{\delta u}{\delta x}$ = cos x$\frac{\delta v}{\delta x}$ = – sin x

Substituting the above term into equation (1)

(1) becomes: $\frac{dy}{dx} = \frac{\cos x\,.\cos x - sinx.-sinx}{cosx\,.\cos x}$

$\frac{\partial y}{\partial x} = \frac{cos^2x+sin^2x}{cos^2x}$ (where $\sin^2$ A + $\cos^2$ B = 1)

$$\frac{dy}{dx} = \frac{1}{cos^2x}$$

$$= sec^2x$$

If y = tan x, $\frac{\delta y}{\delta x} = sec^2x$

Hence the other standard derivative is:

Y	$\delta y / \delta x$
x^n	$n.x^{n-1}$
c	0
$a.x^n$	$an.x^{n-1}$
sin x	cos x
cos x	- sin x
tan x	$\sec^2 x$
e^x	e^x
a^x	a^x .lna
e^{kx}	ke^{kx}
a^x	a^x . lna
lnx	1/x
Cot x	- $\text{cosec}^2 x$
Sec x	Sec x tan x
Cosec x	- Cosec x .Cotx
Sinh x	cosh x
Cosh x	sinh x

4.7 Exercises

A. Differentiate using first principle.

i. y = x^3 ii. y = $5x^2 + 2$ iii. y = $6x^2 - 1$ iv. y = $4x^3$

B. Find the derivative of the following.

i. $y = 6x^3 + 4x^2 - 7x + 2$ ii. $y = 15x^3 - 6x^2 + 10$ iii. $y = 10x^5 + 7x^3 + 2x$

4.8 Summary

i. Differentiation is the action of computing a derivative. The derivate of a function f(x) of a variable x is a measure of the rate at which the changes with respect to the change of the variable.

ii. Derivate of powers of x

a) y = c (constant), $\frac{\delta y}{\delta x} = 0$

b) $y = x^n$, $\frac{\delta y}{\delta x} = nx^{n-1}$ c) $y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$

iii. Gradient of a straight line graph (m) = $\frac{\delta y}{\delta x}$

iv. Differentiation of polynomial means differentiating each term in turn

4.9 References

1. Additional Mathematics by Godman and J.F Talbert

2. Calculus An Applied Approach Larson Edwards Sixth Edition

3. Blitzer Algebra and Trigonometry Custom 4th Edition

4. Engineering Mathematics by K.A Stroud.

MODULE TWO

UNIT 2: DIFFERENTIATION TECHNIQUES

Contents

5.1 Introduction

This shows the useful formulas in showing that the derivative is linear. Here we will learn the quotient rule, product, function of a function and implicit function

5.2 Objective:

In this module, you will cover the following subtopics

5.3 Differentiation of products of functions (product rule)
5.4 differentiation of a quotient of two function (quotient rule)
5.5 Function of a function (composite function)
5.6 Implicit function

5.3: Differentiation of Products of Functions (Product Rule)

Let y = uv, where u and v are functions of x

If x → $x + \delta$x, u → u+δu, v →v + ∂v and as a result,y → y+ δy.

The above expression shows that an increment ∂x in x will turn produce increments ∂u in u and also producing a change ∂v in v and a change ∂y in y.

Using first principle:

If y = uv. . . . .(1)

Therefore, y+δy= (u+δu)(v+δv). . . . . . . (2)

Expanding the left hand side in equation (2) gives:

$y + \delta y = uv + u\delta v + v\delta u + \delta u.\ \delta v$ . . . . (3)

Subtracting y = uv in equation (3) gives

$y + \delta y - y = uv + u\delta v + v\delta u + \delta u.\delta v - y$

$\delta y = uv + u\delta v + v\delta u + \delta u.\delta v - y$

$\delta y = uv + u\delta v + v\delta u + \delta u.\delta v - uv$ (where y=uv)

$\delta y = u\delta v + v\delta u + \delta u.\ \delta v$

Divide throughout by δx

$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \delta u.\delta v$ . . . . (4)

Limit if $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{\delta y}{\delta x}, \frac{\delta u}{\delta x} \to \frac{\delta u}{\delta x}, \frac{\delta v}{\delta x} \to \frac{\delta v}{\delta x}$

(NB tends to turns to)

Therefore, (4) now gives:

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} + \frac{0.\delta v}{\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

Product Rule:

If y=uv

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

Example 1

Differentiate with respect to x: $y = x^3 \sin x$

$y = x^3 \sin x$

Let $u = x^3$ and $v = \sin x$

$\frac{\delta y}{\delta x} = \cos x, \frac{\delta u}{\delta x} = 3x^2$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x} \quad = x^3 \cos x + \sin x\ 3x$$

$$= x^3 \cos x + 3x \sin x$$

$$= x\,(x^2 \cos x + 3 \sin x)$$

Example 2

Differentiate $(3x-2)(x^2 + 3)$ with respect to x

$(3x-2)(x^2 + 3)$

Let u=3x-2

v= x^2+3

$\frac{\delta v}{\delta x}$ = 2x, $\frac{\delta u}{\delta x}$ = 3

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

= (3x-2) 2x+(x^2+3)3

=$6x^2-4x+3x^2+9$

=$6x^2+3x^2-4x+9$

=$9x^2$-4x+9

Example 3

Differentiate with respect to x

y =$x^5 . e^x$

y = $x^5 . e^x$

Let u=x^5 and v =e^x

$\frac{\delta u}{\delta x}$ = $5x^4$, $\frac{\delta v}{\delta x}$= e^x

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

= $x^5 . e^x + e^x . 5x^4$

=$x^4 . e^x(x+5)$

Example 4

Differentiate y= $x^2(2x-5)^4$ *with respect to x*

y =x^2 $(2x-5)^4$

Let u=x^2 and v = $(2x-5)^4$

$\frac{\delta u}{\delta x}$ = 2x

$$\frac{\delta v}{\delta x} = \frac{\delta y}{\delta x} = \frac{\delta y}{\delta x} . \frac{\delta u}{\delta x}$$

Let u=2x – 5 and y =u^4 (as in function of a function (composite function) as will be explained in module 3.3)

$\frac{\delta u}{\delta x} = 2$ $\frac{\delta y}{\delta x}$ = $4u^3$

$\frac{\delta v}{\delta x} = \frac{\delta y}{\delta x}$ = $4u^3 . 2 = 4(2x-5)^3 . 2$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$=x^2 x4[2x-5]^3\, X\, 2+[2x-5]^4\, X\, 2x$

$=2x^2\, X\, 4[2x-5]^3+(2x-5)^4 2x$

Simplify as far as possible by collecting common terms and leaving the result in factors

$= 2x\, [2x-5]^3\, [4x+2x-5]$

$=2x[2x-5]^3[6x-5]$

Example 5

Differentiate $y=(3x-1)^3(x^2+5)$

$y=(3x-1)^3(x^2+5)$

Let $u=(3x-1)^3$ and $v=(x^2+5)$

$\frac{\delta u}{\delta x}=9(3x-1)^2$

$\frac{\delta v}{\delta x}=2x$

$$\frac{\delta y}{\delta x}=\frac{u\delta v}{\delta x}+\frac{v\delta u}{\delta x}$$

$=(3x-1)^3 2x+(x^2+5)9\,(3x-1)^3$

Collecting the common terms and leaving the result in factors

$=(3x-1)^2[2x(3x-1)\quad+9\,(x^2+5)]$

$=(3x-1)^2\,(6x^2-2x+9x^2+45)$

$\frac{\delta y}{\delta x}=\ (3x-1)^2\,(15x^2-2x+45)$

Checkpoints

Differentiate the following with respect to x

1. $y=(x^2-1)\,(x^3+1)$
2. $y=x(x^2-1)^{\,3}$
3. $y=e^x.\sin x$
4. $y=4x^3.\sin x$
5. $y=3x^3.e^3.$

Solution

1. y = $(x^2 - 1)\ (x^3 + 1)$

Let u= $(x^2 - 1)$ and v = $(x^3 + 1)$

Using the formula:

$$\frac{\delta u}{\delta x} = ex, \qquad \frac{\delta v}{\delta x} = 3x^2$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$=(x^2 - 1)(3x^2) + (x^3 + 1)\ 2x$

$$= 3x^4 - 3x^2 + 2x^4 + 2x$$

$$= 3x^4 + 2x^4 - 3x^2 + 2x$$

$=5x^4 - 3x^2 + 2x$

$=x(5x^3 - 3x + 2)$

2. y $=x(x^2 - 1)^3$

u =x, and v = $(x^2 - 1)^3$

$\frac{\delta u}{\delta x} = 1, \qquad \frac{\delta v}{\delta x} = 6x(x^2 - 1)^2$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$= 6x(x^2 - 1)^2 . x + (x^2 - 1)^3 . 1$

$=6x^2(x^2 - 1)^2 + (x^2 - 1)^3$

$=(x^2 - 1)^2\ (6x^2 + x^2 - 1)$

$$= (x^2 - 1)^2(7x^2 - 1)$$

3. y $=e^x . \sin x$

Let u$=e^x$, v=sin x

$$\frac{\delta u}{\delta x} = e^x, \qquad \frac{\delta v}{\delta x} = \cos x$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$= e^x . \cos x + \sin x . e^x$

$=e^{x}(\cos x + \sin x)$

4. $y = 4x^3 . \sin x$

$u = 4x^3$, and $v = \sin x$

$$\frac{\delta u}{\delta x} = 12x^2, \qquad \frac{\delta y}{\delta x} = cosx$$

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$= 4x^3\cos x + \sin x . 12x^2$

$=4x^2(x\cos x + 3\sin x)$

5. $y = 3x^3 . e^{x}$.

$u = 3x^3, v = e^{x}$

$$\frac{\delta u}{\delta x} = 9x^2, \qquad \frac{\delta v}{\delta x} = e^{x}$$

Therefore,

$$\frac{\delta y}{\delta x} = \frac{u\delta v}{\delta x} + \frac{v\delta u}{\delta x}$$

$3x^3 . e^{x} \ + e^{x} . 9x^2$

$3x^2 . e^{x} \ + e^{x}(x + 3)$

Differentiation of a quotient of two function (Quotient Rule)

Let $y = \frac{u}{v}$ where u and v are functions of x.

An increment $\delta x \ in \ x \ \ will \ turn \ to \ produce \ increment \ and \ a \ change \ \delta u \ in \ u$ and a change $\delta v \ in \ v \ and \ a \ change \ \delta y \ in \ y$

If x → x+ $\delta x, u \rightarrow u + \delta u, v \rightarrow v + \delta v \ and \ as \ a \ result, y \rightarrow y + \delta y$

Using the first principle

If $y = \frac{u}{v}$ . . . .(1)

Then $y + \delta y = \frac{u+\delta u}{v+\delta v}$ . . . .(2)

Subtract y from both side of the term in equation (2) above

$$y + \delta y - y = \frac{u + \delta u}{v + \delta v} - y$$

$$\delta y = \frac{u + \delta u}{v + \delta v} \frac{-u}{v} \quad [where \ y = \frac{u}{v} \quad in \ (i)$$

$$\delta y = \frac{v(u + \delta u) - u(v + \delta v)}{(v + \delta v)v}$$

$\delta y = \frac{uv + v\delta u - uv - u\delta v}{\mathrm{v}^2 + v\delta v}$ Simplify by collecting like terms.

$$\delta y = \frac{v\delta u - v\delta u}{\mathrm{v}^2 + v\delta v}$$

$$\delta y = \frac{v\delta u + \ u\delta v}{\mathrm{v}^2 \ + \ v\delta v}$$

$$dividing\ both\ side\ by\ \delta x$$

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} + \frac{u\delta v}{\delta x}}{v^2 + v\,dv}$$

$$if\ \delta x \to 0, \delta u \to 0\ and\ \delta x \to 0, then\ equation\ (3)$$

becomes

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{\mathrm{V}^2 + v(0)}$$

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v^2}$ . . . .(4)

Then equation (4) is the quotient rule of differentiation

Example 1

1. $if\ y = \frac{x^2}{\sqrt{x+1}}\ differentiate\ with\ respect\ to\ x$

Solution

$$y = \frac{x^2}{\sqrt{x+1}}\ since\ the\ function\ above\ is\ the\ form\ y = \frac{u}{v}, we\ will\ use\ quotient\ rule$$

$$let\ u = x^2, v = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$$

$using\ the\ quotient\ rule\ \frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v^2}$

$\frac{\delta u}{\delta x} = 2x, \frac{\delta v}{\delta x} = \frac{1}{2\sqrt{x+1}}\ (i.e.\ differentiating\ u\ and\ v\ with\ respect\ to\ x$

$$\therefore \frac{\delta y}{\delta x} = \frac{\sqrt{x+1}\,.2x - x^2.\,2\,\frac{1}{\sqrt{x+1}}}{(x+1)^{½.2}}$$

$\frac{\delta y}{\delta x}$

$= \frac{\sqrt{x+1}\,.2x - x^2.2\frac{1}{\sqrt{x+1}}}{x+1}$ $using\ indices\ law\ to\ simplify\ the\ denominator\ i.e\ (a^{\frac{1}{n}})^m$

$= \ a^{\frac{m}{n}}$

$$= \frac{\sqrt{x+1}\,.2x - \frac{x^2}{2\sqrt{x+1}}}{x+1} simplifying\ the\ fration\ in\ the\ numerator.$$

$=\frac{2x\sqrt{x+1}\,.2\sqrt{x+1} - \frac{x^2}{2\sqrt{x+1}}.2\sqrt{x+1}}{2\sqrt{x+1}.x+1}$

$$= \frac{2\sqrt{x+1}.\sqrt{x+1}\,.2x - x^2}{2\sqrt{x+1}.(x+1)}$$

$$= \frac{4x.\sqrt{x+1}.\sqrt{x+1} - x^2}{2(x+1)^{½}.(x+1)}$$

Simplifying the numerator $\sqrt{a}\,.\sqrt{a} = a\ and\ the\ numerator\ using\ indices\ a^m.a^n = a^{m+n}$

$$= \frac{4 \times (x+1) - x^2}{2(x+1)^{½+1}}$$

$$= \frac{4 \times (x+1) - x^2}{2(x+1)^{\frac{3}{2}}}$$

$= \frac{4x^2 + 4x - \ x^2}{2(x+1)^{\frac{3}{2}}}$

$= \frac{3x^2 + 4x}{2(x+1)^{\frac{3}{2}}}$

$= \frac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}$

Example 2:

Differentiate with respect to x, if y=$\frac{\sin x}{x^2}$

Let u=sin x, v=x^2

$\frac{\delta u}{\delta x}$ = cos x, $\frac{\delta v}{\delta x}$ =2x

$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v^2} = \frac{x^2.cosx - sinx.2x}{x^{(2)2}}$

$$= \frac{x^2cosx - 2xsinx}{x^4}$$

$$\frac{x^2cosx}{x^4} - \frac{2xsinx}{x^4}$$

$$\frac{xcosx}{x^3} - \frac{2sinx}{x^3}$$

$$\frac{xcosx - 2sinx}{x^3}$$

Example 3:

If y=$\frac{5e^x}{cos^x}$, dfferentiate with respect to x.

Solution

$$y = \frac{4e^x}{cosx}$$

Let u=4e^x and =cos x

$\frac{\delta u}{\delta x} = 4e^x, \frac{\delta v}{\delta x} = -sinx$

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v^2} = \frac{cosx(4e^x)\; - \;4e^x(-sinx)}{cos^2x}$$

$$\frac{cosx(4e^x) + \;4e^x sinx}{cos^2x}$$

$$\frac{4e^x(cosx + sinx)}{cos^2x}$$

Example 4

When $y = \frac{sinx}{cosx}$, differentiate with respect to x

Solution

Let u= sin x and v = cos x

$$\frac{\delta u}{\delta x} = cosx, \frac{\delta v}{\delta x} = -sinx$$

Using quotient rule

$$\frac{\delta y}{\delta x} = \frac{\frac{v\delta u}{\delta x} - \frac{u\delta v}{\delta x}}{v^2}$$

$= \frac{cosx(cosx) - sinx(-sinx)}{cos^2x}$

$=\frac{cos^2x+sin^2x}{cos^2x}$ $(in\ trigonometry\ identety\ sin^2\theta+\ cos^2\theta=1)$

$$=\frac{1}{cos^2x}=sec^2x$$

Checkpoint

Differentiate the following with respect to x:

a) y = $\frac{4x+1}{7x-4}$ (**Answer**: $\frac{\delta y}{\delta x}=\frac{-a}{(7x-4)}$)

b) y = $\frac{x^2-1}{x^3}$ (**Answer**: $\frac{\delta y}{\delta x}=\frac{x(-x^2+3x+2)}{(x^2+)^2}$)

c) $y=\frac{cosx}{sinx}$ (**Answer**: $\frac{\delta y}{\delta x}=-cosec^2x$)

3.5 FUNCTION OF A FUNCTION (COMPOSITE FUNCTION)

When differentiating a function of a function, we will have to introduce the chain rule

$$let\ say, if\ y=\frac{1}{(5x-6)^2}=(5x-6)^{-2}.$$

This cannot be expressed as a polynomial and it leave us to devise a very important techniques to solve such a function. To solve such a function in$\frac{dy}{dx}$, we have the differentiate$(5x-6)^{-2}$ as a **composite function** or as a **function of a function** which can be differentiated from two main functions i.e

(5x-6) by putting u=(5x-6)

And let y =$u^{-2}\ where\ u=5x-6$

Hence,if **x**has increase in dx,**u** will have an increase δu and then **y** will have an increase δy i.e.

$x\rightarrow x+\delta x, u\rightarrow u+\delta u\ and\ y\rightarrow y+\delta y.$

At this stage, the increase $\delta x, \delta u\ and\delta y$ are all finite values and therefore we can say that $\frac{\delta y}{\delta x}=\frac{\delta y}{\delta x}.\frac{\delta u}{\delta x}$ Because the $\delta u\ in\frac{\delta y}{\delta u}$ cancels the δu in $\frac{\delta u}{\delta x}$.

Also $\frac{\delta y}{\delta x}\rightarrow\frac{\delta u}{\delta x}$ and the previos statement now becomes:

$$\boxed{\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}.\frac{\delta u}{\delta x}}$$

This is the **chain rule** and is very useful in determining the derivatives of function of functions of a function.

Example 1

$$if\ y = \frac{1}{(6x-5)^2}, differentiate\ with\ respect\ to\ x$$

Solution

y = $\frac{1}{(6x-5)^2}$ $\quad = (6x-5)^{-2}$

let u = (6x – 5), therefore y = u^2

$\frac{\delta u}{\delta x}$ = 6 and $\frac{\delta y}{\delta x}$ = $-2u^{-3}$ = $\frac{-2}{u^3}$ (Indices $a^{-n} = \frac{1}{a^n}$)

Using chain rule $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$

= $\frac{-2}{u^3}$. 6 = $\frac{-12}{u^3}$

$\frac{\delta y}{\delta x}$ = $\frac{-12}{u^3}$ (replacing value of u)

$\frac{-12}{(6x-5)^3}$ (replacing back the value of u)

Example 2

If y = $(2x + 8)^2$, find the value of $\frac{\delta y}{\delta x}$.

Solution

y = $(2x + 8)^2$ . . . .(1)

let u = (2x + 8) . . . . (2)

Therefore, y = u^3 . . . .(3) (substituting u into equation(1))

Differentiating u with respect to x and differentiating y in equation(3) with respect to u.

$\frac{\delta u}{\delta x}$ = 2, $\frac{\delta y}{\delta u}$ = $3u^2$ = $3u^2$

Using $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$ (Chain rule)

$\frac{\delta y}{\delta x}$ = 2 x $3u^2$ = $6u^2$

$\frac{\delta y}{\delta x}$ = $6u^2$ . . . .(4)

Replacing back the value of u into equation(4)

$\frac{\delta y}{\delta x}$ = $6(2x + 8)^2$

Example 3

If y = sin(5x – 6), determine $\frac{\delta y}{\delta x}$.

Solution

y = sin(5x – 6)

let u = (5x – 6), therefore, y = sin u

$\frac{\delta u}{\delta x}$ = 5, $\frac{\delta y}{\delta u}$ = cos u

$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} . \frac{\delta y}{\delta u}$ = 5cos u

= 5 cos(5x – 6)

Example 4

Determine $\frac{\delta y}{\delta x}$ when y = tan(3x + 2).

Solution

Y tan(3x + 2)

Let u = 3x = 2, therefore, y = tan u

$\frac{\delta u}{\delta x}$ = 3 and $\frac{\delta y}{\delta u}$ = sec^2u

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} . \frac{\delta y}{\delta u}$$

= $3sec^2u$

= $3sec^2(3x =2)$

Checkpoint

Differentiate the following with respect to x:

i. $y = (2x + 5)^3$ [**Answer:** $6(2x + 5)^2$]

ii. ii. y = $\sqrt{4x-3}$ [**Answer:** $\frac{2}{(4x-3)^{\frac{3}{2}}}$]

iii. y = sin(4x + 3) [**Answer:** 4cos(4x + 3)]

Exercises

Differentiate the following with respect to x:

i. $y = (2x + 5)^3$ ii. y = $\sqrt{4x-3}$ iii. y = sin(4x + 3)

5.6 IMPLICIT FUNCTIONS

When functions that are differentiated which in the form y= x^3 + 5x -3 are called an **explicit function** of x. i.e. y is stated directly in terms of x.

A relationship in which x and y is more involved may not be possible to separate y completely on the left hand side.

For example xy + cosy = 5, in this case, y is called an **implicit function**of x, because a relationship of the form y = f(x) is implied in the given equation.

Steps for Differentiating Implicit function

When differentiating implicit function, it is important to determine the derivate y with respect to x and while doing so, it is the derivative of the function. It is very important to multiply the differential function of y by the derivative of the function. It is very important to also notice that derivative of constant number is zero.

Example 1

If $2x^2 + 3y^2 = 16$, find $\frac{\delta y}{\delta x}$

Solution

$2x^2+3y^2 =16$

Differentiating the above term with respect to x

4x + 6y $\delta y/\delta x$ = 0

Subtracting L. H. S. from R. H. S. by 4x

4x + 6y $\delta y/\delta x$– 4x = 0 - 4x

6y $\delta y/\delta x$= -4x

Making $\delta y/\delta x$ subject of the formula by divding both side by 6y

$\delta y/\delta x = \frac{-4x}{6y}$

$\delta y/\delta x = \frac{-4x}{6y}$

Example 2

If $2x^2+ 3y^2 - 4x – 3y + 6 = 0$, find $\delta y/\delta x$ and ${d^2y}/{dx^2}$ at x = 3, and y = 2

Solution

$2x^2+ 3y^2 - 4x – 3y + 6 = 0$

Differentiating the above expression with respect to x, 4x + 6y$^{dy}/_{dx}$ – 4 - 3$^{dy}/_{dx}$ + 0 = 0

Collecting like terms

6y$^{dy}/_{dx}$ – 3$^{dy}/_{dx}$ + 4x – 4 = 0

6y$^{dy}/_{dx}$ – 3$^{dy}/_{dx}$ = 4 – 4x

$^{dy}/_{dx} = \frac{4-4x}{6y-3} = \frac{4(1-x)}{3(2y-1)}$

∴ at (x, y) = (3, 2);

$^{dy}/_{dx} = \frac{4-4(3)}{6(2)-3} = \frac{4-12}{12-3} = \frac{-8}{9}$

∴ $^{dy}/_{dx} = \frac{-8}{9}$ [Substituting the values of x and y into values of $^{dy}/_{dx}$]

To calculate $^{d^2y}/_{dx^2}$, since $^{dy}/_{dx} = \frac{-4-4x}{6y-3}$, we have to differentiate $^{dy}/_{dx}$ once to get

$^{d^2y}/_{dx^2}$.

Then $^{d^2y}/_{dx^2} = \frac{d}{dx}\{\frac{4-4x}{6y-3}\}$

Using quotient rule to differentiate since the function is the form $\frac{u}{v}$

Let u = 4 – 4x and v = 6y – 3, $\frac{du}{dx}$ = -4, $\frac{dv}{dx} = 6\frac{dy}{dx}$

Hence, $\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{(6y-3).-4-(4-4x)6\frac{dy}{dx}}{(6y-3)^2}$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{-4(6y-3) - (4-4x)6\frac{dy}{dx}}{(6y-3)^2}$$

Therefore, at (x, y) = (3, 2) and $\frac{dy}{dx} = \frac{-8}{9}$

Substituting the above terms into $\frac{d^2y}{dx^2}$,

$= \frac{-4(6y-3)-(4-4x)6\frac{dy}{dx}}{(6y-3)^2}$

$= \frac{-4(6(2)-3)-(4-4(3))6\left(\frac{-8}{9}\right)}{(6(2)-3)^2}$ $\quad = \frac{-4(12-3)-(4-12)6\left(\frac{-8}{9}\right)}{(12-3)^2}$

$= \frac{-4(9)-(8)6(\frac{-8}{9})}{(9)^2}$ $\quad = \frac{-36-(-8)2(\frac{-8}{3})}{81}$

$= \frac{-36-(-16)(\frac{-8}{3})}{81}$ $= \frac{-36-(\frac{128}{3})}{81}$

$= \frac{\frac{-108-128}{3}}{81}$ $= \frac{\frac{-236}{3}}{81}$ $= \frac{-236}{3} \cdot \frac{1}{81}$ $= \frac{-236}{243}$

Example 3

Find $\frac{dy}{dx}$, if $x^3 + y^3 = 2xy$

Solution

$x^3 + y^3 = 2xy$

Differentiating with respect to y, we have to treat 2xy as product of the function.

$= 3x^2 + 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$

Collecting like terms (i.e. common terms)

$3y^2\frac{dy}{dx} – 2x\frac{dy}{dx} = 2y – 3x^2$

$\frac{dy}{dx}(3y^2 – 2x) = 2y – 3x^2$

$\frac{dy}{dx} = \frac{2y-3x^2}{3y^2-2x}$ [dividing the both sides by $3y^2 – 2x$)

Example 4

Find the equation of the tangent(s) where x = 2on the curve $x^2+y^2= 2x +y =6$.

Solution

Differentiating the expression with respect to x

$2x + 2y\, \delta y/\delta x – 2 + \delta y/\delta x = 0$

$2y\, \delta y/\delta x + \delta y/\delta x = 2 - 2x$

$\delta y/\delta x(2y + 1) = 2 - 2x$ (collecting like term and common term)

$\delta y/\delta x = \frac{2-2x}{2y+1}$

Substituting the value of x=2 in the original equation of the curve, we find:

$x^2 + y^2 - 2x + y = 6$

$(2)^2 + y^2 - 2(2) + y = 6$

$4 + y^2 - 4 + y = 6$

$y^2 + y - 6 = 0$

$(y + 3)(y - 2) = 0$

$y = -3 \text{ or } 2$

There are two points on the curve where x=2. (2,-3) and (2,-2)

Gradient at point (2,-3)

m = $\frac{\delta y}{\delta x} = \frac{2-2(2)}{2(-3)+1} = \frac{2-4}{-6+1} = \frac{-2}{-5} = \frac{2}{5}$

Equation of the tangent = $y - y_1 = m(x - x_1)$

Where$y_1 = -3,\ x_1 = 2,$ m = $\frac{2}{5}$

$y—(-3) = {}^{2}\!/_{5}\,(x-2)$

$y + 3 = {}^{2}\!/_{5}\,(x-2)$, $y + 3 = {}^{2}\!/_{5}\,x - {}^{4}\!/_{5}$

Multiplying throughout by 5

$5 \times y + 5 \times 3 = 5 \times {}^{2}\!/_{5}\,x - {}^{4}\!/_{5} \times 5$

$5y + 15 = 2 \times -4$

$5y - 2x + 15 + 4 = 0$

$5y - 2 \times +19 = 0$

Gradient at point (2,-2)

m= $\frac{\delta y}{\delta x} = \frac{2-2x}{2y+1} = \frac{2-2(2)}{2(2)+1} = \frac{2-4}{4+1} = \frac{-2}{5}$

m= ${}^{-2}\!/_{5}$

Equation of tangent = $y - y_1$= m(x-x_1)

$y - 2 = {}^{-2}\!/_{5}\,(x-2)$

$y - 2 = {}^{-2}\!/_{5}\,x + \frac{4}{5}$

Multiplying throughout by 5

$5y - 10 = -2x + 4$

$5y + 2x - 10 - 4 = 0$

$5y + 2x - 14 = 0$

Checkpoint

I. If $x^2 + 2xy + 3y^2 = 4$ find $\frac{\delta y}{\delta x}$ Ans $\frac{\delta y}{\delta x} = \frac{-x-y}{x+3y}$

II. if $x^3 + y^3 + 3xy^2 = 8$ find $\frac{\delta y}{\delta x}$ Ans $\frac{\delta y}{\delta x} = \frac{-(x^2+y^2)}{y^2+2y}$

1.7 Exercises

Exercises D

I. If $x^2 + y^2 - 2x + 2y = 23$, find $\frac{\delta y}{\delta x}$ and $\frac{\delta^2 y}{\delta x}$ at the point where x= -2, y= 3

II. Find $\frac{\delta y}{\delta x}$ for these following implicit functions:

a. $xy - x^3 = 6$ b. $x^2 + y^2 = 8$ c. $x^2 + 6xy = 4y$

d. $x^3 + y^3 - x - y = 3$

Exercise A

Differentiate the following with respect to x simply as far as possible, leaving your answer in factor

I. $(2x-1)(x+4)^2$ II. $3x^3(x^2+4)^2$ III. $\sqrt{x}(x+3)^2$

IV. $y = 5x^3. \sin x$ V. $y = \cos x. \sin x$VI. $y = e^x. \cos x$

VII. $y = 2x^5.\cos x$

Exercise B

Differentiation the following with respect to x:

I. $y = \frac{5x^2}{\cos x}$ II. $y = \frac{6e^x}{\sin x}$ III. $y = \frac{\cos x}{x^5}$

IV. $y = \frac{x^3+1}{x-1}$ V. $y = \frac{2x^2-x+3}{2x-5}$

Exercise C

Differentiate the following with respect to x

I. $y = (7x-2)^7$ II. $y = (5x^3 - 2)$ III. $y = \frac{1}{x^2+2x-3}$

IV. $y = \cos(7x+3)$ V. $y = e^{2x-3}$

1.8 Summary

I. If $y = \frac{u}{v}, \frac{\delta y}{\delta x} = \frac{V\frac{\delta U}{\delta X} - U\frac{\delta V}{\delta X}}{V^2}$

II. If$y = uv, \frac{\delta y}{\delta x} = U\frac{\delta V}{\delta X} + V\frac{\delta U}{\delta X}$

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MODULE TWO

UNIT 3: INTEGRATION

Contents

6.1Introduction

Integration is a way of adding slices to find the whole. Integration can be used to find areas, volumes, central points and many useful things. Integration is like filling a tank from a tap. The input (before integration) is the flow rate from the tap. Integrating the flow (adding up all the little bits of water) gives us the volume of the water in the tank.

Hence the processes of integration are used in many applications.

(1) The Petronas Tower in Kuala Lumpur, experience high forces due to winds. Integration was used to design the building for its strength
(2) It is used in finding areas under curve surfaces, centre of mass – displacement and velocity, fluid flow, modelling the behaviour of object under stress (e.g. Car engine block, piston, curvature beam of houses, bridges e.t.c)

6.2 Objective:

In this module, the student will learn the following topics and its application:

6.3 Integration

6.4 Notation of Integration

6.3 Integration

Integration simply means the inverse operation to differentiation. Let say when y = $x^{n,}$ the derivate i.e. $\frac{\delta y}{\delta x}$= n x^{n-1}. If we differentiate $\frac{1}{n+1}x^{n+1} = \frac{x^{n+1}}{n+1}$ with respect to x.

$\therefore \frac{\delta y}{\delta x} = n + 1.\ \frac{1}{n+1}.\ x^{n+1-1}$.

$$\frac{\delta y}{\delta x} = x^n$$

Therefore, when $\frac{\delta y}{\delta x}$ = x^n, then y =$\frac{x^{n+1}}{n+1}$, that is to say the integral of x^n with respect to x is $\frac{x^{n+1}}{n+1}$ (where n≠ 1).

When integrating, it is very important that you show a constant term called an **arbitrary constant c**. The reason is that when we integrate $3x^2 - 1$, it might be for the following function as their differentiatione.g. $x^3 - x + 5, x^3 - x$in each case. When integrating$3x^2 - 1$, the constant is not always recovered. In order to show there is a constant term in the integral, the arbitrary constant is added.

6.4 Notation for Integration

$\int$ is theSymbolof integration (integral sign) and both $\int$ and δx must be written. **Integrand** is the function to be integrated and it is place in between the$\int$ and δx. $\delta x\ is$ written to illustrate that the integrand is to be integrated.

If $\frac{\delta y}{\delta x}$ = f(x), v=$\int f(x)\, dx$ + c where c is any constant.

Example1

Integrate the following:

a. $\int(x^3 + 3x^2 + 2x + 4)\, dx$ b.$\int t^3 + 4t^2 - 2\ dt$c.$\int(s^3 + 4s)ds$

Solution

Using the general formula: $\int x^n\ dx = \frac{x^{n+1}}{n+1} + c$

1. $\int(x^3 + 3x^2 + 2x + 4)\, dx$

$=\frac{x^{3+1}}{3+1}+\frac{3x^{2+1}}{2+1}+\frac{2x^{1+1}}{1+1}+\frac{4x^{0+1}}{0+1}$ +c **In indices law: a^0 = 1, i.e. $4x^0 = 1$.**

$=\frac{x^4}{4}+\frac{3x^3}{3}+\frac{2x^2}{2}+\frac{4x}{1}$ +c **Simplifying.**

$=\frac{x^4}{4}+x^3+x^2+4x$ + c

2. $\int(s^3+4s)ds$

$=\frac{s^{3+1}}{3+1}+\frac{4s^{1+1}}{1+1}$ + c

$=\frac{s^4}{4}+\frac{4s^2}{2}$ + c

$=\frac{s^4}{4}+2s^2$ + c

3. $\int(t^3+4t^2-2)dt$

$=\frac{t^{3+1}}{3+1}+\frac{4t^{2+1}}{2+1}-\frac{2t^{0+1}}{0+1}$ + c

$=\frac{t^4}{4}+\frac{4t^3}{3}$ – 2t + c

6.4 Standard Integral.

I. $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ + c (provided n$\neq$ 1)

II. $\int\frac{1}{x}dx$ =ln x +c

(3) $\int e^x\,dx = e^x + c$

(4) $\int e^{kx}dx = \frac{e^{kx}}{k}$ + c

(5) $\int \sin x dx = -\cos x + c$

(6) $\int cosxdx = \sin x + c$

(7) $\int \sec^2 xdx = \tan x$ +c

(8) $\int \sinh xdx = \cosh x + c$

IX. $\int \cosh xdx$ =sinh $x + c$

X. $\int\frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}x + c$

XI. $\int\frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + c$

XII. $\int\frac{1}{\sqrt{1+x^2}}\,dx = \tan^{-1}x + c$

XIII. $\int\frac{1}{\sqrt{x^2+1}}\,dx$ =$\sinh^{-1}x + c$

XIV. $\int\frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + c$

XV. $\int\frac{1}{1-x^2}dx = \tanh^{-1}x + c$

XVI. $\int a^x dx = \frac{a^x}{\ln a}$ + c

6.5 Methods of Integration

Functions of a Linear function of x

In functions of a linear function, we say that an alphabet should replaced by a linear function. If the alphabet stands for the linear function, the integral becomes $\int$ " an alphabet" δx and before we complete the operation, we must change the variable.

Hence: $\int$ " an alphabet" δx =$\int$ " alphabet" $\frac{dx}{d\text{"}alphabet\text{"}}$. $\delta(alphabet)$

Example 1

$$\int (3x - 2)^6 \delta x$$

Solution

Let z = 3x – 2

$\therefore \int (3x\ \text{-}2)^6 \delta x = \int z^6\ \delta x = \int z^6 . \frac{\delta x}{\delta z} . \delta z$

$= \int z^6 . \frac{\delta x}{\delta z} . \delta z \left[\boldsymbol{\frac{\delta x}{\delta z} . \delta z = \ \delta x} \right]$.

Now $\frac{\delta x}{\delta z}$ can be found from the substitution z = 3x – 2.

For $\frac{\delta z}{\delta x}$ = 3 **Differentiating z with respect to x.**

$\therefore \frac{\delta z}{\delta x} = \frac{1}{3}$. **Since $\boldsymbol{\frac{\delta z}{\delta x} = \frac{3}{1} \leftrightarrow \delta z. 1 = 3. \delta x \leftrightarrow \frac{\delta x}{\delta z} = \frac{1}{3}}$.**

The integral becomes:

$\int z^6\ \delta x = \int z^6 . \frac{\delta x}{\delta z} . \delta z = \int z^6 . (\frac{1}{3}) . \delta z = \frac{1}{3} \int z^6 . \delta z$**Integrating w.r.t.x**

$= \frac{1}{3} . \frac{z^7}{7} + c$

Finally, replacing the value of z in its terms of its original variable, x, so that: $\int (3x\ \text{-}2)^6 \delta x$ = = $\frac{1}{3} . \frac{z^7}{7}$ + c = = $\frac{1}{3} . \frac{(3x-2)^7}{7}$ + c

$= \frac{(3x-2)^7}{21} + c$

Example 2

$$\int \cos(6x + 4)\,\delta x$$

Solution

Let z = 6x + 4

$\int cos(6x + 4)\delta x = \int \cos z\,\delta x$

$= \int \cos z.\frac{\delta x}{\delta z}.\delta z$

$\frac{\delta z}{\delta x} = 6$

$\therefore \frac{\delta x}{\delta z} = \frac{1}{6}$

$\int \cos z.\frac{\delta x}{\delta z}.\delta z = \int \cos z.\frac{1}{6}.\delta z = \frac{1}{6}\int \cos z.\delta z$

$= \frac{1}{6}\sin z + c = \frac{1}{6}\sin(6x + 4) + c$

$= \frac{\sin(6x+4)}{6} + c$

Example 3

$$\int sec^2 8x\,\delta x$$

Solution

Let z = 8x

$\int sec^2 8x\,\delta x = \int sec^2 z\,\delta x = \int sec^2 z.\frac{\delta x}{\delta z}.\delta z$

$\frac{\delta z}{\delta x} = 8$

$\therefore \frac{\delta x}{\delta z} = \frac{1}{8}$

$\int sec^2 z.\frac{\delta x}{\delta z}.\delta z = \int sec^2 z.\frac{1}{8}.\delta z = \frac{1}{8}\int sec^2 z.\delta z$

$= \frac{1}{8}\tan z + c = \frac{\tan z}{8} + c = \frac{\tan 8x}{8} + c$

Integrals of the form $\int \frac{f'(x)}{f(x)} \delta x$ and $\int f(x).f'(x)\, \delta x$

Integral in the form$\int \frac{f'(x)}{f(x)}$ is any in which the numerator is the derivative of the denominator.

It is of the kind $\int \frac{f'(x)}{f(x)} \boldsymbol{\delta x}$ = $\ln\{f(x)\}$ +c .

Example 1

Let us consider the integral $\int \frac{2x-5}{x^2-5x+6} \boldsymbol{\delta x}$

We notice that when we differentiate the denominator, we will have the expression in the numerator.

Let z = $x^2 - 5x + 6$

$\therefore \frac{\delta z}{\delta x}$ = 2x – 5

δz = (2x – 5) $\boldsymbol{\delta x}$ **Making $\boldsymbol{\delta z}$ subject of the formula.**

δz = (2x – 5) δx... (1)

$\therefore$ The given integral can be rewritten in terms of z.

$\int \frac{2x-5}{x^2-5x+6} \boldsymbol{\delta x} = \int \frac{2x-5}{z} \boldsymbol{\delta x}$

Substituting equation (1) into the above term

$\int \frac{2x-5}{z} . \boldsymbol{\delta x} = \int \frac{\delta z}{z} = \int \frac{1}{z} . \delta z$**Where** $\boldsymbol{\frac{\delta z}{z} = \frac{1}{z} . \delta z}$

$= \int \frac{1}{z} \delta z = \ln z + c$

Replacing back the value of z.

$\ln z + c = \ln(x^2 - 5x + 6) + c$

Example 2

Integrate $\int \frac{3x^2}{x^3-6} \boldsymbol{\delta x}$

Solution

Let z = $x^3 - 6$

$\frac{\delta z}{\delta x}$ = $3x^2$, δz = $(3x^2)$ $\boldsymbol{\delta x}$

$\therefore \int \frac{3x^2}{x^3-6} \boldsymbol{\delta x} = \int \frac{3x^2}{z}.\ \delta x = \int \frac{\delta z}{z} = \int \frac{1}{z}\ \delta z = \ln z$ + c

= $\ln(x^3 - 6)$ + c

Example 3

$$\int \frac{4x-8}{x^2-4x+5} \boldsymbol{\delta x}$$

Solution

$$\int \frac{4x-8}{x^2-4x+5} \boldsymbol{\delta x}$$

= $\int 2.\ \frac{2x-4}{x^2-4x+5}$ $\boldsymbol{\delta x}$

Note:When differentiating the denominator $\boldsymbol{x^2 - 4x + 5}$, it gives 2x− 4. Looking at the numerator, collecting common terms give 2(2x−4).

= $2\int \frac{2x-4}{x^2-4x+5} \delta x$**Placing the constant number before the integral sign.**

Let z = $x^2 - 4x + 5$,

$\frac{\delta z}{\delta x}$ = 2x – 4, δz = (2x – 4) δx

= $2\int \frac{2x-4}{x^2-4x+5} \delta x$= $2\int \frac{\delta z}{z}$ = $2\int \frac{1}{z}\ \delta z$ = 2ln z + c

= $2\ln(x^2 - 4x + 5)$ + c

Example 4

$$\int \cot x \,\boldsymbol{\delta x}$$

Solution

From trigonometric function $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ **where** $\boldsymbol{\tan x = \frac{\sin x}{\cos x}}$

Substituting $\cot x$ as $\frac{\cos x}{\sin x}$.

$\int \cot x \,\boldsymbol{\delta x} = \int \frac{1}{\tan x} \boldsymbol{\delta x} = \int \frac{\cos x}{\sin x} \,\boldsymbol{\delta x}$

Let z = $\sin x$

$\frac{\delta z}{\delta x} = \cos x$, $\delta z = \cos x \ \delta x$

$\therefore \int \frac{\cos x}{\sin x} \,\boldsymbol{\delta x} = \int \frac{\cos x}{z} \,\boldsymbol{\delta x} = \int \frac{\delta z}{z} = \int \frac{1}{z} \ \delta z$

= $\ln z$+ c

= $\ln \sin x$+ c

Integration of Product (Integration by parts).

If u and v are functions of x, then we known that $\frac{\delta}{\delta x}(uv)$ = u$\frac{\delta v}{\delta x}$ + v$\frac{\delta u}{\delta x}$. Integrating both side with respect to x. We noticed on the left, we get back to the function from which we started.

uv = $\int u \frac{\delta v}{\delta x}.\, \delta x + \int v \frac{\delta u}{\delta x}\,.\, \delta x$

Rearranging the terms, we have:

$\int u \frac{\delta v}{\delta x}.\, \delta x$ = uv$- \int v \frac{\delta u}{\delta x}\,.\, \delta x$**Clearing the common factors**.

$\int u\delta v$ = uv $- \int v \,\delta u$.

Example1

$$\int x^2. \ln x \,\delta x$$

Solution

Let u = $\ln x$, v = x^2. **Since $\ln x$ is not in the list of standard integrals.**

$\frac{\delta u}{\delta x} = \frac{1}{x}$, $\delta u = \frac{1}{x} . \delta x$

$\frac{\delta v}{\delta x} = x^2$

Integrating it $\int \frac{\delta v}{\delta x} = \int x^2 \ \delta x$

v = $\frac{x^3}{3}$**Omitting integration constant because of evaluation in the middle.**

$$\int x^2 . \ln x \, \delta x$$

Using $\int u\delta v$ = uv $- \int v \, \delta u$.

$= \ln x \left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} . \frac{1}{x} \delta x$

$= \ln x \left(\frac{x^3}{3}\right) - \int \frac{1}{3} . x^3 . \frac{1}{x} \delta x$

$= \ln x \left(\frac{x^3}{3}\right) - \frac{1}{3} \int x^3 . \frac{1}{x} \delta x$**Simplifying $\boldsymbol{x^3 . \frac{1}{x} = x^2}$.**

$= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \ \delta x$

$= \frac{x^3}{3} \ln x - \frac{1}{3} . \frac{x^3}{3}$ + c **Integrating $\boldsymbol{\int x^2 \ \delta x = \frac{x^3}{3}}$ + c .**

$= \frac{x^3}{3} \left[\ln x - \frac{1}{3}\right]$ + c **Collecting common terms.**

Example 2

$\int x^2 e^{3x} \ \delta x.$

Solution.

Using $\int u\delta v$ = uv $- \int v \, \delta u$.

Let u = x^2, v = $\frac{e^{3x}}{3}$

$\int u\delta v \ = x^2.\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}.2x\,\delta x$

$= x^2.\frac{e^{3x}}{3} - \int \frac{1}{3}\,.\,e^{3x}.\,2x\,\delta x$

$= x^2.\frac{e^{3x}}{3} - \frac{2}{3}\int e^{3x}\,.\,x\delta x$

The integral $\int e^{3x}\,.\,x\delta x$ will also have to be integrated by part. So that:

$$\int e^{3x}\,.\,x\delta x$$

Let u =x ,δv = $e^{3x}\delta x$

$\frac{\delta u}{\delta x}$ = 1, δu= δx

v= $\frac{e^{3x}}{3}$

$$\int e^{3x}\,.\,x\delta x$$

Using $\int u\delta v$ = uv $- \int v\,\delta u$.

$=$ x.$\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\ \delta x$

$\therefore \frac{x^2e^{3x}}{3} - \frac{2}{3}\int e^{3x}.\,x\ \delta x$becomes:

$= \ \frac{x^2e^{3x}}{3} - \frac{2}{3}\left\{x\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\ \delta x\right\}$

$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left\{x(\frac{e^{3x}}{3}) - \frac{1}{3}\int e^{3x}\delta x\right\}$

$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\int e^{3x}\delta x$

$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\left[\frac{e^{3x}}{3} + c\right] = \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{9}\,.\,\frac{e^{3x}}{3}$ +c

$= \frac{e^{3x}}{3}\left[x^2 - \frac{2x}{3} + \frac{2}{9}\right]$ +c

Integration by partial fractions

1. **Integration by partial fraction for a irreducible denominator.**

Example 1

$$\int \frac{x+1}{x^2-3x+2}\delta x$$

When we clearly look at this, the numerator is not the derivative of the denominator. This is not an example of standard integral.

$$\int \frac{x+1}{x^2-3x+2}\delta x$$

Since the denominator is irreducible, we will have to factorize it

$\int \frac{x+1}{x^2-3x+2}\delta x = \int \frac{x+1}{(x-2)(x-1)}\delta x.$

Resolving into partial fraction

$\frac{x+1}{(x-2)\,(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)}$

x +1 = A (x – 1) + B (x – 2)

To get A, put x = 2

2+1 = A (2 – 1) + B (2 – 2)

3 = A

∴ A= 3

Similarly to get B, put x = 1

1 + 1 = A (1 – 1) + B (1 – 2)

2 = − B

B = −2

∴ $\int \frac{x+1}{(x-2)(x-1)}\delta x = \int \left(\frac{A}{x-2} + \frac{B}{x-1}\right)\delta x$

Replacing back the values A and B respectively into the term above.

$= \int \frac{3}{x-2}\delta x - \int \frac{2}{x-1}\delta x$

$= 3\int \frac{1}{x-2}\,\delta x - 2\int \frac{1}{x-1}\,\delta x$

$= 3\ln(x-2) - 2\ln(x-1) + c$

Integration of Partial Fraction by repeated rule.

Example 2:

Determine $\int \frac{x^2}{(x+1)(x-1)^2}\,\delta x$

Solution.

$\int \frac{x^2}{(x+1)(x-1)^2}\,\delta x = \int \left(\frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right)\delta x$

Resolving into partial fraction

$\frac{x^2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{c}{(x-1)^2}$ **Multiplying LHS and RHS throughout by the LCM of the denominator.** $\mathbf{(x+1)(x-1)^2}$ **.**

x^2 = A $(x-1)^2$ + B(x + 1)(x – 1) + C(x+1)..(1)

To get C, put x = 1.

$x^2 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$

$1^2 = 2C$

C= ½

To get A, put x = -1in equation (1)

Equation (1) becomes:

$(-1)^2 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$

$1 = A(-2)^2$

1=4A, A= ¼

To get B, using equation (2) taking the coefficient of x^2.

Coefficient of [x^2]: 1= A+B

1=¼ +B

B= 1- ¼ = ¾

B= ¾

$\therefore \frac{x^2}{(x+1)(x-1)^2} = \frac{¼}{x+1} + \frac{¾}{x-1} + \frac{½}{(x-1)^2}$

$= ¼.\frac{1}{x+1} + ¾.\frac{1}{x-1} + ½.\frac{1}{(x+1)^2}$

$\therefore \int \frac{x^2}{(x+1)(x-1)^2}\,\delta x = ¼\int \frac{1}{x+1}\,\delta x + ¾\int \frac{1}{x-1}\,\delta x + ½\int \frac{1}{(x+1)^2}\,\delta x$

$= ¼\ln(x+1) + ¾\ln(x-1) + \frac{1}{2(x-1)} + c$

Example 3:

Determine $\int \frac{x^2+1}{(x+2)^3}\,\delta x$

Solution

Resolving $\frac{x^2+1}{(x+2)^3}$ into partial fraction.

$\frac{x^2+1}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$

Multiplying throughout by the LCM of the denominator.

$x^2 + 1$ = A$(x+2)^2$ + B$(x+2)$ + C ....(1)

To get C, put x = -2.

Equ (1) becomes:

$(-2)^2 + 1 = A(-2+2)^2 + B(-2+2) + C$ **Simplifying.**

4+1=C

C=5

Expanding equation 1 becomes:

$x^2 + 1$ = A(x^2+ 4x+ 4) + B(x+2) +C

$x^2 + 1$ = A$x^2 +\ 4Ax$ + 4A +Bx + 2B+ C

$x^2 + 1$ = Ax^2 + 4Ax + Bx +4A+2B+C **Collecting like terms**

Taking the coefficient of x^2

Coeff of $[x^2]$: 1= A

A =1

Taking coefficient of constant term.

Coeff of [CT]: 1= 4A+2B+C...(2)

Substituting the value of A and C into equation 2 to get B.

Eqn(2) gives: 1= 4(1) +2B+5

1= 4+ 5+ 2B

2B =1$-$9

2B = $-$ 8

B = $\frac{-8}{2}$ = -4

$\therefore \frac{x^2+1}{(x+2)^3} = \frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{5}{(x+2)^3}$

Hence:

$\int \frac{x^2+1}{(x+2)^3} \delta x = \int \frac{1}{x+2}\ \delta x - \int \frac{4}{(x+2)^2}\ \delta x + \int \frac{5}{(x+2)^3}\ \delta x$

$= \ln(x+2) - 4\int \frac{1}{(x+2)^2}\ \delta x + 5\int \frac{1}{(x+2)^3}\ \delta x$

$= \ln(x+2) - 4\int (x+2)^{-2}\ \delta x + 5\int (x+2)^{-3}\ \delta x$ **From indices law:** $\boldsymbol{\frac{1}{a^n} = a^{-n}}$

Let a= x+2

b=x+2

$= \ln(x+2) - 4\int a^{-2}\ \delta x + 5\int b^{-3}\ \delta x$

$= \ln(x+2) - 4[a^{-1}] - \frac{5}{2}[b^{-2}]$ +c

Replacing back the value of a and b in the expression above.

$= \ln(x+2) + \frac{4}{x+2} - \frac{5}{2(x+3)^2}$ +c

Example 4

Determine $\int \frac{x^2}{(x-2)(x^2+1)} \delta x$

Solution

Resolving $\frac{x^2}{(x-2)(x^2+1)}$ into partial fraction.

$\frac{x^2}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+c}{x^2+1}$

Multiplying throughout by the LCM of the denominator.

x^2= A(x^2+1) + (Bx+C)(x – 2)...(1)

To get A, put x =2

Eqn(1) becomes: 2^2= A(2^2+1) + [B(2) + C][2 – 2]

4=5A

A= $\frac{4}{5}$

To get Band C, expanding equation and taking the coefficient

x^2 = Ax^2+A+Cx +Bx^2 - 2Bx +Cx -2C

x^2= Ax^2+ Bx^2-2Bx+Cx +A – 2C

x^2 = x^2(A+B) – x(2B – C) + A – 2C

$To\ get$ B, take the coefficient of x^2

$coeff[x^2]$: 1 = A+B

$1 = \frac{4}{5} + B$

$B = 1 - \frac{4}{5} = \frac{5-4}{5} = \frac{1}{5}$

$B = \frac{1}{5}$

To get C, taking the coefficient of CT

Coeff of CT: 0= A- 2C

$0 = \frac{4}{5} - 2C$

$2C = -\frac{4}{5}$

$C = -\frac{4}{5} \times \frac{1}{-2}$

$C = \frac{2}{5}$

$\therefore \frac{x^2}{(x-2)(x^2+1)} = \frac{\frac{4}{5}}{x-2} + \frac{\frac{1}{5}x + \frac{2}{5}}{x^2+1}$

$= \frac{4}{5} \cdot \frac{1}{x-2} + \frac{1}{5} \cdot \frac{1}{x^2+1} + \frac{2}{5} \cdot \frac{1}{x^2+1}$

$\therefore \int \frac{x^2}{(x-2)(x^2+1)} \delta x = \int \frac{4}{5} \cdot \frac{1}{x-2} \delta x + \int \frac{1}{5} \cdot \frac{1}{x^2+1} \delta x + \int \frac{2}{5} \cdot \frac{1}{x^2+1} \delta x$

$= \frac{4}{5} \int \frac{1}{x-2} \delta x + \frac{1}{5} \int \frac{1}{x^2+1} \delta x + \frac{2}{5} \int \frac{1}{x^2+1} \delta x$

$= \frac{4}{5} \ln(x-2) + \frac{1}{5} \int \frac{\frac{1}{2}(2x)}{x^2+1} \delta x + \frac{2}{5} \tan^{-1} x$

$= \frac{4}{5} \ln(x-2) + \frac{1}{5} \cdot \frac{1}{2} \int \frac{2x}{x^2+1} \delta x + \frac{2}{5} \tan^{-1} x$

$= \frac{4}{5} \ln(x-2) + \frac{1}{10} \ln(x^2+1) + \frac{2}{5} \tan^{-1} x + c$

6.6 Exercises

A. Find the following:

i. $\int(3x-1)\,\delta x$ iii. $\int(x+1)(x-2)\ \delta x$

ii.$\int(3t+4t^2)\ \delta t$iv. $\int\frac{2(x+3)}{x^3}\delta x$

B. Integrate the following:

i. $\int(2x-7)^3\ \delta x$ v.$\int\sin(3x+8)\ \delta x$

ii.$\int\cos(7x+2)\ \delta x$vi.$\int\sec^2 4\mathrm{x}\ \delta x$

iii.$\int\sec^2(3x+1)\,\delta x$vii.$\int\frac{1}{5x+4}\ \delta x$

iv.$\int(5x-4)^6\ \delta x$viii.$\int 4^{3x}\ \delta x$

C. Integrate the following:

i.$\int\frac{3x^2}{x^3-4}\ \delta x$ v. $\int\frac{\sec^2\mathrm{x}}{\tan x}\ \delta x$

ii. $\int\frac{2x+3}{x^2+3x-5}\ \delta x$vi. $\int\tan x\ \delta x$

iii. $\int\frac{2x+4}{x^2+4x-1}\ \delta x$vii.$\int\frac{2x-3}{x^2+3-7}\ \delta x$

iv. $\int\frac{x-3}{x^2-6x+2}\ \delta x$viii.$\int\frac{9x^2}{x^3-7}\ \delta x$

D. Integrate the following by part:

i.$\int e^{3x}\sin x\ \delta x$

ii. $\int x\ln x\ \delta x$

iii. $\int e^{5x}\sin 3x\ \delta x$

iv. $\int x^3 e^{2x}\ \delta x$

6.7 Summary

1.$\int$ is the symbol of integration (integral sign)

2. Integration simply means the inverse operation to differentiation

3. $\int x^n \, \delta x = \frac{x^{n+1}}{n+1} + C$ {provided $n \neq 1$}

4. $\int u \, \delta v = uv - \int v \, \delta u$

5. $\int \frac{f'(x)}{f(x)} \boldsymbol{\delta x} = \ln\{f(x)\} + c$.

6.8 References

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MODULE 8:

VOLUME OF SOLIDS OF REVOLUTION BY DEFINATE INTEGRAL

Contents.

8.1 Introduction:

Volume of solids of revolution by definite integrals deals with volume of solid such as cone, cylinder, sphere used in construction of mega structure in beams of houses, sky scrapper, for bridges to be able to withstand load and stress.

8.2 Objectives:

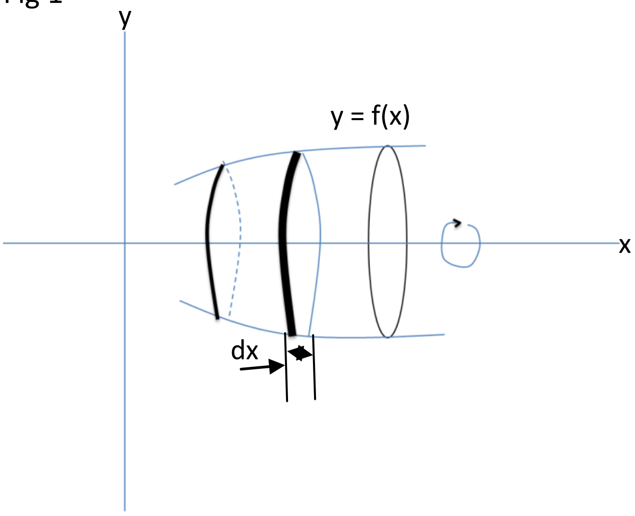
In this module, students are expected to know how to use definite integral in calculating volume of solids such as sphere, cylinder, cone etc.

8.3 Volume of Solids of Revolution

A solid figure generated by revolving a given curve around an axis of revolution is called the **solid of revolution**.

The volume generated by the region of the coordinate plane bounded by the segment of the curve y=f(x) between x=a, y=b and the x- axis, revolving around the x-axis, is shown in the figure 1 below:

Fig 1



Note that the infinitesimal volume of the cylinder representing an element of the integration

$V_x = \pi \int_a^d y^2 \delta x$

The volume generated by the segment of a curve x=g(y) between y=c and y=d, revolving around the y-axis is shown in the figure 2 below

Fig 2

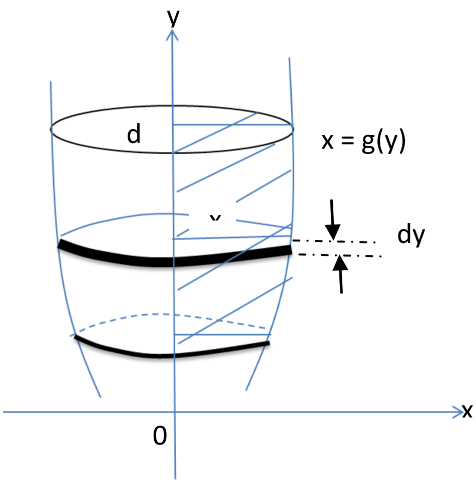


Fig 2

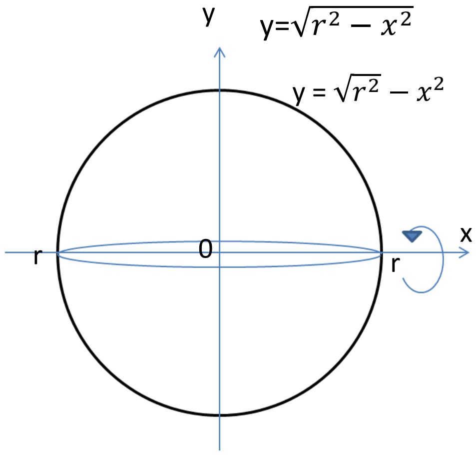
$V_y = \pi \int_o^d x^2 \ \delta y$

8.4 The Volume of a sphere

Find the volume of a sphere generated by a semicircle $y=\sqrt{r^2 - x^2}$ revolving around the x-axis.

Solution.

Since the end points of the diameter lying on the x-axis and –r and r as shown below, the



Since y=$\sqrt{r^2-x^2}$

Squaring both side

$y^2 = (r^2 - x^2)^{½ \times 2}$

$y^2 = r^2 - x^2$

$v_x = \pi \int_a^b y^2\, \delta x \ = \pi \int_{-r}^{r} (r^2 - x^2)\, \delta x \ = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r}$

$= \pi[(r^2.r - r^3/3) - (r^2. -r - (-r)^3/3)]$

$= \pi[(r^3 - r^3/3) - (-r^3 + r^3/3)] = \pi \left[\frac{(3r^3 - r^3)}{3} - \frac{(-3r^3 + r^3)}{3} \right]$

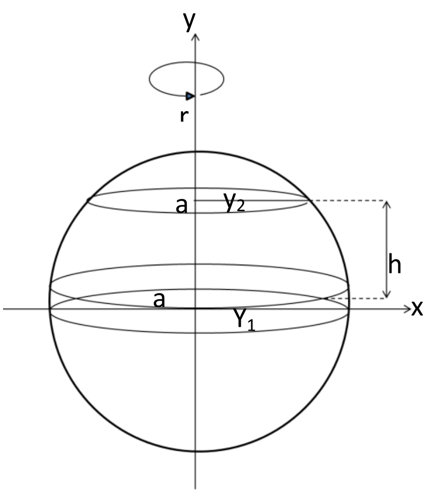
$= \pi \left[\left(\frac{2r^3}{3} \right) - \left(\frac{-2r^3}{3} \right) \right] = \pi \left[\frac{2r^3}{3} + \frac{2r^3}{3} \right] \ = \ \pi \left[\frac{2r^3 + 2r^3}{3} \right]$

$= \pi \,.\, 4r^3/3 \ = 4\,\pi r^3/3$

$V_x = 4\,\pi r^3/3$

8.5 The Volume of a Spherical segment

Example: Find the volume of a spherical segment generated by the portion of the right semi circle between y=a and y=a+h, revolving around the y-axis, is as shown in the below figure:



Solution

Since the right semi circle equation $x = \sqrt{r^2 + y^2}$ then

$V_y = \int_c^d x^2 \delta y = \int_a^{a+h} (r^2 - y^2) \delta y$

$= \pi \left[r^2 y - \frac{y^3}{3} \right]_a^{a+h}$

$V_y = \pi \left[r^2(a+h) - \frac{1}{3}(a+h)^3) - (r^2 . a - \frac{1}{3}(a^3) \right]$

$= \pi[(r^2a + r^2h - \frac{1}{3}(a^3 + 3a^2h + 2ah^2 + h^2a + h^3) - (r^2a - \frac{1}{3}a^3)]$

$= \pi[r^2a + r^2h - \frac{1}{3}a^3 - a^2h - \frac{2}{3}ah^2 - \frac{1}{3}h^2a - \frac{1}{3}h^3 - r^2a + \frac{1}{3}a^3]$

$= \pi[r^2a + r^2h - \frac{1}{3}a^3 - a^2h - \frac{2}{3}ah^2 - \frac{1}{3}h^2a - \frac{1}{3}h^3 - r^2a + \frac{1}{3}a^3]$

Collecting like or common terms

$= \pi\ [r^2a - r^2a + r^2h - \frac{1}{3}a^3 + \frac{1}{3}a^3 - a^2h - \frac{2}{3}ah^2 - \frac{1}{3}ah^2 - \frac{1}{3}h^3]$

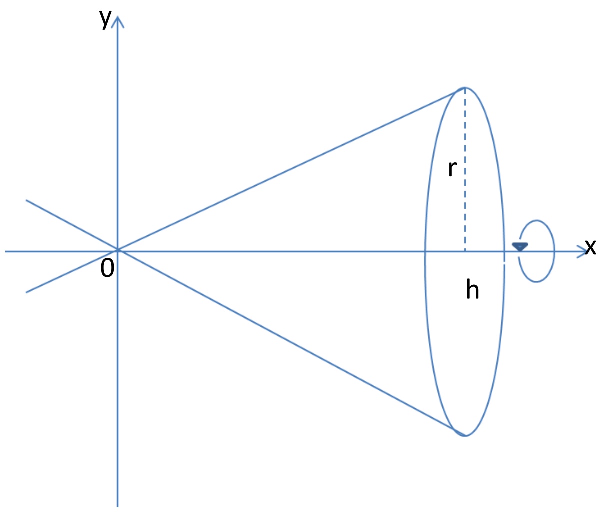
$= \pi[\ r^2h - a^2h - ah^2 - \frac{1}{3}h^3]$

$v_y = \pi h\ \ [r^2 - a^2 - ah - \frac{1}{3}h^2]$

8.6 The volume of a cone

Example 1

Find the volume of a right circular cone generated by the line (segment) passing through the origin and the point (h,r), where h denotes the height of the cone and r is the radius of its base, revolving around the x-axis , as shown in the figure below.



Solution

The equation of the generating line

$y = mx$ .... (1)

$m = \frac{\Delta x}{\Delta y} = \frac{r}{h}$....................................... (2)

Substituting (2) into (1)

(1) becomes: $y = \frac{r}{h}x$......................... (3)

$V_x = \pi \int_a^b y^2 \delta x$ .... (4)

sub (3) into (4)

$= \pi \int_o^h (\frac{r}{h}x)^2 \delta x \quad = \quad \pi \int_o^h \frac{r2x2}{h2} dx$

$= \pi \frac{r^2}{h^2} \int_o^h x^2 \delta\, x \;=\; \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h$

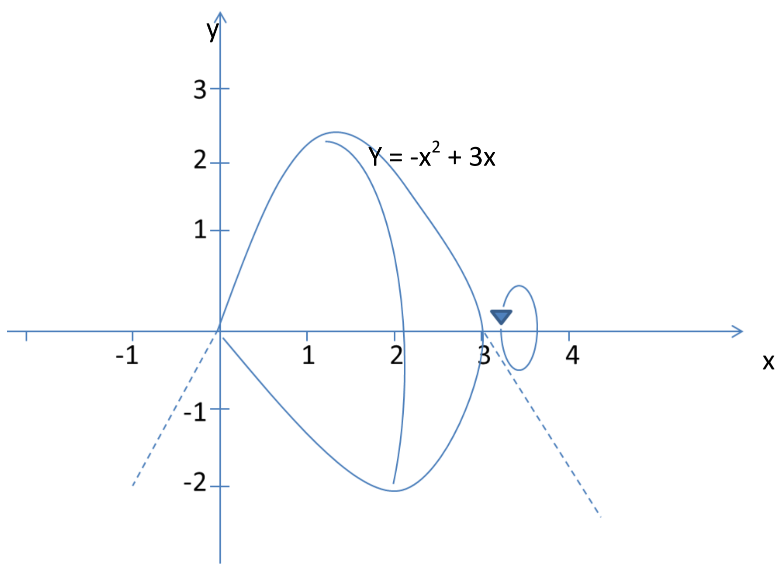
$= \frac{\pi r^2}{h^2} \left[\frac{h^3}{3}\right] \quad = \quad \frac{\pi r^2 h^3}{3h} \;=\; \frac{\pi r^2 h}{3}$

$V_x = \frac{\pi r^2 h}{3}$

$V_{cone} = \frac{1}{3} \pi r^2 h$

Example 2

Find the volume of a solid of revolution generated by a plane bounded by the segment of a curvey = $-x^2$ + 3x and the x – axis, revolving around the x – axis, as shown in the figure below:



Solution

When $y = -x^2 + 3x$, let y = 0

The limits of the integration: $-x^2 + 3x = 0$

Factorizing the term above

$x(-x + 3) = 0$

$x = 0$ or $-x+3 = 0$

$x=0$ or $x=3$

$\therefore$ $x_1 = 0$ and $x_2 = 3$

$V_x = \pi \int_a^b y^2 \delta x = \pi \int_0^3 (-x^2 + 3x)^2 \delta x$

Expanding $(-x^2 + 3x)^2 = (-x^2 + 3x)(-x^2 + 3x)$

$= x^4 - 3x^3 - 3x^3 + 9x^2$

$= x^4 - 6x^3 + 9x^2$

$:-v_x = \pi \int_0^3 (-x^2 + 3x)^2 \delta x$

$= \pi \int_0^3 (x^4 - 6x^3 + 9x^2) \delta x$

$= \pi \left[\frac{x^5}{5} - \frac{3x^4}{2} + 3x^3\right]_0^3$

$= \pi \left[\frac{3^5}{5} - \frac{3(3)^4}{2} + 3\,(3)^3\right]$

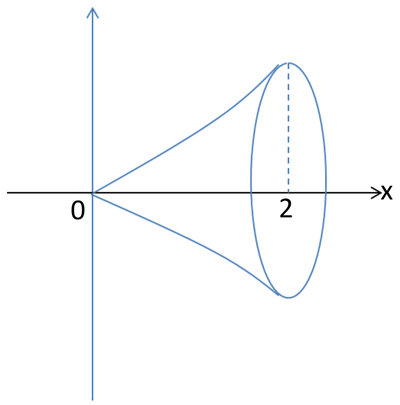
$= \pi \left[\frac{243}{5} - \frac{243}{2} + \frac{81}{1}\right] = \pi \left[\frac{486 - 1215 + 810}{10}\right]$

$= \pi \left[\frac{81}{10}\right] = \frac{81\pi}{10}$

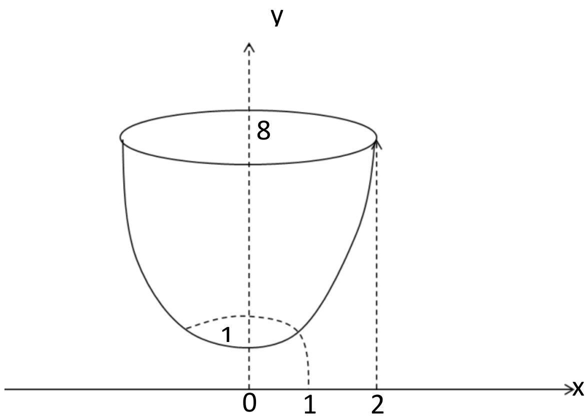
8.7 TMA

1. The portion of the curve $y=x^2$ between x=0 and x=2 is rotated completely round the x-axis. Find the volume of the solid created.

y



2. The part of the curve $y=x^3$ from x=1 to x=2 is rotated completely round the y – axis. Find the volume of the solid generated.



3. The finite area enclosed by the line 2y = x and the curve $y^2 = 2x$ is rotated completely about x-axis. Calculate the volume of the solid produced.

4. Find the volume generated by rotating the curve $y=3x^2 + 1$ from x=1 to x=2 completely round the x – axis.

5. If the area enclosed between the curves $y=x^2$ and the line y=2x is rotated around the x – axis through four right angles, find the volume of the solid generated.

6. Find the volume of a solid generated by rotating a curve y=2sinx between x=0 and x= $\pi/2$ around the x-axis.

8.9 References:

1. Engineering Mathematics by K. A Stroud 5th Edition.
2. Additional Mathematics by Godman& Y. F Talbert.
3. Calculus An Applied Approach Larson Edwards Sixth Edition.
4. Blitzer Algebra and Trigonometry Custom 4th Edition.