NATIONAL OPEN UNIVERSITY OF NIGERIA

SCHOOL OF MANAGEMENT SCIENCES

COURSE CODE: MBF 839

COURSE TITLE: QUANTITATIVE TECHNIQUES FOR BANKING & FINANCE

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Published by:
National Open University of Nigeria 2008

First Printed 2008

ISBN: 978-058-365-3

Printed in Nigeria by:
Daily Graphics Nigeria Ltd.,
Tel: 0803-3108965, 02-7522693, 0804-4104608,
0803-3449198, 0805-2281869,
E-mail: dailygrafix@yahoo.co.uk
For:
National Open University o ľ Nigeria.

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MODULE 1

UNIT 1 INTRODUCTION TO BASIC MATHEMATICS

CONTENTS

1.0 INTRODUCTION

This unit is to assist you recapitulate your knowledge of numbers and basic arithmetic operations of addition, subtraction, multiplication and division of numbers. You are expected as Banking and Finance student to be reasonably grounded in numerical studies. As you go through the examples and exercises, you need a calculator to assist you in facilitating your calculation. This unit will give you the needed foundation and to appreciate mathematics of finance as it is applicable to business in later units.

As mathematics is full of symbols e.g. x, y, z and expressions involving powers, roots and the basic arithmetic operations, you will go through methods of manipulating such expressions in order to present them in their simplest forms. There will be combination of symbols to obtain algebraic expressions (basic arithmetic operations).

In this units too, ratio, proportion as well as variation will be revised. These are basic concepts which appear frequently in business and financial calculations. For example, liquidity ratios, profitability ratios and activity ratios are normally used for determining how well a company's fund is managed etc. etc.

2.0 OBJECTIVES

On completion of this unit, you should be able to:

- perform basic arithmetic operations on numbers like conversion between fractions, decimals and percentages;
- simplify numerical expressions involving exponents and roots;
- identify like terms in algebraic expressions and perform basic algebraic operation including long division;
- distinguish between variations (direct and indirect).

3.0 MAIN CONTENT

3.1 Numbers and Basic Arithmetic Operations

3.1.1 Numbers

Numbering in word and expressing them correctly in figures is important in daily business. For example, if you write out a cheque to draw N10,000.00, the amount in words is ten thousand naira. In the figure, each digit has a place value which enables the correct magnitude to be attached e.g. 16,790,406 — "sixteen million, seven hundred and ninety thousand, four hundred and six".

There are ten digits (base 10 or decimal system) namely 0, 1, 2, 3, 4, 5, 6, 7, 8 & 9, all integers can be formed using these ten digits.

There are positive as well as negative integers. Numbers such as 10, 20, 35, 78 and 300 are positive integers while numbers such as — 50, —38, and —137 are negative. A number without a sign is always assumed a positive number. Numbers can be arranged in word form or figures as below:

- Billions—. Hundred Millions—. Ten Millions—> Millions—> Hundred Thousands Ten Thousands—> Thousands—> Hundreds—. Tens—> Ones.

3.1.2 Basic Arithmetic Operations

The basic arithmetic operations are addition (+), subtraction (—), multiplication (x) and division (+). When these are present in a single mathematical expression, they are performed in a specific order:

(i) perform all operations within brackets (parentheses);
(ii) perform all operations with exponents (e.g. 2^3);
(iii) multiply, divide, add or subtract in the order that numbers are presented (left to right).

Example 1

Simplify 15 + 3+2^3 x 1 —4 x 2 (3 + I)

Step I: open the brackets:
15+3 +2^3x 1-4 x2 x 4
Step II: the term with exponents:
15+3 + 8 x 1-4 x 2 x 4

Step III: multiply and divide in the order they appear:
(15 + 3)+(8 x 1)—(4 x 2 x4)

5 + 8 — 32 (Add and subtract as they appear)

13—32 = **—19**

Fractions and Percentages

Proper and Improper Fractions

Fractions are common phenomenon when working with numbers. Fraction helps you in terms of parts of the whole.

For example; One-third, meaning one out of three, written 'A or 1/3, while one-quarter, meaning; one-fourth, written as 'A or 1/4. In general, a fraction is an expression of the form a or a/b where a and b are integers. This definition obviously excludes the case b = 0 (why?) Imagine dividing a number by 0 (if is allowed), what will be the values of, say 5/0 or 18/0? Suppose 5/0 = k; then one must have 5 = Ox k i.e. 5 = 0, which is absurd.

In m/n fraction, m is the numerator while the number n is denominator. The fraction min is a proper fraction if m < n (m is less than n) and known as an improper fraction if m n (i.e. if m is greater than or equal to n). E.g. %, 4/7 and % are proper fractions while 7/3, 8/8 and 4/3 are improper fractions.

7/3 may be written as 2% (mixed number).

Note: Every mixed number is the sum of an integer and a proper fraction. Conversely, every mixed number is an improper fraction.

Finally, note that the reciprocal of a fraction is obtained by simply interchanging the numerator and the denominator.

Every answer in mathematics in the form of fraction is presumed to be expressed in the lowest term (form) or simplest or reduced form, unless otherwise directed.

Example 2

$$\frac{70}{105} = \frac{14}{21} = {}^2\!A \quad \text{(why?)}$$

is the lowest form.

SELF ASSESSMENT EXERCISE 1

(i) Convert 9/4 to a mixed number.

(ii) Convert $4\frac{3}{5}$ to an improper fraction.

(iii) A store carries 22 different styles of girl's shoes, 16 different styles of boy's shoes, 12 different styles of men's shoes and 24 different styles of women's shoes. What fraction of all shoe styles is for men?

Arithmetical Operations with Fractions

Addition and Subtraction

When two fractions are with a common denominator, they can be added by simply adding their numerators;

$$\frac{a}{c}+\frac{b}{c} = \frac{a+b}{c} \quad \text{e.g. } \frac{4}{5}+\frac{3}{5} = \frac{4+3}{5} =$$

A similar rule applies for subtraction

a—b = a—b e.g. 5— 3 = 5-3 = 2 = 1

cc c 12 12 12 12 6

However, when you must add or subtract two fractions with unequal denominators, then the fractions must first be re-written with a common denominator, (usually the least common multiple (LCM) of the unequal denominator or simply the product of the unequal denominator).

Example 3

5 + 3 =
64

Approach **I:**

10 + 9 (since LCM of 6 and 4 is 12)
12 12

10+9 = 19 = 1 7
12 12 12

Approach II (Alternatively):

5 + 3 20 + 18 (24 is the product of 6 and 4)
64 24 24

20 + 18
24

= 38 = 19 = 1 1
24 24 12

For mixed fractions, deal with the whole numbers first and then treat the fractional part as in example below:

Example 4

Simplify 4% + 21/4

Solution:

Step **I:** Deal first with whole numbers:
4 + 2 = 6

Step II: ²A + % = 8 + 3 = $\frac{8 + 3}{12}$ = 11
12 12 12

4% + 2% $6\frac{11}{12}$

Multiplication and Division

axe = ac
b d bd

Irrespective of whether the fractions are proper or improper, it is always the product of the numerators divided by the product of the denominators.

Example 5

lx 5 = $\frac{1 \times 5}{3 \times 9}$ = 5
3 9 27

To divide one fraction by another, multiply the first fraction by the reciprocal of the second.

a±c = axd = ad
b d b c bc

3 x 3 =

Example 6

$\frac{3x5}{7 \times 3}$ = 15 5
7 5 21 7

Decimal Fractions

Another common expression of fraction is in decimal form like:

$\frac{3}{10}$ = 0.3; $\frac{27}{100}$ = 0.27

Example 7

3= 3 ÷ 8 = 0.375
8

Percentage

Percent means of one hundred and is denoted by the symbol %. For example, 5% is read as five percent, meaning 5 of one hundred, or five hundredths.

Percentage also indicates proportion; it is a fraction with a denominator of 100 i.e. 75% is equal to $\frac{75}{100} = \frac{3}{4}$..

Example 8

In a firm of 1,250 employees, 575 are male. Find the percentage of female employees.

Solution:

Number of Female employee 1,250 less 575
675

Percentage of female = $\frac{675}{1250} \times \frac{100}{1}$

54%

SELF ASSESSMENT EXERCISE 2

(i) Convert the following percentages to decimal numbers: 6.5%, $12^{3}/4$%, 147%.

(ii) Convert 80% to a fraction reduced to its lowest terms.

3.2 Exponents and Roots

Exponents

The product x. x. x... = x^3 because there are three (3) of the x multiplied together. The 3 in x^3 is the exponent (power), while x is known as the base. In general:

n times

E.g. 2^5 = 2x2x2x2x2 = 32

i.e. '2 raised to power 5 is equal to 32'.

State 3 in descending order as follows:

$3^3=$	27	$(3^1 x\ 3^1 x\ 3^1) =$	3^{1++1}			
3^2	9	(3'x35	3^{1+1}			
	3	(3^1)				
3^0	1	(3 + 3) = 1 =	3 I-I			
3^{-1}	1 31	(1 + 3) = 1 = 31	$3^{1.1}$			
3^{-2}		(1 + 3) =1= 3^1 3^2	lx1 3^1 31	=	1 32	
3^{-3} =	1 33	(1+3) = 1 = 32 3^3	lx lx 1 3^1 3^1 3^1	=	1 3^3	

And so on.

This pattern leads one to a definition of x" when n is zero or a negative integer:

$$x^0 = 1 \text{ if } x \neq 0$$

$$\frac{-1}{x^n} \qquad \frac{1}{x.\ x.\ x.\ \ldots\ldots\ldots\ldots} \qquad \text{for } x\ 0$$

Example 9

(a) $1.075^3 =$ 1.075 x 1.075 x 1.075 = 1.2423

Use calculator with the key (y") to calculate easily.

(c) $\frac{1}{4^5}$ 45 = 4 x4 x4 x4x 4 = 1,024

(d) $\frac{1 - 1.02^{-3}}{0.02}$ 2.88 $\frac{1 - 1/1.02^3}{0.02}$ = $\frac{1 - 0.942}{0.02}$ =

Roots

Roots are the opposite of exponents. If r" = x, the r is the nth root of x to be written as r = x.

E.g; 5^2 =25;

So 5 is a second root (normally called the square root of 25). (-5)+ = 25 also, therefore —5 is also a square root of 25. (Generally, the square of any real number is expected to be positive).

Conventionally, ²4x is often written as ■ix without the 2, e.g. ²4 36 = 4 36 = 6. If x is a positive number, then the expression q■ixP means the same as x^{pig} where p and q are integer and q is positive (i.e. q> 0).

Example 10

(a) X 3 1
4

(b) $9^{-1/4}$ 24 9 1 = 41/9= V3

(c ~~7~~ = —3

SELF ASSESSMENT EXERCISE 3

From what you have learnt, so far, simplify the following expressions:

(i) $\frac{x^8}{x^2}$ (ii) $(2x^3)^3$

3.3 Algebraic Expressions/Operations

In algebraic expression, numbers are expressed in symbols like x, y and z They are combined by operations of addition, subtraction, multiplication, division on extraction of roots. For example, $2x^2 - 3x + 7$ is an algebraic expression in the variable x while $3x^2y - 2x + 1$ is in the variables of x and y.

In the example $2x^2 - 3X \pm 7$, there are three terms: $2x^2$, $- 3X$ and 7. In the term $2x^2$, 2 is called the coefficient of the literal part x^2. Similarly, in the term $- 3x$, -3 is the coefficient of the literal part x. The third term 7 does not have a literal part. Try and be familiar with the following properties of addition and multiplication, cumulative property, associative property and distributive property before studying the combinations of algebraic expression.

Cumulative Property:

x + y y + X andx+(—y)— —y+x,

E.g. 4+5 = 5 + 4 = 9; 4 + (-5) = —5+4 = —1

3 x 6 = 6x3 = 18; 3 x(-6) = (-6)x 3 = —18

Associative Property:

x-f(y+z) = (x+y)+z and xx(yxz) = (xxy)xz

E.g. 2 + (3 + 5) = (2+3)+5 = 10; 2 x(3 x 5) = (2 x 3)x 5 =30

Distributive Property:

X + + = xxy+X+z;
X x (y z) = xxy—xxz

Multiplication is distributive over addition, i.e. multiplication can be distributed over addition. More so, multiplication is also distributive over subtraction. In view of the cumulative law, x x (y + z) is the same as

(y + z) x x. Thus, (y +z) X =y x X + z x X = X + y + X + z a s well.

E.g. (2 + 3) x 4 = 2 x 4 + 3 x 4 = 8 + 12 = 20.

—2(3 —5) = (-2) (3) — (-2) (5) =-6 +l6=4

3.3.1 Addition and Subtraction of Exprosion

When adding or subtracting algebraic expressions, the terms must be arranged such that all like terms are grouped together. That simplifies computation.

Example 1.11

Simplify 5a + 7b — 3 + 313 — 2a + 8

Solution: Re-arrange the terms:

$5a + 7b - 3 + 3b - 2a$
$(5-2)a \pm (7+3)b + (-3 + 8)$
$3a + 10b + 5$

3.3.2 Multiplication Expressions

This is straight forward in algebraic expressions. Just use the distributive property repeatedly and put like terms together as shown below:

Example 12

(a) $-3(x - 2y + 7z^3) = (-3)x + (-3)(-2y) + (-3)(7z^3)$
$= -3x + 6y - 21z^3$

(b) $x^2(x^2 + 3x - 5y^3) = x^2 y x^2 + x^2 y\, 3x + x^2 y(-5y^3)$
$= x^4 y\ 3x^3 y\ 5x^2 y^4$

(c) $(3x - 4)(6x^2 - 5x + 2)$

$- \quad (3x - 4)(6x^2 - 5x + 2) - 4(6x^2 - 5x + 2)(3x)(6x^2) +$
$(3x)(-5x) + (3x) - 3(z) + (-4)(6x^2) + (-4)(-5) + (-4)(2)$

$- \quad 18x^3 - 15x^2 + 6x - 24x^2 + 20x - 8$

Like terms together:

$- \quad 18)(^3 - 15x^2 + 6x - 24x^2 + 20x - 8$
$18x^3 - 39^2 + 26x - 8$

3.3.3 Division Expressions

Recall that $\underline{a + b} = a + b$
cc

This will be used when dividing one algebraic expression by another.

Example 13

(a) $\frac{2x^2 + 6x}{2x} = \frac{2x^2}{2x} + \frac{6x}{2x} = x + 3$

(b) $\frac{25(1^3 — 12q^2 + 15q - 6}{3q}$

$\frac{25q^3}{3q} — \frac{12q^2}{3q} + \frac{15q}{3q} — \frac{6}{3q} = \frac{25^2}{3} — 4q — 2$

In the division of one algebraic expression by another, the expression that is being divided is the dividend and the expression by which we are dividing the dividend is called the divisor. If the power or degree) of the divisor is less than or equal to the power of the dividend, then the long division method can be used to carryout the division as will be shown below in

Example 14

Note: The result of an algebraic division normally consists of two parts — quotient and remainder — written thus:

$$\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Reminder}}{\text{Divisor}}$$

Divide 2)(3— 14x —5 by x —3

Solution:

Here, the dividend $2x^3$ —• 14x — 5 and the divisor is x — 3 using long division method. Therefore, to avoid errors, we arrange both dividend and divisor in decreasing powers of x and fill in missing powers with zero coefficients.

X^3 - 14x —5 (quotient)

divisor x 3 2)(3-0— 14x —5 (dividend)

$2x^3$-6x

$6x^2$ — 14x

— $6x^2$ — 14x

4x —5

— 4x — 12

— 7 (remainder)

So $\frac{2x^3 - 14x - 5}{x-3} = 2x^2 + 6x + 4 + \frac{7}{x-3}$

SELF ASSESSMENT EXERCISE 4

In the algebraic expressions below, carry out the operations indicated and simplify your answers:

L $7x - 4y + 3 \text{ -4- } 3x - 2y - 1$
ii. $(x^2 - 3)(x^3 - 4x^2 - 4x + 1)$
iii. $\frac{t^3 + 7t^2 - 5t + 4}{\text{i. } t^2}$

3.4 Ratio and Proportion

Ratio

Business information often compares related quantities as ratio. For example, if someone invests a total of N22,000 in two popular stocks in the stock market — stock A and stock B, in the stock market, investing N6,000 in stock A and N16,000 in stock B, then

$$\frac{\text{Investment in stock A}}{\text{Investment in stock B}} \quad \frac{\text{N6,000}}{\text{N16,000}} = \frac{6}{16} = \frac{3}{8}$$

is known as the ratio of investments in the two stocks.

There are three ways to express ratios:

(1)	three to eight or 3 to 8	(word/figure description)
(2)	3:8	(colon)
(3)	3/8	(fraction)

However, if you are comparing more than two quantities at the same time, then use colon e.g. compare 6kg, 2kg and 7kg would be written as 6 : 2 : 7.

Note: such comparisons must be made only if the quantities being compared are expressed in the same unit. For example, if the ratio A : B where A is 1 hour and B is 50 minutes, then first convert B to hours; then

A : B = 1 : 50/60

Ratios can also be obtained when comparing quantities relating to different things but expressed in the same units. For example, suppose a product is in terms of cost components, made up of N200 of material, N100 of direct labour and N25 of overhead. The ratio of the cost elements is then 200:100:25.

Now, just as fractions are reduced to their lowest terms, ratios can also be reduced to their lowest terms; simply divide out by their common factor.

Example 15

200 : 100 : 25 = 8 : 4 : 1

27 : 45 = 3 : 5

Ratio is also tool for allocation.

Allocation problems require the division of a whole into a number of parts according to a specified ratio. The number of parts into which the whole is divided is the sum of the terms of the ratio.

Example 16

A business suffers a fire loss of N210,000. It is covered by three insurance policies and the claims these insurance companies have to pay are in the ratio 1/3 : 1/8 : 5/12. Find the amount paid by each company.

Solution:

Since ratios are normally expressing using integers, it is necessary to convert 1/3 : 1/8 : 5/12 to a ratio involving integers only. This is done by multiplying the fractions by the LCM of their denominators, namely 24. Thus,

1/3 : 1/8 : 5/12 1/3 (24) : 1/8 (24) : 5/12 (24)

The ratio now 8 : 3 : 10

The sum of the terms of the ratio is 8 + 3 + 10 = 21; then divide N210,000 into 21 parts in the ratio i.e. N80,000, N30,000 and N100,000 respectively:

(8/21 of 210,000) N80,000,

(3/21 of 210,000) N30,000

(10/21 of 210,000) N100,000

Proportion

A proportion is formed when two ratios are equal.

For example: 2 : 3 = 8 : 12
2/3 = 8/12 = 4/6

1 : 3 : 9 6 : 18 : 54

Example 17

Solve for x if 5/16: 14/3 = x : 21/10

Solution:

Expressing both ratios as fraction, we have

5
16 = x (multiply both sides by 21/10)
14 21
3 10
– 5
16 x 21
14 10
3

– **9**

64 Example

18

A grocery store charges N100 for 5 packs of oranges. Assuming there is a no quantity discount, how much will be charged for 18 packs of oranges?

Solution:

Let x represent what is charged for 18 packs of oranges.

That is: = 1 0 0
18 5

100 x 18
5 1

Thus, the grocery shop will charge N360 for 18 packs of oranges.

3.5 Variation

Variation is related to ratio and proportion because it is also concerned with the comparison of two (or more) variables. Two kinds of variation are most common: direct or indirect variations. In direct variations, the ratio y : x is constant i.e. y/x = k, or y = kx where k is a constant.

In indirect (or inverse) variation, the product yx is constant i.e. yx = k for a constant k. Note that in direct variation, y = kx implies that as x increases in value, y also increases and as x decreases in value, y decreases as well.

However, in indirect or inverse variation, yx = k implies that as x increases in value, y decreases.

Conversely, as x decreases in value, y increases. In example 1.18, as the number of packs of oranges increases, the amount charged increases; this is direct variation. Actually, the price for one pack of oranges remains constant; in this case at N100/5 = N20 per pack. This is why we obtain:

360 = 180x x 360=-
18 180 18 1

We can express this as y = kx, where y is the amount charged, x is the amount charged, x is the number of packs of oranges and k = 20 is the price per pack.

Example 19

If it takes 12 days for eight workers to complete a job, how long does it take 15 workers to complete the same job? Assume that the ability of each worker is the same.

Solution:

With more workers — the job will be done in less time.

So this is the case of inverse variation. Let y be the number of days needed and let x be the number of workers. The inverse variation equation is yx = k for some constant k. Now, when x = 8, y = 12, since it took 8 workers 12 days to complete the job. Therefore, from yx = k, it follows that k = 12, 8x = 96, and it is in order to state that yx = 96.

If there are now 15 workers, i.e. x = 15, we obtain the corresponding value of y, y x 15 = 96.

$$96/15 = 32/5 = 6\frac{2}{5}$$

Thus, it will take 15 workers 62/5 days to complete the job.

SELF ASSESSMENT EXERCISE 5

(i) Solve for x if 4 : 14 = x : 9.

(ii) The scale of a map is 1.5 cm for 300 km. How many kilometers are represented by 5 cm.

3.6 Advantages of Quantitative Skills for Managers

Executives at all levels in business and industry make decisions at every course of daily activities. Quantitative methods provide scientific basis for decision-making to the executive and enhance his ability to make long range plans and to solve everyday problems of running a business and industry with greater efficiency and confidence.

The advantages of the study of quantitative methods include:

(1) **Definiteness:** general statements are presented in a precise and definite form.

(2) **Condensation:** it simplifies large data and presents meaningful information from them.

(3) **Comparison:** it enables comparisons between past and present results with a view to ascertaining the reasons for change.

(4) **Formulation of policies:** provides the basic material for framing policies in business, etc.

(5) **Formulating and testing hypothesis:** useful in formulating and testing hypothesis or assumption or statement and to develop new theories.

(6) **Prediction:** for framing suitable policies or plans and then for implementation it is necessary to have the knowledge of future trends.

4.0 CONCLUSION

The understanding of the basics of the nature of numbers and the basic operations of addition, subtraction, multiplication and division determine the efficiency in quantitative skills. It is then necessary to become familiar with algebraic expressions and operations in order to cope with problems involving unknown quantities. This marks the conclusion of our review of the basic concepts of mathematics involving arithmetic and algebra, which you will encounter in business, banking and finance calculations.

5.0 SUMMARY

This unit is a review of numbers and various ways numbers are combined with the basic mathematical (arithmetic) operations of addition, subtraction, multiplication and division. It covers fraction and percentages, algebraic operations, the use of symbols in place of figures, and establishes relationship between ratios, fractions, proportion and variations. These are all the tools for business calculations.

6.0 TUTOR - MARKED ASSIGNMENTS

(1) 3 / 7 3 / 5
(2) $9^{-1/4}$
(3) Allocate N600 in the ratio of 3 : 2 —
(4) $(x^2 + 2)(x^2 - 2) =$

7.0 REFERENCES AND FURTHER READINGS

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UNIT 2 FUNCTIONS

CONTENTS

1.0 INTRODUCTION

For decision problems which use mathematical tools, one would first identify or formally define all significant interactions or relationships among variables relevant to the problem. These relationships usually are stated in equation form (or sets of equations) or in-equations. Such type of simplified mathematical relationships help the manager — the decision-maker to understand (any) complex management problems. For example, the decision-maker knows that demand of an item is not only related to price of that item but also to the price of the substitutes. Thus, if specific mathematical relationship (model) exist and can be defined, then the demand of the item in the near future can be forecasted. In this unit, you will study mathematical relationships (functions) in the context of managerial problems.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Explain functional relationships in business calculations;
- Define the terminologies relative to function;

3.0 MAIN CONTENT

3.1 Definitions of Terminologies

From unit 1, you were exposed to basic mathematic expressions that will enhance your business and financial calculations. You correlate your previous knowledge vis-à-vis this exposition to more dimension.

Variable

A variable is something whose magnitude can vary or which can assume various values. The variables in applied business mathematics include: sales, price, profit, cost, etc. Since magnitude of variables can vary, therefore, are represented by symbols (such as x, y, z etc.) instead of specific number.

P	=	price;
9	=	quantity;
c	=	cost;
s	=	saving or sales
d	=	demand etc. as the case may apply.

Variables can be classified in a number of ways. For example, a variable can be discrete (bias to counting e.g. 2 houses, 3 machines, etc.) or continuous (bias to measurement, e.g. temperature, height, etc.)

Constant and Parameter

A quantity that remains fixed in the context of a given problem or situation is called a constant.

An absolute (or numerical) constant such as V², 7t, etc. retains the same value in all problems whereas an arbitrary (or parametric) constant or parameter retains the same value throughout any particular problem but may assume different values in different problems, such as wages rates of different category of labourers in an industrial unit.

Function

There are situations in which two or more variables are related to each other. For example, demand (D) of a commodity is related to its price (P) mathematically expressed as:

$$f(P) \qquad (1)$$

i.e. "Demand is function of price" or simply "f of p". It is not D equals f times P. The function has two variable D and P. These are known as variables because they can take on different numerical values.

Let us consider a mathematical relationship that contains three variables. Assume that the demand (D) of a commodity is related to the price (P) per unit of the commodity, and the level of advertising expenditure (A). Then the general relationship among these variables can be expressed as:

$$D = f(P, A) \qquad (2)$$

The functional notations of (1) and (2) above are meant to give a general idea that certain variables are related. However, for making managerial decisions, we need a specific and explicit, not a general and implicit relationship among selected variables. For example, for the purpose of finding the value of demand (D), we make the general relationship (2) more specific as shown in (3) below:

$$4 + 3p — 2pA + 2A2 \ldots\ldots\ldots\ldots\ldots\ldots(^3)$$

Note: For any given values of p and A, the value of D can be calculated using the relationship (3). This means that the value of D depends on the values of p and A. D is the dependent variables and p and A are called independent variables. That is, a rule of correspondence established between the dependent variable and independent variable(s). As soon as values are assigned to the independent variable(s) the corresponding unique value for the dependent variable is determined by the given specific relationship. This is why a function is sometimes defined as a rule of correspondence between variables.

Other examples of functional relationships are as follows:

(I) Sales volume (V) of the commodity is a function of price (P) i.e. V = f (P);

(H) Total inventory cost (T) is a function of order quantity (Q), i.e. T =f (Q);

(iii) The net present value (Y) of an investment is a function of net cash flows (C_r) in different time periods; project's initial cash outlay (B), firm's costs of capital (P) and the life of the project (N), i.e. y = f (C_t, B, P, N).

To understand the nature of model, (mathematical relationship) between independent variable(s) and dependent variable you must be familiar with such terms as parameter, constants and variables.

Example 1

Suppose an industrial worker gets N25 per day. If he works for 26 days in a particular month, then his total wage for this month is N25 x 26 = N650. During some other month, he may have worked a total of only 25 days, and then he would have earned N625. Thus, the total wages of the worker, assuming no overtime; is calculated as follows:

Total wages	,	25 x number of days worked
T	=	Total wage
D	,	number of days worked
then T	=	25D

This represents the relationship between total wages and number of days worked. In general, the above relationship can also be written as:

T = KB

where k is a constant for a particular class of worker(s), to be assigned or determined in a specific situation. Since the value of k can vary for a specific situation, problem or context (parameter).

3.2 Types of Function

They include: linear polynomial, absolute value, inverse step, algebraic and transcendental (exponential and logarithm).

(1) Linear Functions: A linear function is one in which the power of independent variables is 1, the general expression of linear function having only one independent variable is:

$$y= f(x) = a + bx$$

where; a and b are given real numbers and x is an independent variable taking all numerical values in an interval. A function with only one independent variable is also called single variable function. A single-variable function can be linear and non-linear. For example:

$y = 3 + 2x$, (linear single-variable function) and

$y = 2 + 3x + 3x — 5x^2 \pm x^2$ (non-linear single-variable function)

A linear function with one variable can always be grouped in a two dimensional plane (or space). This graph can always be plotted by giving different values to x and calculating corresponding values of y. The graph of such functions is always a straight line.

Example 2

Plot the graph of the function, $y = 3 + 2x$

For plotting the graph of the given function, assign various values to x and then calculate the corresponding values of y as shown in the table 2.1 below:

x	0	1	2	3	4	5
Y	3	5	7	9	11	13

Table2.1

The graph of the given function is shown in figure 1:

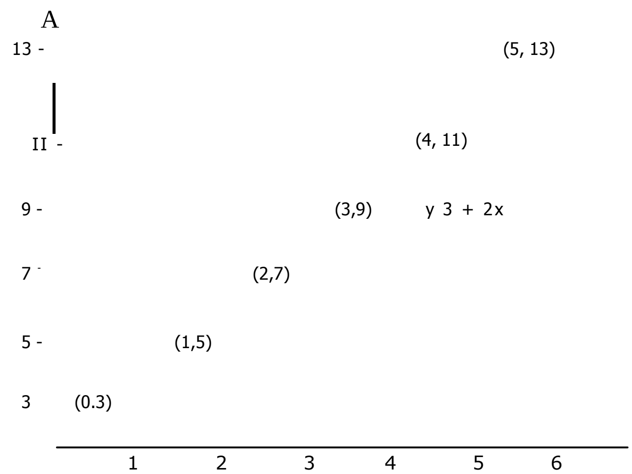

Figure .1

A function with more than one independent variable is defined, in general form, as:

$$y = f\,(x_i,\ x_2,\ \ldots\ldots\ldots\ldots\ x_n) = a_n,\ a_l x,\ a_2 x_2 \pm\ \ldots\ldots\ ax_{,,}$$

Where a_n, a_l, a_2, ..., a_n are given real numbers and x_l, x_2, $x_{,,}$ are independent variables taking all numerical value in the given intervals. Such functions also called multivariable functions. (It can be linear and non-linear). For example:

$y = 2 + 3x_1 + 5x_2$ (linear multi-variable function) and

$y = 3 + 4x_1 + 15x_{,}x_2 + 10x_2^2$ (non-linear multivariable function)

Multivariable function may not be graphed easily because these require three-dimensional plane or more dimensional plane for plotting the graph. In general, a function with n variables will require (n + 1) dimensional plane for plotting its graph.

2. Polynomial Functions: A function of the form:

$$y = f(x) = a_l x'' + a_2 x''^{-I} + \qquad + ax' +$$

where a,'§ (i = 1, 2, ..., n + 1) are real numbers, ... (4)

al # 0 and n is a positive integer is called a polynomial of degree n.

(a) If n = 1, then the polynomial function is of degree 1 and is called a linear
function. This is, for n = 1, function (4) can be written as:

$$y = a_l x^l + a_2 x^0 \; (a_l \# 1)$$

This is usually written as:

$$y = a + bx \; (.' \qquad = 1)$$

where 'a' and 'b' symbolize a_2 and a_l respectively.

(b) If n = 2, then the polynomial function is of degree 2 and is called a quadratic function. That is, for n = 2, function (4) can be written as:

$$y = a_1 x^2 + a_2 x^1 + a_3 \; (a, \# 0)$$

This is usually written as:

$$y -= ax^2 + bx + c$$

where $a_l = a$, $a_2 = b$ and $a_3 = c$

(3) Absolute Value Functions: By absolute value, we mean that whether x is positive or negative, its absolute value remains positive.

For example /7/ = 7 and 1-6/ = 6.

(4) Inverse Function: Take the function y = f (x). Then the value of y, can be uniquely determined for given values of x as per the functional relationship. Sometimes, it is required to consider x as a function of y, so that for given values of y, the value of x can be

uniquely determined as per the functional relationship. This is called the inverse function and also denoted by $x = f^1$ (y). For example, consider the linear function:

Expressing this in terms of x

$$x = \frac{y - b}{a} = \frac{y}{a} - \frac{b}{a} = cy + d$$

where c = 1/a, and d = —b/a

This is also a linear function and is denoted by $x = f^{-1}(y)$

(5) Step Function: For different values of an independent variable x in an interval, the dependent variable y = f (x) takes a constant value, but takes different values in different intervals. In such cases, the given function y = f (x) is called a step function. For example:

y = f (x) y_k if 0 x<50
Y2, if 50 x < 100
y_3, if 100 < x < 150

This can be shown graphically in figure 2.2 below for y_3 <Y2 < Yi:

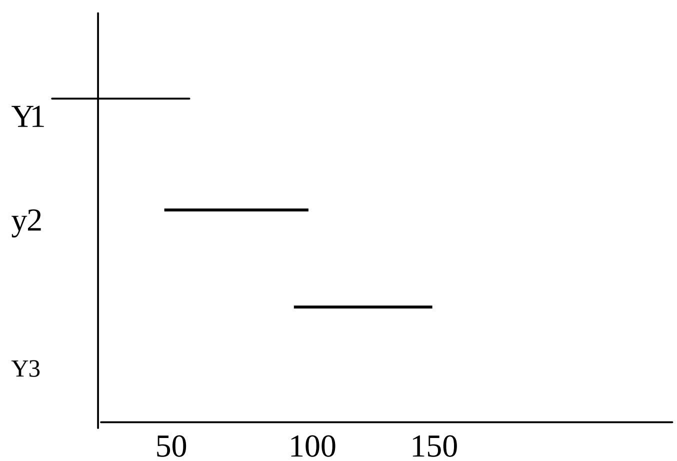

Figure 2

(6) Algebraic and Transcendental Function:

In unit 1, you were treated to functions with respect to mathematical operations (addition, subtraction, multiplication, division, power (exponents) and roots) associated with functional relationship between dependent variable and independent variable(s). When only finite numbers of items are involved in a functional relationship and variables are affected only by the mathematical operations, then the function is called an algebraic function, otherwise transcend function.

The sub-classes of transcendental functions are as follows: Exponential and logarithm i.e. functions. The exponential has been introduced in unit 1, you will now be introduced to logarithm function — expressed as:

$$y = \log_s x$$

where a 1 and > 0 is the base. It is read solution of functions.

The value(s) of x at which the given function f (x) becomes equal

to zero are called the roots (Or zeros) of the function f (x) for the linear function:

$$y \; ax + b$$

the roots are given by:

$$ax + b = 0 \text{ or } \quad x = \frac{-b}{a}$$

Thus, if x = —b/a is substituted in a given linear function y = ax + b, then it becomes equal to zero.

In the case of quadratic function

$$y = ax^2 + bx + c,$$

to solve the equation ax^2 bx + c = 0; a 0 to find the roots of y. The general value of x for which the given quadratic function will become zero is given by:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, in general, there are two values of x for which y becomes zero. One value is:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and other value is:

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- It is very important to note that the numbers of roots of the given function are always equal to the highest power of the independent variable.

In supply and demand functions, cost, profit, revenue, production and utility functions etc. are applied in mathematical operation, examples will be constructed to show these functions an their solutions.

Example 3

A company sells x units of an item each day at the rate of N50 per unit. The cost of manufacturing and selling these units is N35 per unit plus a fixed daily overhead cost of N1000. Determine the profit function.
Solution:

The revenue received by the company per day is given by:

Total Revenue (R) = price per unit (P) x number of items sold (S)

– 50 x x (50.x)

The total cost of manufactured items per day is given by:

Total Cost (C) = variable cost per unit (V) x number of items manufactured (N) + fixed daily overhead cost (FOC)

– 35.x+ 1000

Thus, total profit (p)= Total revenue (R) — Total cost (C)

- 50.x — (35.x + 1000) — 15.x— 1000

Example 4

Let the market supply function of an item be q = 160 + 8p, where q

Solution:

Total profit function can be constructed as follows:

Total profit (P) = Total revenue — Total cost

Price per unit x Quantity supplied

p.q — c.q

= (p — c).q

Given that c = Nq and q = 160 + 8p, then total profit function becomes:

$$P = (p - 4)(160 + 8p)$$
$$= 8p^2 + 128p - 640$$

If p = 500, then = $500 = 8p^2 + 128p - 640$

Thus becomes = $8p^2 + 128p - 1140 = 0$

Writing as $ax^2 + bx + c = 0$ (the quadratic equation formula)

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-128 \pm \sqrt{(128)^2 - 4 \times 8 \times (a-1140)}}{2 \times 8}$$

$$m \frac{-128 \pm 229.92}{16}$$

6.36 or —22.37

Since negative price has no economic meaning, therefore, the required price per unit should be N6.36.

SELF ASSESSMENT EXERCISE

(i) Solve the equation $2x^2$ — I lx + 22 = 10 using the quadratic equation formula.

(ii) Explain the role of variables in describing a business transaction and give four (4) examples of each terminology.

4.0 CONCLUSION

In this unit, effort has been put in place to explain the unique role of functional relationship among decision variable. It will facilitate your understanding of basic mathematics in business calculation.

5.0 SUMMARY

We started with the mathematical concept of function and defined terms such as constant, parameter, independent and dependent variable. Various examples of functional relationships are mentioned. Types of functions normally used in managerial decision-making are sited along with suitable examples, their graphs and solution procedure, with some examples.

6.0 TUTOR MARKED ASSIGNMENTS

(1) Name various types of functions used in calculus (mathematical calculations) and explain two you know in detail.

(2) If the profit function of business is $p^2 - 4p + 10 = 6$, what is the required price per unit?

7.0 REFERENCES AND FURTHER READING

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UNIT 3 SEQUENCES AND SERIES

CONTENTS

1.0 INTRODUCTION

Sequence of numbers — when a set of numbers is arranged in a recognizable pattern so that there is a first, second and third number and so on, then the set of numbers is called a sequence of numbers, e.g. 2, 4, 6, 8, ... Such numbers are arranged according to a definite rule and each number in the sequence is called a term of the sequence. The terms are usually expressed as x_l, x_2, x_3... x„ ... where x, is the first term, x_2 is the second term, etc. The nth term x„, is the general term. The two kinds of sequences useful for business applications, especially in financial and banking (interest) calculations are: arithmetic progression (arithmetic sequence) and geometric progression (geometric sequence).

2.0 OBJECTIVES

After studying this unit, you should be able to:

- define the arithmetic and geometric progression;
- calculate sequences of numbers in arithmetic or geometric progression in business;

3.0 MAIN CONTENT

3.1 Sequences

If for every positive integer (number) n, there corresponds a number an is related to n by some rule, then the terms a_l, a_2..., a_n ... are said to form a sequence. A sequence is denoted by bracketing its nth term i.e. (a_n).

Example of some sequences:

(i) If an = n^2, then sequence (a_n) is 1, 4, 9, 16, ... n^2, ...
(ii) If an = 1/n, then sequence (a„) is I, 1/4, 1/4, 1/4, ..., 1/n ...
(iii) If an = n^2/n+1, then sequence (a_n) is 1/4, 4/3, 9/4, ..., n^2/n+1,

Note: The concept of sequence is very useful in finance and banking — simple and compound interest, annuities, present value and mortgage payments.

3.2 Series

A series is formed by connecting the terms of a sequences connoting plus or minus sign. Thus, if a_n is the nth term of a sequence, then: a, + a_2 ... a_n is a series of n terms.

3.3 Arithmetic Progression (AP)

A progression is a sequence whose successive terms indicate the growth or progress of some characteristics. An arithmetic progression is a sequence whose term increases or decreases by a constant number — common difference of an A.P, denoted by d. That is, each term of the arithmetic progression after the first is obtained by adding a constant d to the preceding term. The standard form of an A.P is written as:

$$a, a + d, a + 2d, + a + 3d.$$

where 'a' is called the first term. Thus, the corresponding standard form of an arithmetic series becomes:

$$a + (a + d) + (a + 2d) + (a + 3d) +$$

Example I

Suppose we invest N100 at a simple interest of 15% per annum for 5 years. The amount at the end of each year is given by 115, 130, 145, 160, 175.

This forms an arithmetic progression.

The nth term:

The nth term of an A.P is also called the general term of the standard A.P. It is given by:

T„ = a + (n — 1) d; n = 1, 2, 3... sum of the first n terms of an A.P.

Consider the first n terms of a, a + d, a + 2d, a + 3d ..., a + (n — 1) d.
The sum, S,, of these terms is given by:

= a + (a + d) + (a + 2d) + (a + 3d + + a + (n — 1) d

= (a -I a) ... 1- a) F d {1 + 2 -I- 3 I ....± (n — 1){

= n.a + d $\frac{n (n-1)}{2}$ } (using formula for the sum of first (n— 1) natural numbers)

$\frac{n \{ 2a (n-1) cl\}}{2}$

Example 2

Suppose Mr. Sola repays a loan of N3,250 by paying N20 in the first month and then increases the payment by N15 every month. How long will be taken to clear his loan?

Solution:

Since Mr. Sola increases the monthly payment by a constant amount, N15 every month, therefore d = 15 and first month installment is, a = N20. This forms an A.P. If the entire amount be paid in a monthly installment, then we have:

$$\frac{n\{ 2a+(n-1)d\}}{2}$$

$$\text{or } 3{,}250 = \frac{n \{ 2 \times 20 + (n-1) 15\}}{2}$$

$$3{,}250 \times 2 = n \{ 2 \times 20 + (n-1) 15\}$$

$$6{,}500 = n\{25 + 15n \}$$

$15n^2$ -I- 25n —6500 = 0 a quadratic equation in n.

Find n applying formula as previously discussed in unit 2.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-25 \pm \sqrt{4(25)^2 - 4 \times 15 \times (-6500)}}{2 \times 15}$$

$$\frac{-25 \pm 625}{30} = \quad 20 \text{ or } -21.66$$

The value n = —21.66 (negative, hence meaningless). Mr Sola will pay the entire amount in 20 months.

SELF ASSESSMENT EXERCISE 2

(i) Find the 15th term of an A.P. whose first term is 12 and common difference is 2.

(ii) A retailer's revenue and costs exactly balanced on December 31, 2001. During the following year, his costs rose by N150 each month and his revenue rose by N200 each month. What were; (a) the cost incurred and (b) the revenue received during the month of December 2002?

3.4 Geometric Progression (G.P)

A geometric progression (G.P) is a sequence when each terms increases or decreases by a constant ratio called common ratio of G.P and is denoted by r. In other words, each term of G.P is from the first by multiplying the preceding term by a constant r. The standard form of a G.P is expressed as a, ar, ar^2..., where `a' is called the first term.

Thus, the corresponding geometric series in standard form is

$$a + ar + ar^2 +$$

Example 3
Suppose we invest 14100 at a compound interest of 12% per annum for three years. The amount at the end of each is calculated as follows:
(i) Interest at the end of first year:

- 100 x 12 =- 1412
  100

Amount at the end of first year:

– Principal + Interest

– 100 + 100 (12/100)

- 100 0 + 12/100)

(Principal at the beginning of second year) = { **1 +** 12 }
100

- 100 (1 + 12/100) {1 + 12/100)

100 $\{1 + 12/100\}^2$

Amount at the end of third year:

– 100 {1 + 12/100) {1 + 12/100) {1 + 12/100}

- 100 $\{1 + 12/100\}^3$

Thus, the progression giving the amount at the end of each year is:

– 100 {1 ±12/100); 100 (1 ±12/100)2; 100 {1 +12/1O0}; ....

This is a G.P with common ratio r = {1 -I- 12/100)

In general, if P is the principal and the compound interest rate per annum, then the amount at the end of first year becomes P (1 + 1/100). Also, the amount at the end successive years forms a G.P.

= P(1 + 1/100); P $(1 + 1/100)^2$; .... with r = P (1 -I- 1/100)

The nth term of:

The nth term of G.P. is called the general term of the standard G.P. It is given by

$T_n \quad ar^{n-1} \quad n = 1, 2, 3....$

Note here that the power of r is one less than the index of T_n, which denotes the rank of this term in the progression.

Sum of the First n Terms in G.P

Consider the first n terms of the standard form of G.P.

$a, ar, ar^2, \quad ar^{n-1}$

The sum, S,, of these terms is given by:

$$S_n \quad a + ar + ar^2 \quad arn^{2} + a?'^{I} \ldots\ldots \textbf{(1)}$$

Multiplying both sides by r, we get:

$$rS_n = ar \text{-F} ar^2 ar^3 + \quad + arn^{-2} \text{-I- } a? \qquad (2)$$

Subtracting (2) from (1), we have:

$$S_n - rS_n \quad a -$$

$$S_n (1 - r) \quad a (1 - ?)$$

Or $S = \dfrac{a (1 -}{(1 - r)}$; r #1 and < 1

Changing the signs of the numerator and denominator, we have:

$$S_n = \frac{a (r^n - 1)}{(r - 1)} ; r \#1 \text{ and } > 1$$

(a) If r = 1, G.P becomes a, a, a, ... so that S,, in this case is S_r, = n.a.

(b) If number of terms in a G.P are infinite, then:

$$S_n = \frac{a}{(1 - r)} ; r < 1$$

For r I, the sum tends to infinity.

Example 4

A car is purchased for N80,000. Depreciation is calculated at 5% per annum for the first 3 years and 10% per annum for the next 3 years. Find the money value of the car after a period of 6 years.

Solution:

(i) Depreciation for the first year:

$$\frac{80{,}000 \times 5}{100}$$

The depreciated value of the car at **the** end of first year is:

$$80{,}000 - \frac{80{,}000 \times 5}{100}$$

$$80{,}000 \left(1 - \frac{5}{100}\right)$$

(ii) Depreciation for the second year:

(Depreciated value at the end of first year) x (rate of depreciation for second year)

$$80,000\left(1 - \frac{5}{100}\right)\frac{5}{100}$$

Thus, the depreciated value at the end of the second year is:

(depreciated value after first year) — (depreciation for second year)

$$80,000\left(1 - \frac{}{100}\right) - 80,000\left(1 - \frac{j}{100}\right)\frac{/51}{100}$$

$$80,000\left(1 - \frac{5}{100}\right)\left(1 - \frac{5}{100}\right)$$

$$80,000\left(1 - \frac{5}{100}\right)^2$$

Calculating in the same way, the depreciated value at the end of three years is:

(depreciated value after second year) — (depreciation for third year)

$$80,000\left(1 - \frac{5}{100}\right) - 80,000\left(1 - \frac{5}{100}\right)\left(\frac{5}{100}\right)\left(\frac{5}{100}\right)$$

$$80,000\left(1 - \frac{5}{100}\right)\left(1 - \frac{5}{100}\right)\left(1 - \frac{5}{100}\right)$$

$$80,000\left(1 - \frac{}{100}\right)^3$$

(iii) Depreciation for fourth year:

$$80,000\left(1 - \frac{}{100}\right)^3\left(\frac{10}{100}\right)$$

Thus, the depreciated value at the end fourth year is:

(depreciation value after three years) x (depreciation for fourth year)

80,000(1 _)3 — 80,000 (1 –)3 (L))
100 100 100

Calculating in the same way, the depreciated value at the end of six years becomes:

= 80,000(1 –5f — 80,000 (1 _)3 (1 0)3
100 100 10

N49,980.24

SELF ASSESSMENT EXERCISE 3

(i) Find the 6th term of the geometric progression 72, —24, 8, ...

(ii) You decide to increase your saving by N200 per annum. If you save N500 in the first year, how long will it take to save N5,300?

4.0 CONCLUSION

In this unit, you can define sequences of numbers in arithmetic or geometric progression and examples of their application in some business calculations.

5.0 SUMMARY

This unit has directed you to define, distinguish and establish managerial applicability of arithmetical and geometrical progressions in business operations.

6.0 TUTORMARKED ASSIGNMENTS

(1) Find the general term and the sum of first 20 terms, of the arithmetic progression 2, 6, 10 .....

(2) Suppose that people spent 0.9 and save 0.1 of any money received. If N1,000 is received, then N900 is spent and N100 saved. The people receiving the N900 will spend N810 and save N90. This process will continue. Find the maximum sum of a geometric progression.

7.0 REFERENCES AND FURTHER READINGS

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UNIT 4 INVESTMENT APPRAISAL I

CONTENTS

1.0 INTRODUCTION

In unit 3, you were led through the progressions (arithmetic and geometric) as they are need in business and financial calculations. This unit will introduce geometric progression relationship with investment appraisal through compound interest problems solving. The compound interest rate used for discounting cash flows is also called the discount rate.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- Identify how close geometric progression is to solving compound interest problems;
- Solve problems in investment appraisals.

3.0 MAIN CONTENT

3.1 Compound Interest

Compound interest is the interest that is received on the original amount (principal) but not withdrawn during earlier periods. Compounding is the process of finding the future values of each cash flow by applying the concept of compound interest.

As stated above, geometric progressions are closely linked to compound interest calculation as could be stated below.

Suppose N100 is invested at 5% per annum, then the interest earned after the first year is:

$$\frac{N100 \times 5}{100} = N5$$

and the value of the investment at the end of the first year would be

N100 + N5 =14105

If the rate of interest is expressed as a proportion rather than a percentage, the value of the investment can be calculated thus:

14100 -i 100 x 0.05

N100 (1 + 0.05) = 14105

Expressing the rate of interest as a proportion is more convenient than expressing it as a percentage. Instead of a rate of interest of r% per annum, we shall let the rate of interest be 100r% per annum. This may appear a little difficult.

If the rate of interest is 7% per annum, then

100r = 7

$$\frac{7}{100} = 0.07$$

Suppose NP is invested at 100e/0 per annum compound interest. At the end of the first year, the value of investment is

P + Pr = P (1 Fr)

At the end of the second year, the investment is

P 0 10 ± rP (1 r)

– (1+r) (P + rP)

– (l+r) (1 +r) P

– P (1+0²

At the end of the third year, the value of the investment is

$(1+0^2$ P (1+0

P $(l+r)^3$

It should be obvious to you from the pattern emerging that the value of the investment at the end of the nth year is

P (1 + r)" (1)

This formula involves calculating P r)" and from the basic knowledge of basic mathematics in unit I and 2, where exponential function mare discussed, you would be able to calculate with ease and with the use of scientific calculator. In most books that involves mathematics of finance, discount table are normally provided to assist. It gives values of (I It 0 $^{-IT}$ which is a more convenient form than (I I-On. The discount table is in table I and gives sufficient vahaes for all calculation related to investment appraisal (compounding),. Rates of interest are measured horizontally and time is measured vertically. (Most books of logarithms include a reciprocal table. The reciprocal of a number is the number divided into unity:

The reciprocal of x =1 = x^{-1}

The reciprocal of x^{-1} = I =
x-t

Refer to unit 1, hence reciprocal of (1 + r) = (1 +

Example 1

Suppose N1,000 is invested at rA per annum compound interest. The value of the investment at the end of the tenth year is

$1000\ (1 + 0.07)^{10}$

= $1000\ (1.07)^{10}$

1000 x(1.967)

N1,967.00

OR

$1000\ (1 + 0.07)^{10}$ (From discount tables $(1 -I- 0.07)^{-1°}$ is 0.5083)

$1 + 0.07)^{1°} = \frac{1}{0.5083}$

1.967

Hence, the value of the investment is 1000 x 1.967 = N1,967.00.

Example 2

Suppose that N1,000 are placed in the savings account of a bank at 5 percent interest rate. How much shall it grow at thc end of the three years? It will grow as follows:

$= \quad 1000\,(1 + 0.05)^3$

$1000\,(1.05)^3$ In the scientific calculator, you enter 1.10,

press y" key, press 5 and then equal to key

1000 (1.1576) (---) etc.

N1,157.60

OR Breakdown

Year 1 = 1000 + 1000 x 5%

- 1000 + 50

−N1,050.00

Year 2 = 1050± 1050 x 5%

−1000 + 52.50

- N1,102.50

Year 3 - 1102.50± 1102.50 x 5%

1000 + 55.10

N1,157.60

3.2 Increasing Investment

Suppose that you decided to invest N1,000 on the first of January in a certain year, and to invest a further NI00 at the end of each year. If interest is compounded at 10% per annum, we can deduce the following:

The amount invested at the end of the first year is

$$1000\ (1 + 0.10) + 100$$

The amount invested at the end of the second year is

$$1000\ (1 + 0.10)^2 + 100\ (1 + 0.10) + 100$$

The amount invested at the end of the nth year is

$$1000\ (1\ 0.10)" + 100\ (1\ F \qquad \pm 100\ (1 + 0.10r^2 + 100$$

At this point, it is useful to have a general function. If an amount P is invested at the beginning of a year, and a further amount a is invested at the end of each year, and ifs is the sum invested after n years then:

$$P\ (1 + r)" + a\ (1 + \qquad + a\ (1 + rf^2 + ... + a$$

If the first term of the right hand side is ignored, then the remainder forms a geometric progression with a first term a **(1 + r)** $^{-I}$ and a common ratio 1/1+r. Substituting in the G.P formula, the series can be summed for n years:

$$P_0 \pm {}_0 n + {}_a 0 + \qquad + \frac{\{1 \_ \quad 1\quad r}{1 - \frac{1}{\mathbf{1} + r}}$$

Now to tidy up this expression, writing the denominator as a single fraction gives:

$$\frac{\underline{\mathbf{1 +} r}}{1 - \frac{1}{1 + r}}$$

OR

$$P\ (1 + \qquad + \underline{a\ (1 + rr^i + (1\ \_ (1)"\ 1\ (1\ \text{-F}r)}$$

Using the rules of indices, in mathematical expressions (calculation):

– **P** (1 +r)"+a(1 +0'1+ (1 —(1)"} (1+4

– PO +0"+a(1 +r)" —a

Removing the common factor (1 + rf

P (1 + r)" + (1 + r)" — a .... (2)

Example 3

You can now use the formula to solve the original problem. Suppose you wish to know the sum invested after 4 years:

Then S = (moo + oc) + (1 + 0.0^4 — oo
0.1 0.1

– 2000 (1.1)^4— 1000

– 2000 (1.464)

N1,928.00

Often, you wish to know how much must be invested now to give a specified income for a specified period, given the rate of interest. Clearly, the sum invested at the end of the period will be zero as follows:

0 = (P — a) (1 + **r)" +** a

(NOTE: that as withdrawal of fund is made, 'a' will be negative).

0 = P(1 + r)^r— a (1 -I O^n -I- a

P(1 F r)" a(1+ — a

P(1 I r)" a [(1 + r)" — 1 I

—(1 + r) .... (3)

Example 4

Suppose an investment is required to yield N1,500 at the end of each year for five years, and money can be invested at 5% per annum compound interest, then P, the sum that must be invested at the beginning of the first year is:

$$\frac{1500\left[1 - (1 + 0.05)^{-5}\right]}{0.05}$$

$$\frac{1500 \times 0.2165}{0.05}$$

N6,495.00

SELF ASSESSMENT EXERCISE I

(i) Your uncle died and leaves you N1,500 invested in unit trusts. Your financial adviser stated that the value of your investment can be expected to grow by 7% per annum (income and capital growth). Estimate the value of the investment after 10 years.

(ii) A Chief Executive Officer is due to retire at the end of the year, and the board vote that an income of N2,000 per annum be paid to him or his family for 10 years. The accountant is instructed to set aside a sum of money from which the income will be paid. If the fund can be invested at 8% per annum, how much should the account set aside?

4.0 CONCLUSION

In this unit, you have leant the link between the progression (especially geometric) to compound interest calculation and how they are used in investment analysis, appraisals as well as calculations.

5.0 SUMMARY

Attention has been focused on investment, savings, pension fund etc. and how they are calculated. The role of compounding is emphasized; noting how it relies on the concept of compound interest and how they are related to investment appraisal, and calculation.

6.0 TUTOR-MARKED ASSIGNMENT

Suppose N20,000 is invested at the beginning of a year at 5% per annum compound interest, and N2,000 is withdrawn at the end of each year for four years. Note that a is a withdrawal and will be negative.

7.0 REFERENCES AND FURTHER READINGS

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UNIT 5 INVESTMENT APPRAISAL II

CONTENTS

1.0 INTRODUCTION

As a follow up from unit 4, you will be shown how compounding can be used to increase an investor's analytical power to compare cash flows that are separated by more than one period, given interest rate per period. With the compounding technique, the amount of present cash can be converted into an amount of cash of equivalent value in future.

Present value of a future cash flow (inflow or outflow) is the amount of current cash that is of equivalent value to the decision maker.

Discounting is the concept of determining present values of a series of future cash flows.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain present value technique (discounting) in financial decisions;
- calculate present and future values of investment appraisals;

3.0 MAIN CONTENT

3.1 Present Value Concept

Were someone to offer you the choice of receiving N1,000 cash now, or alternatively, N1,000 cash in twelve months' time, it is highly likely that you would prefer to take the money now, as most people would. People have a strong preference for holding cash now as against the prospect of receiving cash in future. Economist calls this preference, 'liquidity preference'. To the mathematician, who in many aspects of his work is try to quantify human behaviour, the concept implies that people place a higher value on money received now than they do on the same amount of money received in the future (time value of money).

The recognition of the **time value of money** and risk is extremely vital in financial decision-making. If the timing and the risk of cash flows is not considered, the person (manager etc. may make decisions that may allow it to miss its objective of maximizing the owners' welfare. The welfare of owners would be maximized when wealth or net present value is created from making a financial decision.

What is Net Present Value? How is it computed?

We may say that the N1,000 we receive today has a greater value to use than the N1,000 we are going to receive next year. It is still the same value, but the mere fact that you can express a preference is sufficient evidence that these one Naira notes have a different value according to when they are received. Preference alone, however, is of little use in mathematics. If this aspect of human behaviour is to be analysed, and if the analysis is to be of assistance, one must find some means of qualifying the preference. That is of saying by how much we prefer N1,000 now to N1,000 next year or the year after.

Fortunately, the use of the interest formula developed earlier in unit 4 gives us a simple method of assessing the difference in value between money now and money in the future. You know that if we were to invest N1,000 today at say 10% interest, at the end of twelve months it will have grown to N1,100 and at the end of two years to N1,210. It is thus possible to say that N1,000 is the present value both of N1,100 received one year hence, and also of N1,210 received two years hence, when the rate of interest is 10%. Generalizing present value may be defined as follows:

The present value of Na receivable n years hence is that sum of money which, invested at the current rate of interest roA, will amount to Ilia after the expiry of n years.

This concept is more manageable if you consider the formula already developed. You know already that if NP is invested for n years at r% expressed as:

$$a \quad = \quad P(1+r)^n \qquad \text{.............} \qquad (1)$$

From this, we can deduce that

$$\frac{a}{(1+r)^n} \quad = \quad a\,(1+r) \qquad (2)$$

These two equations {(1) and (2)) are the basis of every investment appraisals. You will have realised that they are in reality merely two different forms of the same equation, but with this important difference tells:

(1) The terminal value of a given sum of money invested at interest over time;

(2) What to invest to achieve at a given terminal sum over a given period over a given time at a given interest rate

It shows in other words, what is the present value of a given sum of money received some time in the future.

Example 1

Suppose we wish to find the present value of N50 receivable in one year's time, given the rate of interest as 10%. The present value, P = N50 $(1 + 0.1)^{-1}$ i.e.

$$\frac{50}{1.1} = \text{N}45.45$$

Recall: the meaning of present value, the logic of this result should be obvious.

N45.45 invested at 10% for 1 year would yield interest of N4.545. Thus, the value of the investment after 1 year would be N45.45 + N4.545 = N50.

Example 2

Suppose we wished to calculate the present value of N1,000 receivable in seven year's time, assuming a 5% rate of interest. Substituting in the formula for present value, you have

$= \quad 1000\,(1 + 0.05)^{-7}$

You will remember the discount tables that give values of (1 + r)

$(1 + 0.05)^{-7} = 0.7107$

hence $\quad P = 1000 \times 0.7107$

N7,107.00

SELF ASSESSMENT EXERCISE 1

(i) Calculate from the first principles the present value of N10,000 receivable one year hence given a rate of interest of :

(a) 3%
(b) 20%

(ii) If you were buying a television set, would you prefer to pay:

(a) N200 now in complete settlement;
OR
(b) N100 now and a second installment of N110 in twelve months' time.

(in) What is the present value of Ni receivable one year hence, given a rate of interest of:

(a) 4%
(b) 6%

3.2 Investment Decisions

The investment decision rules may be referred to as capital budgeting techniques, or investment criteria. A sound appraisal technique should be developed to measure the economic worth of an investment project. The essential property of a sound technique is that it should maximize the shareholder's wealth. One of the characteristics of a sound investment evaluation criterion is that "it should consider all cash flows to determine the true profitability of the project".

In the above examples, you have calculated the present value of a future flow of income and of costs met at different points of time. These two calculations form the basis of most investment decisions (cash flow to determine profitability).

So far, as income is concerned, we would prefer to undertake that project which yields revenues having the greatest present value.

Assumption

You will assume that the objective of the firm considering investment projects is the maximization of profit. This assumption may not be true at all times for all firms, but it is a good working approximation. You can say then that the aim of the firm is to maximize the difference between the present value of the future flow of income and the present value of the flow of costs.

Example 3

An equipment costing N1,000 has an expected life of 5 years. It is estimated that the cash flow resulting from the use of the machine will be N400 a year. The rate of return expected from capital of this type is 15%. Is the investment worthwhile?

The appraisal would be as follows:

Year	**Income**	**(1 + 0.15) —n**	**Present Value**
1	400	0.8696	347.84
2	400	0.7561	302.44
3	400	0.6575	263.00
4	400	0.5718	228.72
5	400	0.4972	198.88
			1,340.88

Thus, for an investment of N1,000 now, you will receive a return over the next five years which has a present value of N1,340.88. What does this signify? It can mean one the one hand that to obtain a return of N400 a year for five years, you would have to invest N1,340.88 at 15% and hence the machine is a good investment. Alternatively, you can look at it this way. Had the rate of return to capital been 15% in this case, the present value of the cash flow discounted at 15% would have been **N1,000.**

The fact that we have a surplus return of N340.88 means that the rate of return to this investment is greater than 15%. The machine is yielding a rate of return greater than capital of this type generally.

This appraisal depends on three variables:

(i) the cost of the equipment;
(ii) the expected income;
(iii) the expected rate of returns.

For present purposes, however, it is enough to note that if you are comparing two projects with different costs of purchase, both of which yield a surplus return, you can compare them by expressing the surplus return as a percentage of cost price. Thus, in the above example, the surplus return is N340.88 which is 34.088% of cost.

Note: Ultimate costs depend on each factor as development costs. In real life, these costs could be over or understated and this will influence the rate of return which is normally obtained from the type of capital applied although expectations of industrialists can vary over time. Consideration is made of appropriate discount rate too in certain circumstance.

The unknown factor is the expected flow of income. At times, industry is often too optimistic about cash flow resulting from an investment.

3.3 Investment Projects

In practice, the decision to compare investment projects is not simply to invest or not to invest. Usually, there is a range of investment projects which can be undertaken, and the problem is to find the best

Example 5.4

A firm is faced with two alternative investment plans. Plan I will cost N750 and Plan 2 N950. Both plans involve the purchase of equipment the life of which is four years, and the current rate of return on capital is expected to be 20%. The estimated cash flows resulting from the projects are:

Year	1	2	2	4
Plan 1	N300	N400	N300	N200
Plan2	N500	N400	N300	N300

The present value of these expected returns assuming a rate of return on capital of 20% is:

Year	**Return**		**(1+ 0.2) '**	**Present value**	
	1	**2**		**1**	**2**
1	300	500	0.8333	249.99	416.65
2	400	400	0.6944	277.76	277.76
3	300	300	0.5787	173.61	173.61
4	200	300	0.4823	96.46	144.69
				797.82	**1,012.71**

Both projects will yield a surplus.

With plan 1;

A return with a present value of N797.82 for a current cost of N750, a surplus of N47.82.

Plan 2 will yield N1,012.71 for a current cost of N950 — a surplus of N62.71.

Both are yielding a return greater than 20%.

The only way in which the two projects can be compared easily is by expressing the surplus as a percentage of the cost of the projects.

Project 1 has a surplus percentage return of almost 6.4%, while project 2 has a surplus percentage return of 6.6%. Thus, although there is little in it, project 2 would be the preferred project to undertake.

SELF ASSESSMENT EXERCISE 2

(i) A further examination of the two projects in Example 5.3 shows that the equipment bought for project 1 will have a scrap value of N150 at the end of its life, while the equipment bought for project 2 will have a scrap value of 14100. Assuming the equipment is sold for cash, will the new information affect the investment decision?

(ii) The Government now decides to give an investment grant of 40% of the original cost of the equipment bought, payable one year after purchase. How will this affect the decision?

(a) ignoring scrap value; (b) allowing for scrap values as above?

3.4 Annuities

Annuity is a fixed payment (or receipt) each year for a specified number of years. If you rent a flat and promise to make a series of payments over an agreed period, you have created an annuity. The equal installments loan from the house financing companies or employers are common examples of annuities.

What is the present value of future sums of money? Or what sum of money investment now at r% for n years would accumulate to a given total? Or how much would I have to invest now at a given rate of interest r% in order to receive an income of Na per year for n years?

Refer to example 5.4. It meant that if you invested N797.82 at 20% compound interest, drawing N300 at the end of the first year, N400 at the end of the second year, and so on, by the time you drew N200 at the end of the fourth year you would just have exhausted our original deposit and the interest earned.

Now this is exactly what an annuity is — an investment which while earning interest, enables us to draw a given sum of money for n years, by which time your capital plus interest is just exhausted.

Annuity Formula

Suppose we consider an annuity yielding Na per year when the rate of interest is r%, that is:

Na receivable one year, hence has a present value of $\frac{Na}{1+r}$

Na receivable two year, hence has a present value of $\frac{Na}{(1+r)^2}$

Na receivable n years, hence has a present value of $\frac{Na}{(1+r)^n}$

Thus, the present value of an annuity yielding Na receivable two year, hence has a present value of Na per year for n years is:

$$P.V = \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + \frac{a}{(1+r)^3} + \dots \frac{a}{(1+r)^n}$$

As you can see, this is a geometric progression of which the first term is a/I 1 r and the common ratio 1/1 + r. The sum to n terms of a geometric progression is as you know, Sn = $\frac{a(1—r')}{1 —r}$ where a is the first term and r the common ratio.

Thus, in summing the progression, you have a = $\frac{a}{1+r}$ and r = $\frac{1}{1+r}$

The present value = the sum of the geometric progression

a [1 — (1) 1
1-Er 1-Er
1— 1
1 + r

a — ___ x 1 + r
1 + r ⌊ 1 + r

P.V = a l 1 — (1 + r) l

Notice the semblance of formula derived in a different fashion in Example 4.4 applied to equation (3) in unit r.

Example 5

An annuity yields N100 per annum for 7 years. How much does it cost if the current market rate of interest is 6%?

The present value of the annuity:

$$100\,[1 — (1.06)^{-7}]$$

$$\frac{100\,(1 — (0.6651)}{0.06} = \frac{100 \times 0.3349}{0.06}$$

SELF ASSESSMENT EXERCISE 3

(i) Effa, an employee of We Limited retired from his employment at the age of 65. He received from his employers a terminal gift of NL500 or the alternative of a pension amounting to N400 per year for life. Assume a rate of interest of 5% and that a man aged 65 has a life expectancy of 7 years, which alternative should Effa choose?

(ii) Had Effa adopted another course of action (using his lump sum to buy an annuity), how much income would he have gained or lost per annum?

3.5 Sinking Fund

A related type of problem to annuities is found in the concept of a sinking fund. Sinking fund is a fund, which is created out of fixed payments each period to accumulate to a future sum after a specified period. For example, companies generally create sinking funds to retire bonds (debentures) on maturity.

Obviously, this is a sum of money put aside at regular intervals usually monthly or annually in order to achieve a given sum at the end of a predetermined period. This concept may be found in the provision for replacement of capital, or in, say, unit trust investment of a few pounds per month by the individual saver. You will recall in unit 4 deriving the formula

$$\text{P I. } (1 + \text{O}^{n-1} 1$$

giving us the terminal value of an annual investment of EP at r% for n years.

It can easily be seen that if you are given the required terminal sum and wish to know the annual savings, we can modify the formula into:

$$\frac{Sr}{(1 + r)'' \ 1}$$

Example 6

A firm wishes to make provision for the replacement of certain items of capital equipment which will wear out in 8 years' time. The estimated cost of replacement is N5,000. If the rate of interest is 8%, what annual provision must it make to ensure funds being available?

The annual provision NP = $\frac{5000 \times 0.8}{(1+ \quad — 1}$

$(1 + r)n \quad = \quad \frac{1}{1.851} \quad = \quad \frac{1}{} =$

$(1 + r) — n \qquad 0.5403$

$$\frac{N400}{1\ 851 — 1}$$

N470.00

SELF ASSESSMENT EXERCISE 4

( ) Suppose an investor wants to find out the present value of N50,000 to be received after 15 years. Her interest rate is 9 percent. What is that present value?

(ii) A man invests N100 in a unit trusts and a further N100 at the end of each year thereafter for the next ten years. What will be the value of his holding at that time assuming a rate of growth in the value of the units of 3%?

3.6 Discount Rate

In this unit, you have been shown elements of Mathematics of Finance — the two most common methods of adjusting cash flows for time value of money:

(i) Compounding — the process of calculating future values of cash flows, and
(ii) Discounting — the process of calculating present values of cash flows.

Choosing an Appropriate Discount Rate

It should be quite obvious by now that in any of the techniques discussed the result obtained will depend largely on the rate of discount applied to future receipts. If this is not appropriate the results can not be a guide to policy. The precise rate you choose must of course depend on the problem in hand.

If we are evaluating capital investment, the most generally useful rate is the rate of return generally expected from capital of this nature. That is, capital investment is just worthwhile if when you discount cash flows, the present value of future cash is just equal to the cost of the investment. If there is a surplus above this, it implies that the rate of return is greater than the discount rate used. That is, the capital is earning a greater percentage return than you would expect. If there is a deficit, it is earning less than comparable capital elsewhere.

On the other hand, if you are dealing with the lending or borrowing of cash funds, you can usually find some well established market rate of interest for a loan of that type and that degree of risk which will give a more than useful guide you with a reasonable knowledge of finance, will soon be able to assess comparability of loans and interest rates.

In all these techniques, however, there is nothing like sound common sense. The methods outlined have been proved over time, but no technique, however good, is better than the user.

4.0 CONCLUSION

In this unit, you have learnt how investments can be analysed to compare cash flows at different time value — present and future. Illustrations of the two most common method of adjusting cash flows for time value of money. Compounding and discounting were used as appropriate.

5.0 SUMMARY

It is hoped that this unit has made you to be able to apply the present value technique in financial and investment decision. The calculations of the present and future values methods are well presented and explained to enable you apply them practically in business problems.

6.0 TUTOR-MARKED ASSIGNMENTS

(1) KC Bank pays 12 percent and compounds interest quarterly. If N1,000 is deposited initially, how much shall it grow at the end of 5 years?

(2) A firm purchases a machinery for N800,000 by making a down payment of N150,000 and remainder in equal installments of NI 50,000 for six years. What is the rate of interest of the firm?

7.0 REFERENCES / FURTHER READINGS

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MODULE 2

UNIT! VALUATION OF BONDS AND SHARES

CONTENTS

1.0 INTRODUCTION

In this unit, you shall be led through the explanation of fundamental characteristics of ordinary shares, preference shares and bonds (or debentures).

Assets can be real of financial; securities like shares and bonds are called financial assets, while physical assets like plant and machinery are called real assets. The concepts of return and risk, as the determinants of value, are as fundamental and valid to the valuation of securities as to that of physical assets. The unpredictable nature of the security prices is, in fact, a logical and necessary consequence of efficient capital markets. This will help the financial manager to know the variables which influence the security prices. The bottom-line is that it should be appreciated that ordinary shares are riskier than bonds (or debentures) and also that some shares are more risky than others.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- explain the fundamental characteristics of ordinary shares, preference shares and bonds (or debentures); earnings and dividends;
- apply the present value concepts in the valuation of shares and bonds;
- use price-earnings ratio

3.0 MAIN CONTENT

3.1 Concepts of Value

Earnings per share (EPS) and price-earnings (PIE) ratios are the most frequently used concepts by the financial community. From our previous unit on investment, it is emphasized that <u>the present value is the most valid and true concept of value.</u> There are many other concepts of value that are used for different purposes. They are named as follows:

- Book Value, Replacement Value, Liquidity Value, Going Concern Value, Market Value

3.2 Features of a Bond

A bond is a long-term debt instrument or security. Government bonds do not have any risk default as it is always honoured. The private sector companies also issue bonds (or debentures) — secured or unsecured. Their interest is generally fixed and known to investors. The principal of a redeemable bond or bond with a maturity is payable after a specific period called maturity period.

The main features of a bond or debenture are: Face Value, Interest Rate, Maturity, Redemption Value and Market Value.

3.3 Bonds Values and Yields

It is relatively easy to determine the present value of a bond since its cash flows and the discount rate can be determined. If there is no risk of default, then there will be no difficulty in estimating the cash flows associated with a bond. The expected cash flows consist of annual interest payments plus repayment of principle. The appropriate capitalization or discount rate would depend upon the risk of the bond.

Bonds are classified into three, namely:

- Bond with maturity;
- Pure Discount Bonds, and
- Perpetual Bonds.

(a) Bonds with Maturity:

The government and companies issue bonds that specify the interest rate (coupon) and the maturity period. The present value of bond (debenture) is the discount value of its cash flows, that is, the annual interest payments plus bond's terminal, or maturity value. The discount rate is the interest rate that investors could earn on bonds with similar characteristics. By comparing the present value of a bond with its current market value, it can be determined whether the bond is over-valued or under-valued.

Example 1 — Value of Bond with Maturity

Suppose an investor is considering the purchase of a five-year NI,000 per value bond, bearing a nominal rate of interest at 7% per annum. The investor's required rate of return is 8%. What will he be willing to pay now to purchase the bond if it matures at par?

The investor will receive cash of N70 as interest each year for 5 years and N1,000 on maturity (i.e. at the end of the fifth year).

The present value can be determined as follows:

$$B_0 = \frac{70,}{(1.08)^1} + \frac{70,}{(1.08)^2} + \frac{70,}{(1.08)^3} + \frac{70,}{(1.08)^4} + \frac{70,}{(1.08)^5} + \frac{1,000}{(1.08)^5}$$

Observation:

N70 is an annuity for 5 years and N1,000 is received as a lump sum at the end of the fifth year.

Using the present value tables in appendix, given at the end of this book, the present value of bond is:

$$B_0 = 70 \times 3.993 + 1,000 \times 0.681$$

$$– \quad 279.51 + 681$$

This implies that N1,000 bond is worth N960.51 today if the required rate of return is 8 percent. The investor would not be willing to pay more than N960.51 for bond today. Note that N960.51 is a composite of the present value of interest payments, N279.51 and the present value of the maturity value N681.00.

Since most bonds will involve payment of an annu9ity (equal interest payments each year) and principal at maturity, we can use the following formula to determine the value of a bond.

Bond value = Present value of interest + Present value of maturity value

$$B_o = \frac{INT_1}{(1+K_d)^1} + \frac{INT_2}{(1+K_d)^2} + \frac{INT_n}{(1+K_d)^n} + \frac{B_n}{(1+K_d)^n}$$

$$= \sum \frac{INT_t}{(1+K_d)^t} + \frac{B_n}{(1+K_d)^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Note: B_o is the present value of a bond (debenture),
INT, is the amount of interest in period t (from year 1 to n),
Kd is the market interest rate or the bond's required rate of return,
Bn is bond's terminal or maturity value in period in n and
n is the number of years to maturity.

In equation (1), the right-hand side consists of an annuity of interest payments that are constant (i.e. $INT_1 = INT_2 = INT$) over the bond's life and a final payment on maturity. Thus, an annuity formula can be used to value interest payments as shown below:

$$B_o = \frac{INT}{(1+K_d)^t} \times INT\left[\frac{1}{(K_d)} - \frac{1}{K_d(1+K_d)^n}\right] + \frac{B_0}{(1+K_d)^n} \ldots\ldots \quad (2)$$

Yield to Maturity (YTM) — the measure of a bond's rate of return that consists both the interest income and any capital gain or loss. YTM is bond's internal rate of return. The yield-to-maturity of 5 year bond, paying 6 percent interest on the face value of N1,000 and currently selling for N883.40 is 10 percent as shown below:

$$883.4 = \frac{60}{(1+YTM)^1} + \frac{60}{(1+YTM)^2} + \frac{60}{(1+YTM)^3} + \frac{60}{(1+YTM)^4} + \frac{60+1,000}{(1+YTM)^5}$$

It is, however, simpler to calculate a perpetual bond's yield-to-maturity. It is equal to interest income divided by the bond's price.

For Example:

If the rate of interest on N1,000 per value perpetual bond is 8 percent, and its price is N800, its YTM will be:

$$B_0 = \sum_{t=1}^{n=\infty} \frac{INT}{(1+K_d)^t} = \frac{INT}{K_d}$$

$$K_d = \frac{INT}{B_0} = \frac{80}{800} = 0.10 = 10\% \qquad (3)$$

Bonds Value and Semi - Annual

In practice, some companies pay interest on bonds (or debentures) semi-annually. The formula for bond valuation can be modified in terms of half-yearly interest payments and compounding periods as given below — ref. equation (1) giving it semi-annual approach.

$$B_0 = \sum_{t=1}^{2\times n} \frac{INT_t/2}{\left(1+\frac{k_d}{2}\right)^t} + \frac{B_n}{\left(1+\frac{k_d}{2}\right)^{2\times n}}$$

$$B_0 = \sum_{t=1}^{20} \frac{60}{(1.06)^t} + \frac{1000}{(1.06)^{20}}$$

60 x Annuity factor (6%, 20) + 1000 x PV factor (6%, 20)

60x 11.4699+ 1,000 x 0.3118

688.20 + 311.80

N1,000.00

(b) Pure Discount Bonds:

Pure Discount Bonds do not carry an explicit rate of interest. It provides for the payment of a lump sum amount at a future date in exchange for the current price of the bond. The difference between the face value of the bond and its purchase price gives the return or YTM to the investor.

For Example:

A company may issue a pure discount bond of N1,000 face value for N520 today for a period of five years. Thus, the debenture has:

(a) purchase price of N520;
(b) maturity value (equal to the face value) of N1,000 of five years

The rate of interest can be calculated as follows:

$$520 = \frac{1,000}{(1+YTM)^5}$$

$$(1 + YTM)^5 = \frac{1,000}{520} = 1.9231$$

1.923l"-1 = 0.14 or 14%

(c) Perpetual Bonds:

Perpetual bonds also known as consols, has an indefinite life and therefore, it has no maturity value. Perpetual bonds or debentures are rarely found in practice. In the case of the perpetual bonds, as there is no maturity, or terminal value, the value of the bonds would simply be the discounted value of the infinite stream of interest flows.

Example:

Suppose that a 10 percent NI,000 bond will pay N100 annual interest into perpetuity? What will be the value of the bond if the market yield or interest rate were 15 percent?

The value of bond is determined as follows:

$$Bo = \frac{INT}{Kd} = \frac{100}{0.15} = N667$$

SELF ASSESSMENT EXERCISE 1

From the above example, calculate the value of the bond if the yield or interest rate were 5%, 10%, 20%, 25% and 30%.

3.4 Valuation of Preference Shares

A company may issue two types of shares: ordinary shares and preference shares. Owners of shares are called shareholders, and capital contributed by them is called share capital.

Preference shares have preference over ordinary shares in terms of payment of dividend and repayment of capital if the company is wound up. They may be issued with or without a maturity period.

- Redeemable preference shares are shares with maturity.
- Irredeemable preference shares are shares without any maturity.

The holders of preference shares get dividends at a fixed rate. With regard to dividends, preference shares may be issued with or without cumulative features. In the case of cumulative preference shares, unpaid dividends accumulate and are payable in the future. Dividends in arrears do not accumulate in the case of non-cumulative preference shares.

Features of preference and ordinary shares include claims, dividends, redemption and conversion.

Example:

Suppose an investor is considering the purchase of 12-year, 10 percent N100 par value preference share. The redemption value of the preference share on maturity is N120. The investor's required rate of return is 10.5 percent. What should she be willing to pay for the share now?

The investor would expect to receive NIO as preference dividend each year for 12 years and N110 on maturity (i.e. at the end of 12 years).

The present value annuity factor can be used to value the constant stream of preference dividends and the present value factor to value the redemption payment.

$$P_0 = 10 \times \left[\frac{1}{0.105} - \frac{1}{0.105 \times (1.05)^{12}} \right] + \frac{120}{(1.105)^{12}}$$

$$= 10 \times 6.506 + 120 \times 0.302$$

$$= 65.06 + 36.24 = N101.30$$

Note that the present value of N101.30 is a composite of the present value of dividend, N65.06 and the present value of the redemption value N36.24. The N100 preference share is worth N101.30 today at 10.5 percent required rate of return. The investor would be better off by purchasing the share for N100 today.

A formula similar to the valuation of bond can be used to value preference shares with a maturity period:

Value of preference similar to the valuation of bond can be used to value preference shares with a maturity period.

$$Po = \left[\frac{PDIV_1}{(1+K_p)^1} + \frac{PD1V_9}{(1+K7} + \frac{+ PD1V_t,}{(1 4\text{-}1(p^n} \right] + \frac{P}{(1+Kpf}$$

$$\text{i.e. } P_o = \left[\sum_{t=1}^{n} \frac{PDIV_1}{(1+K_p)^t} + \frac{\mathbf{P}}{(1+K_p)^t} \right] \ldots\ldots\ldots\ldots\ldots\ldots$$

$PD1V_t$ is the preference dividend per share in period t, K_p the required rate of return of preference share and P_n the value of the preference share on maturity. Since PD1V is an annuity, the equation (4) can also be written as follows:

$$Po = PD1V \left[x \left| \frac{1}{(K_r)} - \frac{1}{IG(1+1G)^1} \right| \right] + \frac{p_n}{(1+Kp)^n}$$

Valuing Irredeemable Preference Share:

Consider that a company has issued N100 irredeemable preference share on which it pays a dividend of N9. Assume that this type of preference share is currently yielding a dividend of 11 percent. What is the value of the preference share?

The preference dividend of N9 is perpetuity. Therefore, the present value of the preference share is:

$$Po = \frac{PD1V}{K_p} = \frac{9}{0.11} = N81.82$$

Yields on Preference Share:

If the price of the preference share is N81.82, what return do investors require? In that case, we will have to solve the following equation:

$$81.82 \quad \frac{9}{K_P}$$

$$K_p = \frac{9}{81.82} = 0.11 \text{ or } \mathbf{11} \text{ percent.}$$

3.5 Valuation of Ordinary Shares

The valuation of ordinary or equity shares is relatively more difficult because of two factors:

(1) The rate of dividend on equity shares is not known (discretionary);
(2) The earnings and dividends on equity shares are generally expected to grow.

Dividend:

The general principle of valuation applies to the share valuation. The value of a share today depends on cash inflows expected by investors and risk associated with those cash inflows (dividend).

Single Period Valuation

Example:

Assuming: an investor intends to buy a share and will hold it for one year. Suppose he expects the share to pay a dividend of N2 next year and will sell the share at an expected price_of N21 at the end of the year. If the investor's opportunity cost of capital or the required rate of return (K_s) is 15 percent, how much should he pay for the share today?

The present value of the expected dividend per share at the end of the first year, $D1V_1$, plus the present value of the expected price of the share after a year, P_I,

$$P_o = \frac{DIAT_I + P_I}{1 + K_s} \qquad (6)$$

$$Po = \frac{2 + 2\,1}{1.15} \qquad N20$$

Note: In practice, there could be a difference between the present value and the market price less than the share's present value. On the other hand, an over-valued share has a market price higher than the share's present value.

It may be seen in the example that theshare value after a year represents an expected growth or capital gain of 5 percent:

$$\frac{21 - 20}{20} = 0.05 \text{ or } 5 \text{ percent}$$

$$\frac{P_i - P_o}{Po}$$

An investor can, thus, represent his expectation with regard to the future share price in terms of expected growth. If the share price is expected to grow at g percent, then we can write P_I as follows:

$$P_1 \quad P_o\,(1 + g)$$

Can be re-written equation (5) as:

$$P_o = \frac{DIV_1 + P_o(1 + g)}{1 + K_e} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Simplify Equation (7), we obtain a simple formula for the share valuation as follows:

$$P_o \quad \frac{}{Ke - g} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

In words, the present value of a share is determined by its expected dividend discounted (divided) by the difference of the shareholders capitalization or required rate of return (K,) and growth rate (g). In the example, if the investor would have expected share price to grow at 5 percent, the value of the share today using equation (8) will be:

$$Po = \frac{2}{0.15 - 0.10} \quad \frac{2}{0.10} = \quad N20$$

3.6 Equity Rate

One must know the expected dividend and the required rate of return (the opportunity cost of capital or capitalization rate) — will depend upon the risk of the share.

In a well-functioning capital market, the market price is fair price of a share. Therefore, the shareholders expect the share to earn a minimum return that keeps the current share price intact. For firms for which dividends are expected to grow at a constant rate indefinitely, and the current market price is given, Equation (9) is modified to estimate the capitalization or the required rate of return of the share:

$$Po \quad \frac{D1V_1}{K_0 - g}$$

$$\frac{DIV_1}{Po} + g$$

Example:

A company's share is currently selling for N50 per share. It is expected that a dividend of N3 per share after one year will grow at 8 percent indefinitely. What is the equity capitalization rate?

The equity capitalization rate is given as follows:

$$K_e \quad \frac{DIV_1}{P_o} + g \qquad \frac{3}{50} + 0.08$$

$$0.14 \quad = \quad 14 \text{ percent}$$

3.7 Linkages between Share price, Earnings and Dividends

Why do investors buy shares? Do they buy them for dividends or for capital gain?
Investors may choose between growth shares or income shares. Growth shares are those which offer greater opportunities for capital gains. Dividend yield (i.e. dividend per shares as a percentage of the market price of the shares) on such shares would generally be low since companies would follow a high retention policy in order to have a high growth rate.

Income shares, on the other hand, are those that pay higher dividends, and offer low prospects for capital gains. Because of the high payout policy followed by companies, their share prices tend to grow at a lower rate. Dividend yield on income shares would generally be high.

Is there a linkage between the share price and earnings and dividends? The following is an example:

Suppose a company estimates its earnings per share after a year (EPS) at N6.67, it follows a policy of paying 100 percent dividend (i.e. its retention ratio, b is zero). Thus, the company's earnings per share (EPS), and its earnings and dividends would not grow since it does not reinvest any earnings.

What would be the price of the company's share if the opportunity cost of capital were 12 percent?

The following formula suffices:

$$P_o \quad DIV_1 = \frac{EPS_1 (1 - b)}{K_e - rb} = \frac{6.67 (1 - 0)}{0.12} = \frac{6.67}{0.12} = N55.58$$

Note that since retention ratio, b, equals to zero, then $D1V_1$ = EPS, and g = rb = 0 and P_o is given by the earnings per share divided by the opportunity cost of capital i.e.

$$Po \quad \frac{EPS,}{K,}$$

Suppose that the company would pay a divided of Rs 4 share in the first year and reinvest the retained earnings (RE) at a rate of return (r = ROE) of 20 percent. What is the company's payout ratio, retention ratio and growth rate?

$$\text{Payout ratio} = \frac{D1V_1}{EPS,} = \frac{4}{6.67} = 0.6 = 60\%$$

Retention ratio =1 — payout 1 — 0.6 = 0.4 = 40%

Growth rate = Retention ratio x ROE

6 x r = 0.4 x 0.2 = 0.08 = 8%

Assuming that the company will follow a constant policy of retaining 40 percent earnings (i.e. payout of 60 percent) at 20 percent rate of return, then its earnings and dividends will grow perpetually at 8 percent (g = rb = 0.2 x 0.4). What would be the price of the company's share? It is calculated as follows:

$$P_o = D1V_1 = \frac{EPS,\ (1 — 1:)}{IC, — rb} = \frac{DIV,}{K, — g}$$

$$\frac{6.67\ (1 — 0.4)}{0.12 — 0.2 \times 0.4} = \frac{4.00}{0.12 — 0.08}$$

N100.00

You may note that without retention of earnings (b = 0), the company has no growth (g = 0) and the price of its share is N55.58. But when 8 percent growth is expected (from reinvestment of retained earnings), the price of the company's share is N100. Thus, the difference — N100 — N55.58 = N44.42 is the value of growth opportunities.

SELF ASSESSMENT EXERCISE I

(i) Calculate the price of a share if EPS = Rs 2.5, b = 0.4, K_e = 0.10 and ROE = r = 0.20. What shall be the price if r = = 0.10?

(ii) The Federal Government is proposing to sell a 5-year bond of N1,000 at 8 percent rate of interest per annum. The bond amount will be amortised (repaid) equally over its life. If an investor has a minimum required rate of return of 7 percent, what is the bond's present value for him?

4.0 CONCLUSION

In this unit, you have been taken through the application of the concept of present value to explain the valuation of bonds shares in quantifiable manner. The specific positions as they affect the value of bonds, shares are variably discussed and applied.

5.0 SUMMARY

In this unit, concept of present value have been applied, to explain the value of bonds — maturity, yield, payments, pure discount and perpetual. Valuation of preference shares, valuation of ordinary shares, equity capitalization rate and linkages between share price, earnings and dividends.

6.0 TUTOR-MARKED ASSIGNMENT

The Chief Executive of a company decides that his company will not pay any dividends till he survives. His current life expectancy is 20 years. After that time, it is expected that the company could pay dividends of N30 per share indefinitely. At present, the firm could afford to pay N5 per share forever. The required rate of this company's shareholders is 10 percent. What is the current value of the share? What is the cost to each shareholder of the managing director's policy?

7.0 REFERENCES /FURTHER READINGS

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UNIT 2 LINEAR PROGRAMMING I

CONTENTS

1.0 INTRODUCTION

Linear programming typically deals with the problem of allocating the limited resources available to an organisation among competing activities in the best possible or optimal way, that is, in a way minimizes the returns from performing them. The problem of resource allocation arises whenever one must select the level of certain activities that compete for scarce resources to perform them.

2.0 OBJECTIVES

After studying this unit, you should be able to:

- Explain the techniques of linear programming (LP);
- Describe the objective functions and the constraint of LP;
- Solve the minimizing problems graphically.

3.0 MAIN CONTENT

3.1 Linear Programming (LP) Allocation, Definition

A variety of business situation exist to which description such as the following applies; the allocation of production facilities to different products, product mix, portfolio selection for the allocation of investment funds, quality control, inspection problems, among others.

Linear programming uses a mathematical model to describe the problem of concern. The term "linear" suggests that all the mathematical functions in the model are required to be linear functions. The word "programming" is not synonymous with computer programming. Rather, it means planning. Thus, linear programming involves the planning of activities to obtain an optimal result, that is, a result that, among feasible alternatives, reaches the specified goal best according to the mathematic model. LP is a mathematical technique concerned with the allocation of scarce resources.

3.1.1 Expressing LP Problems _Limitations

Before considering the detailed methods of solving LP problems, it is necessary to be able to express a problem in a standardized manner. This not only helps the calculations required for a solution but also ensures that no important element of the problem is over-looked. The two major factors are:

- The objectives, and
- The limitations or constraints

Objectives: The first step in LP is to decide what result is required, i.e. the objective. This may be to maximize profit or contribution, or minimize cost or time or some other appropriate measures. Having decided upon the objective, it is now necessary to state mathematically the elements involved in achieving this. This is called the *objective function.*

Example: 1
A factory can produce two products, A and B. The contributions that can be obtained from these products are:

A contributes N20 per unit, B contributes N30 per unit and it is required to maximize contribution.

The objective function for this factory can be expressed as:

$$Maximise\ 20x1 + 30x_2$$

where x_i = number of units of A produced
and x_2 = number of units of B produced

Note: This problem has 2 unknowns, x_i and x_2. These are sometimes known as the *decision variables.* Only a single objective at the time (in the case, to maximize contribution) can be dealt with in a basic LP problem.

The objective function is:

Example: 2
A farmer mixes three products to feed his pigs. Feedstuff M costs 20p per kilo, feedstuff Y costs 40p per kilo and feedstuff Z costs 55p per kilo. Each feedstuff contributes some essential part of the pigs' diet and the farmer wishes to feed

Minimise $20x_1 + 40x_2\ 55x3$

where x, = number of kilos of M
x_2 = number of kilos of Y
x_3 = number of kilos of Z

Alternatively, if the costs were required in N's, the objective function could be expressed as follows:

Minimise $0.2x_1 + O.4x_2 + 0.55x_3$

Note: This example has three decision variables. The number of decision variables can vary from two to many hundreds. For examination purposes, four or five is the maximum that is likely to be encountered. Linearity has been assumed in both examples above and is assumed in al those that follow.

Limitations or Constraints

Circumstances always exist which govern the achievement of the objective. These factors are known as *limitations* or *constraints.* The limitations in any given problem must be clearly identified, quantified, and expressed mathematically. In a problem concerned with the allocation of scarce resources restrictions take the form:

- Resources used< Resources available

The resources used must be expressed in linear form and the resources available form part of the given data.

Limitation Examples:

A factory can produce four products and wishes to maximise contribution. It has an objective function as follows:

Maximise 5.3x1 + 2.7x2 + 6.0x3 +4.1x4

Where xl = number of units of A produced
x2 = number of units of B produced
X3 = number of units of C produced
3(4= number of units of D produced

and the coefficients of the objective function (i.e. 5.5, 2.7, 6.0 and 4.1) are the contributions per unit of the products.

Therefore,

	Products			
	A	B	C	D
Skilled hours	5	3	1	8
Unskilled hours	5	7	4	II

The limitations as regards to labour can be stated as follows:

Skilled:	5x1	●	3x2	●	x3		8x4	●	8000
Unskilled:	5x1	●	73(2	●	4x3	+	1 1 x4	●	6000

In addition a general limitation applicable to all maximising problems is that it is

Notes:
The resource limitations in this maximising problem follow a typical pattern being of the less than or equal to type (1;

The formal statement of the non-negativity constraints on the unknowns (x 1, x2, etc.) has to be made for computer solutions but is normally inferred when solving by manual means;

This above restriction applies to labour hours. Machine hour restrictions would be dealt with in a similar fashion.

3.2 Graphical LP Solution

Graphical methods of solving LP problems can only be used for problems with *two* unknowns or decision variables. Problems with *three* or *more* unknowns must be solved by techniques such as the simplex method. Graphical methods are the simplest to use and should be used wherever possible.

(a) Limitations. Graphical methods can deal with any number of limitations but as each limitation is shown as a line on a graph, a large number of lines may make the graph difficult to read. This is rarely a problem in examination questions.

(b) Types of problems and limitations. Both maximization and minimization problems can be dealt with graphically and the method can also deal with limitations of the 'greater than or equal to' (>) type and the 'less than or equal to' (<) type.

(c) Graphical example. The method of solving LP problems graphically will be described step by step using the following maximising example as a basis.

Example: 3

A manufacturer produces two products, Klunk and Klick. Klunk has a contribution of £3 per unit and Klick £4 per unit. The manufacturer wishes to establish the weekly production plan which maximizes contribution.

Production data are as follows:

	Per unit		
	Machining (Hours)	Labour (Hours)	Material (kgs)
Klunk	4	4	1
Klick	2	6	1
Total available per week	100	180	40

Because of a trade agreement, sales of Klunk are limited to a weekly maximum of 20 units and to hojnour an agreement with an old established customer at least 10 units of RHO(must he sold ner week.

Step 1: Formulate the LP model in the standardized manner described earlier above.

Maximise $3x_1 + 4x_2$

Subject to constraintA $4x_1 + 2x_2 < 100$ (Machining hours constraint)

B $4x_1 + 6x_2 < 180$ (Labour hours constraint)

C x, + $x_2 < 40$ (Materials constraint)

x, < 20 (Klunk sales constraint)

x2> 10 (Klick sales constraint)

x2? 0

where x_i = number of units of Klunk
x_2 = number of units of Klick

Note:

As it is impossible to make negative quantities of the products, it is necessary formally to state the non-negativity constraint (i.e. $x_i > 0$).

The resource and sales constraints include both types of restrictions (i.e. > and <).

This is a problem with only two unknowns (i.e. x_i and x_2), it can be solved graphically. The number of limitations does not exclude a graphical solution.

Step 2: Draw the axes of the graph which represent the unknowns, x_i and x_2 thus:

Figure 1

Note: The scales of the axes are best determined when the lines for the limitations are drawn. Each axis must start at zero and the scale must be constant (i.e. linear) along the axis but it is not necessary for the scales on both axes to be the same.

Step 3: Draw each limitation as a separate line on the graph.

Sales Limitations: These normally only affect one of the products at a time and in Example 1, the sales restrictions were:

$$x_i < 20 \text{ and } x_2 > 10$$

These limitations are drawn on the graph as follows:

The Klunk Sales constraint (i.e. x_i <20).

Constraint D

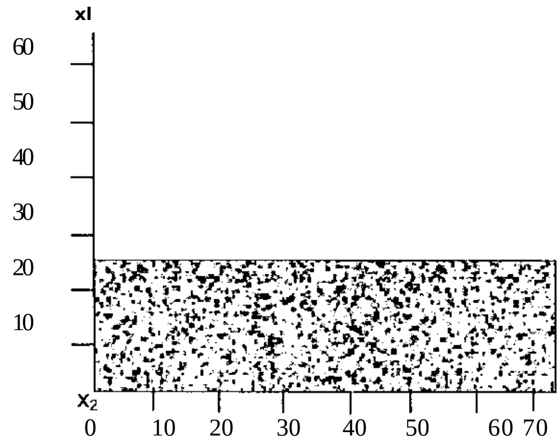

Figure 2

Note: The horizontal line represents $x_1 = 20$ and the batched area below the line represents the area containing all the values less than 20.

The Klick sales constraint (constraint E) (i.e. x_2 ?_ 10) is now entered thus:

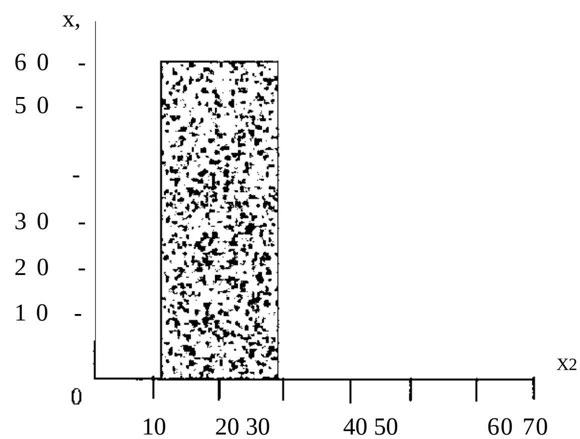

Figure 3

Note: The vertical line represents $x_2 = 10$ and the hatched area to the right of the line represents the area containing all values greater than 10 (i.e. x, 10). These two sales limitations can be shown on the same graph thus:

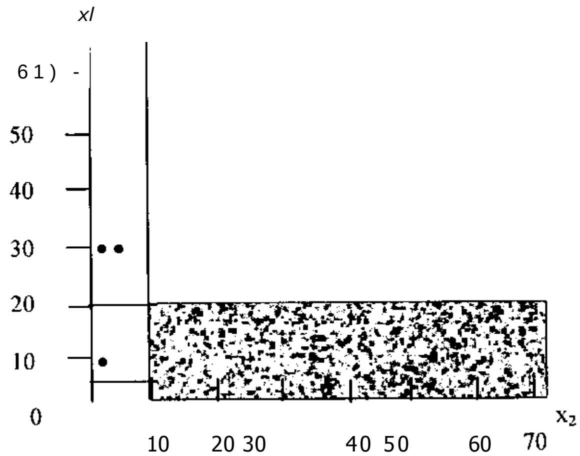

Figure 4

Note: The hatched area represents the area of possible production (i.e. which does not violate the constraints drawn) and is called the *feasible region.* The areas on the graph marked (El) violate one or both of the constraints.

3.2.1 Production and Material Limitations

In a similar fashion to above, the other restrictions should be drawn on the graph. Because these restrictions involve BOTH unknowns they will be sloping lines on the graph and not horizontal or vertical lines like thc sales restrictions.

The three remaining restrictions are all of the same type and area dealt with as follows:

The matching constraint, constraint A, $4x_1$ i $2x_2$:5. 100 is drawn on the graph as $4x_1 + 2x_2 = 100$.

Therefore, when $x_i = 0$, $x_2 = 50$ (i.e. $\frac{100}{2}$)

and when $x_2 = 0$, $x_i = 25$ (i.e. $\frac{100}{4}$)

and so a line can be drawn from 25 on the x_i axis to 50 on the x_2 axis thus:

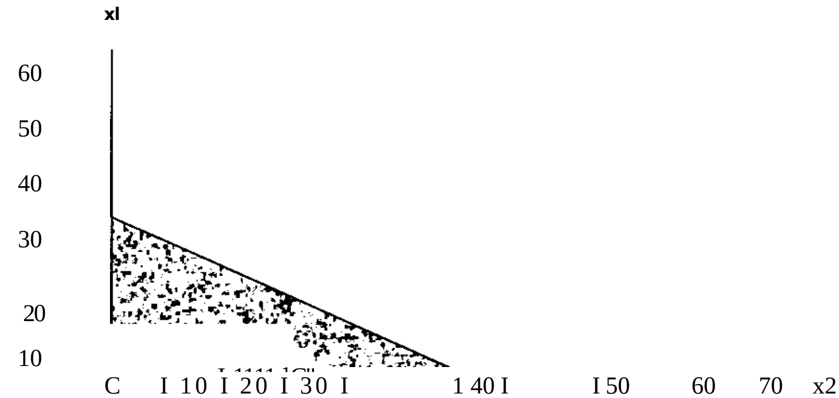

Figure 5

Note: As previously, the batched area represents the area containing the 'less than' values. The other constraints are dealt with in the same manner.

i.e. The labour constraint B $4x_1 + 6x_2$ 180 is drawn on the graph as $4x_1 + 6x_2 = 180$.

Therefore, when $x_i = 0$, $x_2 = 30$ (i.e. $\frac{180)}{6}$

and when $x_2 = 0$, $x_i = 45$ (i.e. $\frac{180)}{4}$

and so a line can be drawn from 45 on the x_i axis to 30 on the x_2 axis.

The materials constraint (C), $x_i + x_2$ 40
when x, = 0, $x_2 = 0$, $x_i = 40$
and a line can be drawn from 40 on the x_i axis to 40 on the x_2 axis.

All of the constraints (sales, production and material), can now be drawn on a single graph and the resultingfeasiNe *region* defined.

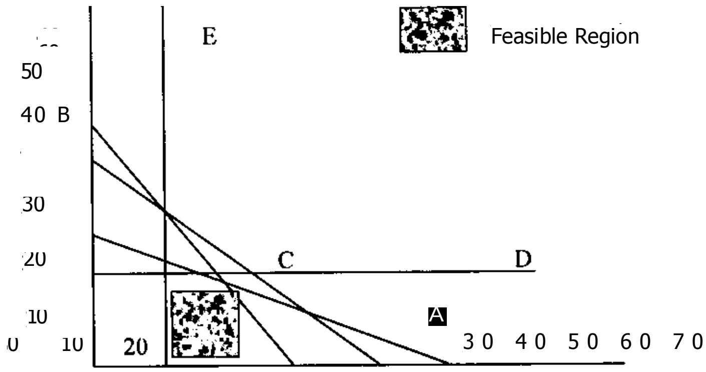

Figure 6

Note:

(a) The *feasible region* is the area which does not contravene any of the restrictions and is therefore the area containing all possible production plans.

(b) The non-negativity restrictions (i.e. x_1 0, x_2 0) are automatically included in the graph because the graph quadrant used in the earlier figure only shows positive values. It should be noted that as more restrictions are plotted, the feasible region usually becomes smaller.

(c) It will noted that the material constraint C (line 40, 40) does not touch the feasible region. This is an example of a *redundant* constraint, i.e. it is non-binding.

Step 4: Now that the feasible region has been defined, it is necessary to find the point in or on the edge of the feasible region that gives the maximum contribution which, it will be recalled, is the specific objective.

This is done by plotting lines representing the *objective function* and thereby identifying the point in the feasible region which lies on the maximum value objective function line that can be drawn. These objectives function or contribution lines are straight lines representing different combinations of Klunk and Klick which yield the same contribution. For example:

- 20 units of Klunk and zero units of Klick yield £60 contribution
- 12 units of Klunk and 6 units of Klick yield £60 contribution
- 8 units of Klunk and 9 units of Klick yield £60 contribution
- zero unit of Klunk and 15 units of Klick yield £60 contribution, etc.

Very many other contribution lines could be drawn and if a number of these lines were drawn on the graph, it would be noticed that:

(a) They are parallel to each other with the same shape, which is determined by the relative contribution of the products.
(b) The further to the right they are drawn, the higher value of contribution they represent.

It therefore follows that the contribution line furthest to the right but still touching the feasible region shows the optimum production plan to provide the maximum possible contribution, thus:

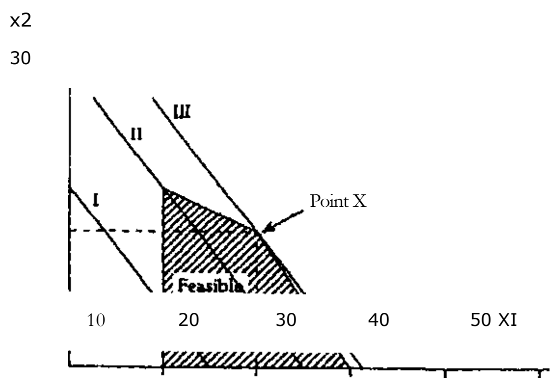

Figure 7

Optimum solution at point X: i.e. 15 units of x_1 and 20 units of x_2 yielding a contribution of £125 (i.e. £3 x 15 + 4 x 20).

Notes:

(a) The lines marked I to III are three of the many contribution lines that could be drawn and represent the following contributions

$$3x_1 + 4x_2 = 60$$
$$3x_1 + 4x_2 = 90$$
$$3x_1 + 4x_2 = 125$$

The contribution line has a slope of 4/3 which is the ratio of the coefficients of x 1 and x2. The intercept on, say, the x 1 axis is found by dividing the contribution by the xl coefficient, and vice versa. The intercepts for line II in figure ..., for example, are as follows:

Intercept on x_1 axis = $\frac{\text{contribution}}{x_1 \text{ coefficient}} = \frac{90}{3} = 30$

Intercept on x_2 axis = $\frac{90}{4} = 22.5$

(b) Only parts of lines **I, II** and III are feasible. As we require a maximise contribution, we are only interested in Point X where line III touches the feasible region. It will be noted that the optimum is at a vertex or corner of the feasible region. This is ALWAYS the case.

(c) Various contribution lines have been drawn on earlier page for instructional purposes. For examinations, it is sufficient to draw only the contribution line representing the optimum position, i.e. in the example above, line **III.**

(d) The contribution lines are sometimes termed **iso-profit** lines.

(e) A simple way to check your answer is actually possible is to insert the values of the unknowns in the constraints and check whether the constraints are satisfied, e.g. the optimum solution of example found from figure .... is

$$x_i = 15 \text{ units}$$
$$x_2 = 20 \text{ units}$$

These values can be inserted into the constraints thus:

Constraint A: (4 x 15) -I (2 x 20) = 100 constraint satisfied, no spare
Constraint B: (4 x 15) + (6 x 20) = 180 constraint satisfied, no spare
Constraint C: (1 x 15) + (1 x 20) = 35 constraint satisfied, 5 below maximum
Constraint D: Sales of x, = 15 Constraint satisfied, 5 below maximum
Constraint E: Sales of x_2 = 20 Constraint satisfied, 10 above maximum

It will be noted that the two constraints which intersect at the optimum vertex (see figure ...) are constraints A and B. These are the only constraints fully satisfied with no spare values. This is a general rule. They are known as **binding constraints.**

Minimization Example

Provided that they only have two unknowns, minimization problems can also be dealt with by graphical means. The general approach of drawing the axes with appropriate scales and inserting lines representing the limitations is the same as for maximizing problems but the following differences between maximizing and minimizing problems will be found.

(a) Normally in a minimizing problem the limitations are of the greater than or equal to type so that the feasible region will be above all or most of the limitations.

(b) The normal objective is to minimize cost so that the objective function line(s) represent cost and because the objective is to *minimize* cost the optimum point will be found from the **cost line further to the left which still touches the feasible region,** i.e. the converse of the method used for maximizing problems.

Example: 4

A manufacturer is to market a new fertilizer which is to be a mixture of two ingredients A and B. The properties of the two ingredients are:

		Ingredients analysis			
	Bone meal	Nitrogen	Lime	Phosphates	Cost/kg
Ingredient A	20%	30%	40%	10%	1.2p
Ingredient B	40%	10%	45%	5%	0.8.

It has been decided that:

(a) the fertilizer will be sold in bags containing 100 kgs
(b) it must contain at least 15% nitrogen
(c) it must contain at least 8% phosphate
(d) it must contain at least 25% bone meal

Solution:

Because of the basic similarity between graphing minimizing and maximizing problems (except for the two differences mentioned above), the detailed intermediate steps given in example ... above will be repeated.

The problem in standardized format is as follows:

Objective Function:

Minimise $12x_1 + 8x_2$ (cost expressed in tenths of a penny) subject to:

Constraints A $x, + x_2 = 100$ (weight constraint)

$0.3x_1 + 0.1x_2 \quad 15$ (nitrogen constraint)

C $0.1x_1 + 0.08x_2 \; 8$ (phosphates constraint)

$0.2x_1 + 0.4x_2 \quad 25$ (bone meal constraint)

x_l ?_ 0 and x_2 ?. 0

Where x_l kgs of ingredient A

x2 = kgs of ingredient B

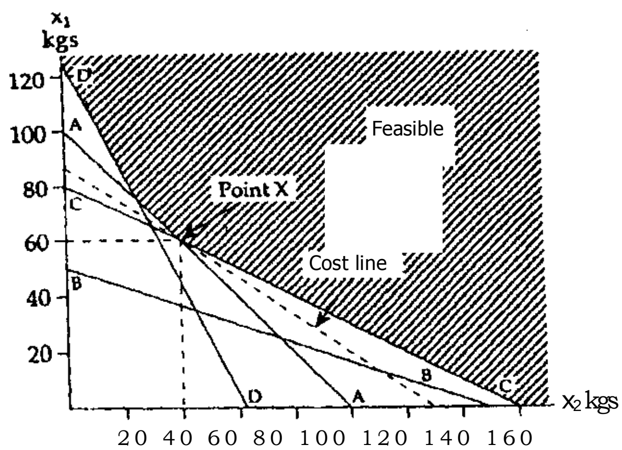

Figure 8

Notes on Figure 7.8 above:

Optimum solution at point X i.e. 60 kgs of x_l and 40 kgs of x_2 giving a cost of:

<u>12 x 60 + 0.8 x 40 = £1.04 per 100 kilo bag.</u>

a) Only one cost line ($12x_1 + 8x_2 = 1{,}040$ in tenths of a penny) has been drawn so as to show the optimum position.

b) Each restriction on the graph is labeled A, B, C, D and can be cross referenced to those stated in the standardized format above.

c) The optimum position in the feasible region is the furthest *point to the left* touched by the cost line. This is because it is a minimizing problem. Note again that the optimum is at a corner or vertex of the feasible region.

Using Simultaneous Equations

An alternative method of finding the solution values of the constraints at any intersection of the graph, including the optimal vertex, is to solve the simultaneous equations of the relevant binding constraints. To illustrate this, the answers to the maximizing problem and the minimizing problem in Example... will be re-calculated using simultaneous equations below;

Reworking Example

The two constraints which intersected at optimum were constraints A and B (see figures ..... and ....) thus:

- Constraint A $4x_1 + 2x_2 = 100$
- Constraint B $4x_1 + 6x_2 = 180$

Solving by deducting A from B gives:

$$4x_2 = 80$$
$$\therefore x2 = 20$$

and by substitution, $x_1 = 15$, which confirms the answer found by reading off the intercepts of the graph.

Reworking example....

The intersecting constraints were A and C (see figure ....) thus:

Constraint A $x_1 + x_2 = 100$
Constraint C $0.1x_1 + 0.05x_2 = 8$

$$A — 10C = 0.5x_2 = 20$$
$$x_2 = 40$$

and substituting x I = 60, again confirming the results from the graph.

If it was known what constraints appeared at the optimum intersection before the graph was drawn, then the problem could be solved by simultaneous equations without having to draw a graph. However, this is unlikely to occur but it is sometimes useful to calculate the exact solution values using the equations rather than use the approximate values obtainable from a graph.

3.2.2 The Valuation of Scarce Resources

The graphing of an LP problem not only provides the optimal answer but also identifies the binding constraints, alternatively called the limiting factors. For example, in example ... machining hours and labour hours are the binding constraints, the shortage of which limits further production and profit. Not all factors in a given problem are limiting factors. For instance, in example ... constraints C, **D** and E are not at their maxima and therefore not scarce or limiting.

It is important management information to value the scarce resources. These valuations are known as the *dual prices or shadow prices* and are derived from the amount of the increase (or decrease) in contribution that would arise if one more (or one less) unit of scarce resource was available. Only a scarce resource can have a positive dual price and the calculated price assumes that there is only a marginal increase or decrease in the availability of the scarce resource and that all other factors are held constant.

Finding the Shadow or Dual Prices

There are two methods of calculating these prices — the arithmetic method and the dual formulation — and both are illustrated using the data and results from Example ..., reproduced below:

Original Problem:

Maximize $3x_1 + 4x_2$

Subject to:

Constraint A	$4x_1 + 2x_2$	100	machining hours
Constraint B	$4x_1 + 6x_2$	180	labour hours
Constraint C	$+ x_2$	40	materials
Constraint D	x_1	20	sales
Constraint E	x_2	10	sales

Where x_1 = units of klunk
$x2$ = units of klick

The solution was Produce $15x_1$ and $20x_2$ giving a contribution of £125.
Constraints A and B are binding

The problem now is to find the shadow prices of the two binding constraints, machine hours and labour hours, i.e. what is the valuation of one more (or less) machine hour and one more (or less) labour hour?

Arithmetic Method of Finding Shadow Prices

Dealing first with machine hours, we assume that 1 more machine hour is available (but labour hours are constant at 180) and calculate the resulting difference in contribution, thus:

The binding constraints become:

Machine hours	$4x_1 \pm 2)(2 = 101$	(i.e. original 100 + 1)
Labour hours	$4x_1 + 6x_2 = 180$	(unchanged)

Solving these simultaneous equations, new values for x_1 and x, are obtained: x_1 = 15.375 and x, = 19.75 and substituting into the objective function gives a new contribution.

3(15.375) + 4(19.75) = £125.125
original contribution = £125

Difference = £0.125

Thus 1 extra machine hour has resulted in an increase in contribution of £0.125 which is the shadow price per machining hour.

A similar process for labour hours is shown below:

New constraints with extra labour hour: (but machine hours constant at 100).

Machine hours	$4x_1 + 2x_2 = 100$
Labour hours	$4x_1 + 6x_2 = 181$

and solving gives x_1 = 14.875 and x_2 = 20.25

New contribution	3(14.875) + 4(20.25)	= £125.625
	Original contribution	= £125
	Difference	= £0.625

shadow price per labour hour = £0.625

Notes:

(a) Similar results would be obtained in each case if 1 less hour had been used in the calculations. Verify this yourself

(b) The shadow prices calculated above only apply whilst the constraint is binding. If, for example more and more machining ours became available, there would eventually be so many machining hours that they would no longer be scarce and some other constraint would become binding. This point is developed further below.

Dual Formulation Method for Shadow Prices

Every LP problem has an *inverse or dual* formulation. If the original problem known as the *primal problem,* is a maximizing one then the dual formulation is a minimizing one and vice versa. Thus, as the Example ...primal formulation is a maximizing problem, its dual is the minimizing problem. Solving the dual problem gives the shadow prices of the binding constraints, hence the alternative term, dual prices.

The stages in finding and solving the dual for the solution to Example... are shown below.

The relevant parts of the original fill formulation which appear in the solution are:

Maximize $3x_1 + 4x_2$ (objective function)

Subject to $4x_1$ + 2x 5. 100 (machine hours)
$4x_1 + 6x_2$ 5 180 (labour hours)

Constraints C, D, E (non-binding so do not appear in the solution).

The minimizing dual problem is formed by inverting the above formulation, i.e. making the columns into rows and the rows into columns thus:

Dual formulation:

Minimize 100M + 180L (i.e. originally the constraint column)

Subject to 4M + 4L 3 (originally the x, column)
2M + 6L 4 (originally the x_2 column)

What are now the constraints can be solved by simultaneous equations?

$$4M + 4L = 3$$
$$2M + 6L = 4$$

Solving gives M = 0.125 and L = 0.625, which will be recognized as the valuations already calculated earlier above.

If these dual prices are inserted into the objective function of the dual, exactly the same value is obtained as in the primal problem.

i.e. 100 (0.125) + 180 (0.625) = £,125

This result is identical to the Primal problem and this will always be so.

Notes:

(a) The dual formulation above is a contraction of the full dual formulation. This is possible here because the problem had already been solved and it was then known that three of the constraints (C, D and E) were non-binding.

(b) The dual formulation is developed further in subsequent chapters.

Interpretation of Shadow Prices

The shadow price of a binding constraint provides valuable guidance because it indicates to management the extra contribution they would gain from increasing by one unit the amount of the scarce resource. As an example, the shadow price of labour hours calculated above is £0.625 per hour. This means that management would be prepared to pay up to £0.625 per hour extra in order to gain more labour hours. If the current labour cost was £8 per hour then management would be prepared to pay up to £8.625 per hour for extra labour hours, perhaps from overtime working.

It is important to realise that only binding constraints have shadow or dual prices. Those that are not binding have zero shadow prices. This accord with common sense for there would be little point in paying more to increase the supply of a resource of which you already have a surplus. In example ... constraints C, D and E have zero shadow prices.

SELF ASSESSMENT EXERCISE 1

What is the 'objective function' of linear programming?

4.0 CONCLUSION

In this unit, you have been introduced to linear programming as a method of solving problems where an objective has to be optimized subject to constraints and how LP is a resource allocation technique.

5.0 SUMMARY

Linear programming is a solution method to problems where an objective has to be optimized subject to constraints. All factors concerned have to be numeric and there must be linear relationships.

LP is a resource allocation technique where some objective, for instance, to maximize contribution, is required to be optimized subject to resource constraints.

LP problems can be maximizing or minimizing problems.

6.0 TUTOR-MARKED ASSIGNMENT

A firm produces two products X and Y with a contribution of N8 and N I 0 per unit respectively. Production data are (per unit):

	Labour hours	Material A	Material B
X	3	4	6
Y	5	2	8
Total available	500	350	800

Formulate the LP model in the standardized way. Solve the model in the above question using the graphical method.

7.0 REFERENCES / FURTHER READINGS

Lucey, T. (2002). *Quantitative Techniques 6th Edition,* (LLST) Educational Low-Priced Sponsored Texts.

Owen, F. and Jones, R. (1975). *Modern Analytical Techniques,* Polytech Publishers Limited, Stockport.

Pandey, I.M. (2005). *Financial Management,* UBS Publisher's Distributors PVT LTD.

UNIT 3 LINEAR PROGRAMMING II

CONTENTS

1.0 INTRODUCTION

When considering linear programming methods, one has so far confined one's attention to production problems — maximizing profit product — mix or minimum cost combination for inputs and blending problems. The time is now to widen the horizons of linear programming techniques and introduce various techniques which will simplify the method under certain conditions.

2.0 OBJECTIVES

After studying this unit, you should be able to:

- use the simplex method for solving maximizing LP problems;
- interpret the simplex tableau.

3.0 MAIN CONTENT

3.1 Simplex Method

A step by step arithmetic method of solving LP problems whereby one moves progressively from a position of, say, zero production, until no further contribution can be made. Each step produces a feasible solution and each step produces an answer better than the one before, i.e. either greater contribution in maximizing problems, or less cost in minimising problems. The mathematics behind the Simplex method are complex and this unit does not try to explain why the method works but it does describe how to use the technique.

Formulating the Simplex Model

To use the Simplex method, it is first necessary to state the problem in the standardised manner. It will be recalled that this results in an objective function and a number of constraints which are inequalities either of the or < type. Having stated the problem in the standardised format, the inequalities must be converted to equations. For example, if a manufacturer made two products A and B, which took 3 and 5 hours respectively to machine on a drilling machine which was available for up to 320 hours per period the constraint would be written in the standardised format as follows:

$$3x_1 + 5x_2 < 320$$

where x_1 units of A
x_2 units of B

This is converted into an equation by adding an extra variable called a *slack variable* thus

$$3x_1 + 5x_2 + x_3 = 320$$

The slack variable, x_3 in this case, represents any unused capacity in the constraint and can thus take any value from 320 hours (i.e. the position of zero production and therefore maximum unused capacity), to 0 hours (i.e. the position of the machine being fully utilised and therefore, zero unused capacity).

Notes:

(a) Each constraint will have its own slack variable;

(b) Once the slack variable has been incorporated into the constraint the Simplex method automatically assigns it an appropriate value at each iteration.

A Simplex Maximizing Example

The following maximizing example will be used to provide a step by step approach to the Simplex method.

Example: 1

A company can produce three products, A, B and C. The products yield a contribution of £8, £5 and £10 respectively. The products use a machine which has 400 hours capacity in the next period.

Each unit of the products uses 2, 3 and 1 hour respectively of the machine's capacity. There are only 150 units available in the period of a special component which is used singly in products A and C.

200 kgs only of a special alloy is available in the period. Product A uses 2 kgs per unit and Product C uses 4 kgs per units. There is an agreement with a trade association to produce no more than 50 units of Product B in the period.

Step 1: Express the problem in the standardised format thus:

Objective function:

maximise	$8x_1$	+	$5x2$ +	$10x_3$		
subject to	$2x_1$	+	$3x2$ +	$x3$	<	400
	$x,$		+	$x3$	<	150
	$2x1$		+	$4x_3$	<	200
			$x2$		<	50

$$x_j\ 0,\ x_2\ 0,\ x_3\ 0$$

=	where x_1	=	no. of units of Product A
	$x2$	=	no. of units of Product B
	$x3$	=	no. of units of Product C

Step 2: Make the inequalities in the constraints into equalities by adding a 'slack variable' in each constraint, thus:

maximize	$8x_1 + 5x_2 + 10x_3$				
subject to	$2x_1$	$+ 3x_2 + x_3 + x_4$			400
	x_1	x_3 +	$x5$	=	150
	$2x_1$	$4x_3$ +	$x6$	=	200
		$x2$	± $x7$	=	50

Note: x_4, x_5, x_6, x_7 are the slack variables and represent the spare capacity in the limitations.

Step 3: Set up the initial Simplex Tableau by arranging the objective function and equalized constraints from Step 2 in the following form.

leau Table 8.1

Solution variable	Products			Slack variables				Solution quantity
	x_t	x_2	x_3	x_4 x_7	x_5		x_6	
x_4	2	3	1	1	0	0	0	400
x_5	1	0	I	0	1	0	0	150
x_6	2	0	4	0	0	1	0	200
x_7	0	1	0	0	0	0	1	50
Z	8	5	10	0	0	Q	0	0

Notes:

(a) It will be seen that the values in the body of the table are the values from the objective function and constraints in Step 2.

(b) The variable 'Z' has been used for the objective function and represents total contribution.

(c) The tableau shows that $x._3 = 400$, $x_6 = 150$, $x_6 = 200$, $x_7 = 50$ and $Z = 0$.

(d) The tableau shows a feasible solution, that of nil production, nil contribution, and maximum unused capacity as represented by the values of the slack variables x_4, x_5, x_5 and x_7.

Although feasible, this plan can obviously be improved and this is done as follows:

Step 4: Improve the previous feasible solution by making as many as possible of the product with the most contribution, i.e. the highest figure in the Z row. The number that can be made will be limited by one or more of the constraints becoming operative thus:

Select highest contribution in Z row — i.e. 10 under x_3.

Divide the positive numbers in the x_3 column into the solution quantity column.

i.e. 400 + 1 = 400
150 + 1 = 150
200 4 = 50
50 + 0 (ignore)

Select the row that gives the lowest answer (in this case the row identified x_6). Ring the element which appears in both the identified column (x_3) and the identified row (x_6), this element is known as the *pivot element* thus:

Initial Simplex tableau reproduced, Table 2

Solution Variable	Products		Slack variables			Solution quantity
	X1 x3	X2	X4 x7	X5	X6	
x_4	2	31		0	0	400
xs	1		0			150
	I	0	0	I	0	200
x_7	1	40				50
	2 0		0	0	1	
	0	1	0			
	0		0	0	0	
			1			
Z	8 10		50 0	0	0	0

Step 5: Divide all the elements in the identified row (x_6) by the value of the pivot element (4) and change the solution variable to the heading of the identified column (x_3) thus:

Old row

X6	2	0	4	0	0	1	0	200

Divide by 4 and re-titling the new row becomes

New row

X3	IA	0	1	0	0	1/4	0	50

Enter this row in a new tableau

Second Simplex tableau Table 3

Row Nos.	Solution variable	Products			Slack variables				Solution quantity
		xi	X2	X3	X4	X5	X6	X7	
' z	x4	2	3	1	1	0	0	0	400
≤	x_5	1	0	1	0	1	0	0	150
	x_6	1/2	0	1	0	0	IA	0	50
	x_7	0	1	0	0	0	0	1	50
5	Z	8	5	10	0	0	0	0	0

Notes:

(a) The row numbers have been included to aid the explanatory material which follows and form no part of the Simplex Tableau.

(b) It will be seen that this second tableau is identical to the first except for row 3 which was calculated above.

(c) Row 3 means that 50 units of x_3 are to be produced.

Step 6: As a consequence of producing 50 units of x3, it is necessary to adjust the other row so as to take up the appropriate number of hours, components etc. used and to show the contribution produced for the 50 units of x3. This is done by repetitive row by row operations using Row 3 which makes all the other elements in the pivot elements column into zeros. To maintain each row as equality, it is, of course, necessary to alter each element along the row on both sides of the equality sign. Using Row 1 in the Second Tableau as an example, the process is as follows:

Row 1 400	x_4	2	3	1	1	0	0	0	
Minus Row 3	x_3	1/2 0	1	0	0	1/4	0		50
Produces a new row	xi:	11/4	01	0	0	—1/4	0	350	

Notes:

(a) This new row will be inserted into a third tableau along with all the other altered rows and Row 3 from the second tableau.

(b) The aim of this row operation was to produce the zero (marked *). In this case, a simple subtraction was all that was necessary but to make a zero in other cases may require further operations using Row 3 as a basis.

The other rows in the second tableau are operated on a similar fashion.

Row 2 x_s	1	0	1	0	10	0	150	
Minus Row 3 x_3	1/2	0	1		0		0	50
Produces a new row	x_5 V2	0	0*	0	1		0	100

Row 4 needs no operation because the element in column x_3 is already zero.

Row 5	Z	8	5	10	0	0	0	0	=0
Minus Rcnn/ 3x1		5	0	10	0	0	21/2	0	= 500
Produces a new row	Z	11/2	0	1	0	0	—21/2	0	=-500

Notes:

(a) To produce the required zero (0 *), it was necessary to multiply the Row 3 by 10 and then subtract from Row 5.

(b) The `-500' at the end of the new Z row is the contribution earned by 50 units of x_3 at £10 i.e. £500. The negative sign is merely a result of the Simplex method and the fact that the contribution is shown as a negative figure can be disregarded.

Step 7: When all the row operations have been done, a third tableau can be produced thus:

Third Simplex tableau, Table 4

Row Nos.	Solution variable	Products			Slack variables				Solution quantity
		xl	x2	x3	x4	x5	x_6	x_7	
6 (Row 1 — Row 3)	x_4	11/2	3	0	1	0	—1/4	0	350
7 (Row 2 — Row 3)	xs	1/2	0	0	0	1	—1/4	0	100
8 (i.e. Row 3)	x_3	1/2	0	1	0	0	1/4	0	50
9 (i.e. Row 4)	x_7	0	I	0	0	0	0	1	50
10 (Row 5 — 10x Row 3)	Z	3	5	0	0	0	—21/2	0	—500

Notes:

(a) All the new rows produced by the row operations in Step 6 have been inserted into the third tableau.

(b) The rows have been consecutively numbered again and a summary of the operations carried out in Step 6 to produce the new lines has been given against the new row numbers. e.g. Row 10 was produced by multiplying Row 3 by 10 and subtracting it from Row 5.

Step 8: To produce subsequent tableaux and eventually an optimum solution, steps 4 to 7 are repeated until no positive numbers can be found in the Z row. From Row 10 it will be seen that the maximum contribution is 5.

x_2 column is chosen

The positive numbers in the x_2 column are divided into the solution quantities and the lowest result selected.

ie.	Row 6	350 ÷ 3	=	**116%**
	Row 7	100 ± 0	=	ignore
	Row 8	50 + 0	=	ignore
	Row 9	50 ÷ 1	=	50

Row 9 is selected and the pivot element identified and the solution variable altered to x_2 thus

Row 9 x_2	0	1	0	0	0	1	=	50

As the pivot element is already 1 no further action is necessary on it but the other elements in the pivot element column (x_2) must be made into zeros by using row operations based on Row 9 thus

Row 6 x_4	1%	3	0	1	0	−1/4	0	=	350
Minus 3 x Row 9 x,	0	3	0	0	0	0	3	=	150
Produces a new row x4	1¼	0 *	0	1	0	_IA	−3	=	200

Note:

It was necessary to multiply Row 9 by 3 to produce the zero in column x_2 (0*) Rows 7 and 8 need no operation because the elements in column x_2 are already zero.

Row 6 Z	3	5	0	0	0	−21/2	0	=	−500
Minus 5 x Row 9 x2	0	5	0	0	0	a	5	=	250
Produces a *new* row x4	3	0*	0	1	0	−21/2	−5	=	−750

The new row produced can now be entered into a fourth tableau as follows:

Fourth Simplex tableau, Table 8. 5

Row Nos.	Solution variable	Products			Slack variables				Solution quantity
		x1	x2	x,	x4	x5	x6	x2	
11 (Row 6 —3x Row 9)	x4	11/2	3	0	1	0	—1/4	—3	200
12 (asRow 7)	x5	1/2	0	0	0	1	—1/4	0	100
13 (i.e. Row 8)	x,	ı/2	0	1	0	0	1/4	0	50
14 (Pivot Row as Row 9)	x2	0	1	0	0	0	0	I	50
15 (Row 10 — 5x Row 9)	Z	3	0	0	0	0	—21/2	—5	—750

Notes:

(a) The above tableau shows that 50 units of Products B and C could be made (x_2 and x_3= 50).

(b) As a result of this amount of production £750 contribution would be gained (Z = —750.

Step 9: Because there is still a positive number in the Z row (3 under column x_i) the iterative process is repeated in precisely the same manner. Column x_l is chosen and the positive numbers in the x_l column are divided into the solution quantities and the lowest number selected.

i.e. Row 11	200 ÷ 11/4	=	1331/4	
Row 12	100 + 1/4	=	200	
Row 8	50 + 'A	=	100	
Row 9	50 ± 0	=	ignore	

Row 13 is selected and the pivot element identified and the solution variable altered to x_l thus

Row 13 x_l 0	0	$\boxed{1/2}$	0	0	0		=	50

The pivot element must be made into a 1 so the whole row is multiplied by 2 thus

Row 13 x_l 0	2	$\boxed{1}$	0	½	100

The rest of the elements in column xl must now be made into zeros by the usual row operations.

Row 11 x4 *Minus 1 'A x*	11/2	0	0	1	0		0	=	200
Row 13 x_i	11/2	0	3	0	0	%	3	=	150
Produces a new row x4	0 *	0	—3	1	0	—1	—3	=	50
Row 12 x5	1/2	0	0	0	1				100

Minus 54 x

Row 13 xi	%	0	1	0	00		=	50
Produces a new row x6	0*	0	—1	0	1	0	=	50

Row 14 needs no operation because the element in column x_1 is already zero.

Row 15 Z	3	0	0	0	0	—21/4	—5	=	—750
Minus 54 x									
Row 13 xi	3	0	6	0	0	—11/2	0	=	300
Produces a new row x5	0*	0	—6	0	0	—4	—5	=	—1050

The new rows produced can now be entered into a fifth tableau thus

tableau Table 6

Row Nos.	Solution variable	Products			Slack variables				Solution quantity
		X,	x2	x_s	x_4	x_s	x_6	x,	
16 (Row 11 — 11/2x Row 18)	x_4	0	0	—3	1	0	—1	—3	50
17 (Row 12 — 1/2 x Row 18)	x_s	0	0	—1	0	1	—1/2	0	50
18 (Pivot Row 2 x Row 13)	X3	1	0	2	0	0	1/2	0	100
19 (as Row 14)	x_2	0	1	0	0	0	0	1	50
20 (Row 15 — 3x Row 13)	Z	0	0	—6	0	0	—4	—5	—1050

As there are no positive values in the Z row the optimum solution has been reached.

Step 10: All that remains is to obtain the maximum information from the fifth tableau.

Dealing first with the solution variables: x_1, x_2, x4 and x_5,
Optimum product mix

XI 100 i.e. produce 100 units of Product A
X2 50 i.e. produce 50 units of Product B

Value of Slack Variables

50 i.e. there are 50 machine hours unused at Optimum

X5 50 i.e. there are 50 components unused at Optimum

Note:

It will be seen that the other two slack variables, x_6 and x_7, do not have values in the Solution Quantity column. Their values are both zero, which means x_6 and x_7, representing the alloy and sales constraints respectively, have no unused capacity at optimum and the constraints they represent are fully utilised.

Contribution and Resource Valuations

It will be seen from Row 20 in Table 6 that Z has the Solution Quantity of —1,050. This means, at optimum, the maximum contribution is £1,050 (this can be confirmed by calculating the contributions of the optimum product mix, i.e. (100A and 50B = (100 x 8) + (50 x 5) = 1,050).

The value of —6 under Product x_3, the product that was not in the optimum plan, means that if any unit of x_3 was produced then overall contribution would fall by £6.

The values of Row 20 for the slack variables are of great importance. These are the valuations of resources and are known as shadow prices. These have the following meanings:

$X4$	=	0	i.e. there are no value to be gained from increasing machine hours
x_5	=	0	i.e. there are no value to be gained from increasing components
x_6	=	—4	i.e. for every extra kilo of alloy available £4 extra overall contribution would be gained
$X7$	=	—5	i.e. for every extra unit of B that was allowed to be produced contribution would increase by £5

It will be seen that only the constraints x_6 and x_7 that are binding, i.e. are fully utilised, have non-zero shadow prices. This is a general rule, for there would be no value in increasing the availability of a resource already in surplus. Thus, in this example, machine hours and the supply of components have shadow prices of zero.

The shadow prices of the binding constraints can be used to confirm again the overall contribution thus:

Alloy availability	200kg x £4 =	£800
Sales constraint	50 units x £5 =	£250
		£1050

When solving LP problems by graphical means, the shadow prices have to be calculated separately. When using the Simplex process they are an automatic by product.

Note:

(a) Each variable in the final solution variable column has a specific meaning which is detailed above.

(b) Using step 10 as a guide interpret the meanings of the solution variables an the valuations in the Z row of the intermediate tableaux.

(c) Alternative names for shadow prices are: shadow costs, dual prices or simplex multipliers.

(d) Take heart. To work through a normal Simplex problem with 3 unknowns is a very quick process once the foregoing steps are mastered.

3.2 Mixed Limitations

The maximizing example given above had constraints all of which were of the 'less than or equal to' type (<). This is a common situation but on occasions the constraints contain a mixture of < and varieties. The usual cause of one or more 'greater than or equal to' (?) constraints is the requirement to produce at least a given number of certain products.

In such circumstances, the simplest approach is to reduce the capacity of the other limitations by the amounts required to make the required number of the product(s) specified. Then maximize in the normal way and add back the quantities which were required to be produced, to the optimum solution found by the normal Simplex method. The method is shown in the following example.

Example Contained Mixed Constraints

Assume that an LP problem had been set up in the usual standardised format as follows:

Example: 2

Objective function

Maximize $5x_1 + 3x_2 + 4x_3$

Subject to a. $3x_1 + 12x_2 + 6x_3 < 660$ (machine hours constraints)

b. $6x_1 + 6x_2 + 3x_3 < 1{,}230$ (labour hours constraints)

c. $6x_1 + 9x_2 + 9x_3 < 990$ (component constraints)

d. $x_3 > 10$ (sales constraints)

where x_1, x_2 and x_3 represent units of Products A, B and C

Only one of the constraints d. is of the variety so we decide to make the minimum quantity possible to satisfy constraint d. i.e. 10 units of x_3. The resource requirements for 10 units of x_3 must be subtracted from the total available in constraints a, b and c thus:

Constraints

a. $3x_1 + 12x_2 + 6x_3 < 660 - (6 \times 10)$

b. $6x_1 + 6x_2 + 3x_3 < 1{,}230 - (3 \times 10)$

c. $6x_1 + 9x_2 + 9x_3 < 990 - (9 \times 10)$

Notes:

(a) In each case the expression in the bracket represents the coefficient of x_3 in each constraint multiplied by the 10 units being made.

(b) An alternative way of dealing with the constraint is to multiply both sides by **—1** and change the inequality sign.

$(x_3\ \mathbf{10}) \times -1$

i.e. $-x_3 - 10$

This constraint can then be used in the normal manner.

Having eliminated constraint D and made the appropriate reductions in the other constraints, the initial Simplex Tableau can be set up.

Initial Simplex tableau Table 8.7

Modified Constraint,	Solution variable	Products			Slack variables			Solution quantity
		x_1	x_2	x_3	x_4	x_5	x_6	
A	x_4	3	12	6	1	0	0	600
B	x_5	6	6	3	0	1	0	1,200
C	x_6	6	9	9	0	0	1	900
	Z	5	3	4	0	0	0	0

Note:

This is now a normal maximizing Simplex model with a slack variable for each constraint and is solved by exactly the same process covered in the earlier part of the chapter.

After carrying out the usual *Simplex procedure,* the final tableau becomes:

Final Sim lex tableau Table 8

Solution variable	Products			Slack variables			Solution quantity
	x,		x_3	X4	Xs	X6	
xt	0	7%	1%	1	0	—%	150
x_5	0	—3	—6	0	1	—1	300
X6	1	11/4	N	0	0	1/6	150
Z	0	0	—6	0	0	—5/6	—750

Solution from final tableau —150 units of x_l producing £750 contribution plus production to satisfy constraint d. 20 units of x_3 producing £40 contribution.

Therefore total solution is 150 units of x_l and 10 units of x_3 giving £790 contribution.

As previously described in Step 10, the tableau can be interpreted as follows:

Spare capacity x_4 = 150 means that there are 150 spare machining hours

x_s = 300 means that there are 300 spare labour hours

Products Not Being Made

From the final Tableau table it will be seen that x_2 and x_3 do not appear in the Simplex solution and they have valuations in the Z row of —41/2 and —31/2 respectively. We already know that 10 units of x_3 will be made and the tableau informs us that the 10 units of x_3 which we have had to make have cost £35 in reduced contribution. Any units of x_2 that were made would similarly reduce contribution by £4.50 per unit.

Shadow Prices

Only one constraint (constraint C, components) is fully utilised and thus has a non-zero shadow price. The tableau shows that for every extra component that could be obtained, contribution would be increased by £216.

As previously, the shadow price of the binding constraint can be used to confirm the overall contribution as follows:

Modified component constraint = 900 x £216 = £750.

An alternative method of dealing with mixed limitations uses what are termed *artificial variables* as well as slack variables. This method is outside the scope of this course material.

3.3 Comparing Simplex and Graphical Solutions

The Simplex method can be used for problems with any number of unknowns — even those with only two unknowns that can also be solved graphically. To illustrate both solution methods, example is shown below using the Simplex method (graphically).

Example reproduced from Unit 2

Maximise $3x_1 + 4x_2$

$4x_1 + 2x_2$ 100 A

$4x_1 + 6x_2$ 180

$x_1 + x_2$ 40

x_i

<20

x_2 10

This problem is inserted into the initial tableau with a slack variable for each of the five constraints and with constraint E multiplied by —1 to reverse the inequality sign.

Initial Simplex tableau Table 9

Solution variable	Products		Slack variables					Solution quantity
	xi	X2	X2	X4	X5	X6	X2	
X1	7 '7		1	0	0	0	0	100
X4	1/4.0		0	I	0	0	0	180
x_5			0	0	1	0	0	40
x_6			0	0	0	1	0	20
x_7	O		0	0	0	0	I	—10
Z	3	40		0	0			

This problem is then solved by the usual Simplex iterations. Each improves on the one before and the process continues until optimum is reached. The following tables show the position after each one is cross referenced to figure ... which is a reproduction of the graphical solution.

First iteration equivalent to point 1 on Table 10

Solution variable	Products		Slack variables					Solution quantity
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	[illegible]		0	0	0	0	—1	10
x_4	[illegible]	0	1	0	0	0	2	80
x_5	[illegible]	0	0	1	0	0	6	120
x_5		0	0	0	1	0	1	30
x_7		0	0	0	0	1	0	20
Z	3	0	0	0	0	0	4	—40

This shows $10x_2$ being produced and £40 contribution. The first four constraints have surpluses of 80, 120, 30 and 20 respectively. Not optimum, as there are still positive values in Z row.

Second iteration equivalent to point 2 on Table 11

Solution variable	Products		Slack variables					Solution quantity
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0.667	1	0	0.167	0	0	0	30
X_4	2.667	0	1	—0.333	0	0	0	40
x_5	—0.333	0	0	—0.167	0	0	0	10
x_6	1	0	0	0	1	1	0	20
x_7	0.333	0	0	0.167	0	0	1	20
Z	0.333	0	0	—0.667	0	0	0	—120

This shows $30x_2$ being produced and £120 contribution. All constraints have surpluses except labour hours. Not optimum as there is a positive value in Z row.

Third iteration equivalent to point 3 on Table 12

Solution variable	Products		Slack variables -					Solution quantity
	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4			0.375	0.125	0	0	0	15
x_4	0	0	—0.25	0.25	0	0	0	20
	0	0	X 0.125	5 0.125	1	0	0	5
x_6	0	0	—0.375	0125	0	1	0	5
x_7	0	0	—0.25	0.25	0	0	I	10
Z	0	0	—0.125	-0.625	0	0	0	—125

This is the optimum and shows all the information obtained previously.

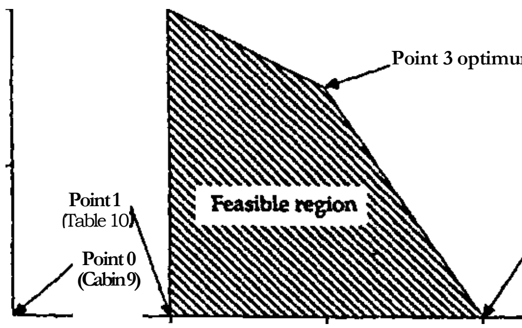

Figure 1 Graphical/Simplex comparison

Solution:

Optimum production:

- $x_1 = 15$ i.e. produce 13 units of Klunk
- $x_2 = 20$ i.e. produce 20 units of Klick giving an overall contribution of £ 125

Shadow prices of binding constraints:

x_3 = £0.125 i.e. each additional machine hour would produce £0.125 extra contribution labour.

X4 £0.625 i.e. each additional labour hour would produce £0.625 extra contribution labour.

The non-binding constraints represented by x_5, x_6 and x_2 have 5, 5 and 10 spare respectively.

Note:

By seeing how the Simplex interactions move from one vertex to an improved vertex on the feasible region, it is possible to gain a better understanding of the process.

Summary of Simplex method

Flow chart 1 (on page ....) provides a concise summary of the Simplex method and is cross referenced to key points in the Unit.

SELF ASSESSMENT EXERCISE 2

(i) What is the dual of a minimizing LP problem?
(ii) How is the final Simplex Tableau interpreted?

4.0 CONCLUSION

This unit introduces special techniques for linear programming e.g. transformation in a simple approach.

5.0 SUMMARY

The simplex flow chart method shows the steps of maximizing. The simple transformation given may be applied to al minimization problems with all constraints of the greater than or equal type. The simplex method can be applied to all combinations but only a restricted range of problem is covered.

6.0 TUTOR MARKED ASSIGNMENT

Form the dual of the following problem:

Minimise: 30A + 608 + 20C

Subject to:	5A +	10B +	15C >	2,000
	2A +	38 +	IC >	300
	8A +	6B +	4C >	650

7.0 REFERENCES / FURTHER READINGS

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UNIT 4 INVENTORY CONTROL I

CONTENTS

1.0 INTRODUCTION

This unit will draw from the business world that irrespective of the nature of a firm, a businessman will need to hold stock. Retailers need stocks of components, raw materials and spare parts of machinery if production is to be continuous. A manufacturer may also need stocks of finished goods to satisfy his customers. Even an Accountant must hold stocks of stationery!

You will be introduced to different types of cost involved in stock (inventory) management like ordering costs, set-up cost, holding cost, stock-out cost, etc. Thus, you will see the need that a policy to minimize inventory costs is a rational objective.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- define the need for and nature of inventory;
- differentiate the four costs associated with inventory;
- identify inventory control systems and periodic review system
- distinguish the principles of re-order or two bin system;

3.0 MAIN CONTENT

3.1 Types / Nature of Inventory

Inventories are stock of the product of a company is manufacturing for sale and components that make up the product. The various forms of inventories that exist in a manufacturing company are conveniently classified into: raw materials, work-in-progress and finished goods.

- Raw materials — the materials, components, fuels, etc, used in the manufacture of products.
- Work-in-progress (WIP) — partly finished goods and materials, sub-assemblies, etc. held between manufacturing stages.
- Finished goods — completed products ready for sale or distribution.

The levels of three kinds of inventories for a firm depend on the nature of business. What would be classified as a finished product for one company might be classified as raw material for another. For example, steel bars would be classified as a finished product for a steel mil and raw material for a nut and bolt manufacturer.

3.2 Reasons For Holding Stocks

The main reasons for holding stocks can be summarized as follows:

(a) to ensure that sufficient goods are available to meet anticipated demand;
(b) to absorb variations in demand and production;
(c) to provide a buffer between production processes;
(d) to take advantage of bulk purchase discounts;
(e) to meet possible shortages in the future;
(0 to absorb seasonal fluctuations in usage or demand;
(g) to enable production processes to flow smoothly and efficiently;
(h) as a necessary part of the production process;
(i) as a deliberate investment policy particularly during inflation or possible shortage.

3.3 Inventory Management Techniques

In managing inventory, the firm's objective should be in consonance with the shareholder wealth maximization principle. To achieve this, the firm should determine the optimum level of inventory. Efficiently controlled inventories make the firm flexible. Inefficient inventory control results in unbalanced inventory and inflexibility. The firm may sometimes run out of stock and sometimes may pile up unnecessary stocks. This increases the level of investment and makes the firm unprofitable. To manage inventory efficiently, answer should be sought to the following two questions:

(1) How much should be ordered?
(2) When should it be ordered?

The first question relates to the problem of determining economic order quantity (EOQ), and is answered with an analysis of costs of maintaining certain level of inventories. The second question arises because of uncertainty and is a problem of determining the re-order point. All these will be discussed as the unit develops and the next unit.

3.4 Stock Costs

Stock represents an investment by the organisation. As with any other investment, the cost of holding must be related to the benefits to be gained. This is done through effective identification of the cost. There are four categories:

- Cost of holding;
- Cost of obtaining stock
- Stock-out cost, and
- The cost of stock itself

3.4.1 Cost of Holding Stock

These costs, are also known as carrying costs including the following:

(a) interest on capital invested in the stock:
(b) storage charge(warehouse — rent, lighting, heating, refrigeration, air conditioning, etc.);
(c) stores staffing, equipment maintenance and running costs;
(d) handling costs;
(c) audit, stocktaking or perpetual inventory costs;
(0 insurance, security;
(g) deterioration and obsolesce;
(h) pilferage, vermin damage.

3.4.2 Costs of Obtaining Stock

Otherwise known as ordering or procurement costs, include the following:

(i) The clerical and administrative costs associated with the purchasing, accounting, and goods received departments;

(ii) Transport costs;
(iii) Where goods are manufactured internally, the set-up and tooling costs associated with each production run.

3.4.3 Stock-Out Costs

These are the cost associated with running out of stock. The avoidance of this cost is the basic reason why this cost is held in the first instance. These costs include the following:

- Lost contribution through the lost sale caused by the stock-out;
- Lost of future sales because customers go elsewhere;
- Lost of customer goodwill;
- Cost of production stoppages caused by stock-outs of W.I.P. or raw materials;
- Labour frustration over stoppages;
- Extra cost associated with urgent often small quantity, replenishment purposes.

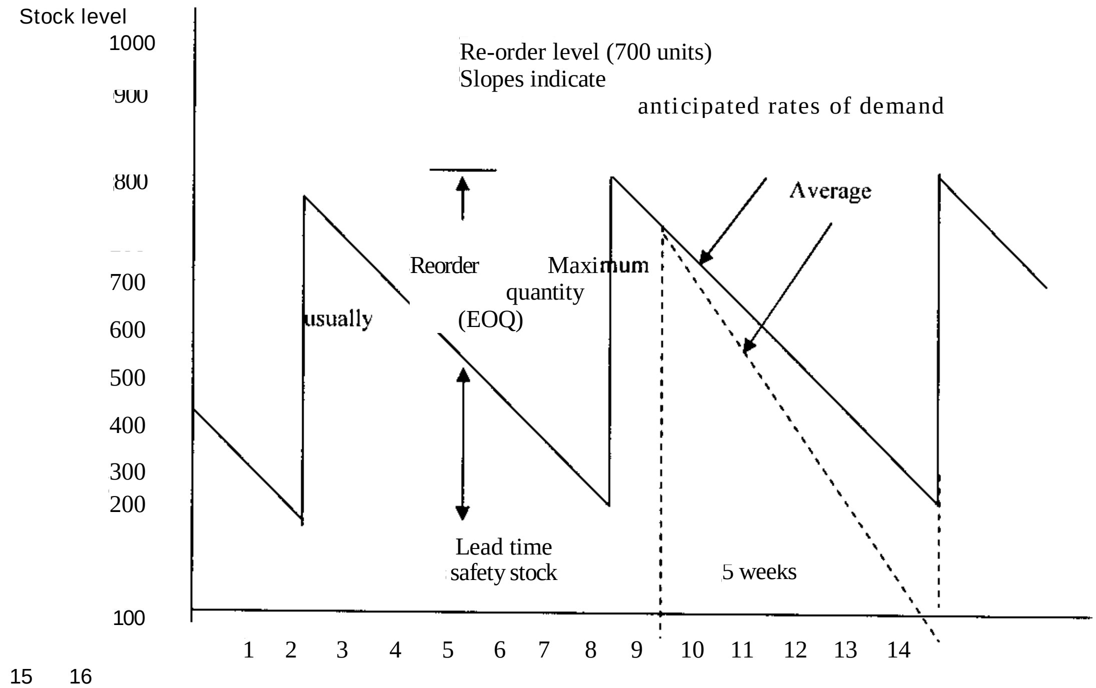
3.4.4 Costs of Stock

These costs are the buying in prices or the direct cost of production. You consider this cost when:

(a) Discounts are available for bulk purchases;
(b) Savings in production costs arc possible with longer batch runs.

Example 1

The following diagram —figure 9.1, shows a stock situation simplified by the following assumptions: regular rates of demand, a fixed lead time, and replenishment in one batch



Time, days or weeks

Stock terminology illustrated, Figure 1

Notes:

- It could be observed from the above diagram **1** that the safety stock is needed to cope with periods of maximum demand during the lead time.
- The lead time is shown as 5 weeks, the safety stock 200 units, and the re-order quantity 600 units.
- With constant rates of demand, as shown, the average stock is the safety stock plus 1/2 re-order quantity. For example in the figure **1,** the average stock is 200 + 'A (600) = 500 units.

Like most other quantitative techniques, two types of models can be used for inventory control — deterministic and stochastic (or probabilistic) models.

- A deterministic model is one which assumes complete certainty. The values of all factors (e.g. demand, usage, lead-time, carrying costs, etc.) are known exactly and there is no element of risk and uncertainty.
- A stochastic model exists where some or all of the factors are not known with certainty and can only be expressed in probabilistic or statistical terms. E.g. if the usage rate or lead-time was

specified as a probability distribution, this would be a stochastic or probabilistic model.

Both types of model will be explained in subsequent units.

3.5 Types of Control System

They include:

(1) Re-order level system
(2) Periodic review system.

3.5.1 Reorder Level System

Otherwise known as the two-bin system with the following characteristics:

(a) A predetermined re-order level is set for each item.
(b) When the stock level falls to the re-order level, a replenishment order is issued.
(c) The replenishment order quantity is invariably the EOQ.
(d) The name `two-bin system' comes from the simplest method of operating the system whereby the stock is segregated into two bins.
(e) Most organizations operating the re-order level system maintain stock records with calculated re-order levels which trigger off the required replenishment order.

Note: Example 1 is a simple re-order level system which is widely used in practice.

Example 2:

The following data relate to a particular stock item:

Normal usage	110 per day
Minimum usage	50 per day
Maximum usage	140 per day
Lead time	25 — 20 days

EOQ (previously calculated) 5,000

Using this data, calculate the various control levels.

Solution:

Re-order Level Maximum Usage x Maximum Lead time

140 x 30

4,200 units

Minimum Level Re-order Level — Average Usage for
Average

Lead Time

4.200 —(110 x 27.5)

1,175 units

Maximum level = Re-order Level F.OQ — Minimum
Anticipated

in Lead Time

4.200 + 5,000 —(50 x 25)

7,950 units

Note: In a manual system, the three levels would be entered on a stock record card and comparisons made between the actual stock and the control levels each time an entry was made on the card.

Re-order level system

Advantages:

(a) Lower stocks on average:
(b) Items ordered in economic quantities via the FOQ calculation:
(c) Somewhat more responsive to fluctuations in demand:
(d) Automatic generation of a replenishment order at the appropriate time by comparison of stock level against re-order level:
(e) Appropriate for widely differing types of inventory within the same firm.

Disadvantages:

(a) Many items may reach re-order level at the same time. thus overloading the re-ordering system:

(b) Items come up for re-ordering in a random fashion so that there is no set sequence;
(c) In certain circumstances (e.g. variable demand, ordering costs etc.), the EOQ calculation may not be accurate.

3.5.2 Periodic Review System

This system is sometimes called the constant cycle system. The system has the following characteristics:

(a) Stock levels for all parts are reviewed at fixed intervals, e.g. every fortnight.
(b) Where necessary, a replenishment order is issued.
(c) The quantity of the replenishment order is not a previously calculated EOQ, but is based upon; the likely demand until the next review, the present stock level and the lead time.
(d) The replenishment order quantity seeks to bring stocks up to a predetermined level.
(c) The effect of the system is to order variable quantities at fixed intervals as compared with the re-order level system where fixed quantities are ordered at variable intervals.

Advantages:

(a) All stock items are reviewed periodically so that there is more chance of obsolete items being eliminated;
(b) Economies in placing orders may be gained by spreading the purchasing office load more evenly;
(c) Larger quantity discount may be obtained when a range of stock items are ordered at the same time from a supplier;
(d) Because orders will always be in the same sequence, there may be production economies due to more efficient production planning being possible and lower set up costs.

Disadvantages:

(a) In general, larger stocks are required, as re-order quantities must take account of the period between reviews as well as lead times;
(b) Re-order quantities are not at the optimum level of a correctly calculated EOQ;
(c) Less responsive to changes in consumption. If the rate of usage change shortly after a review, a stock-out will occur before the next review;
(d) Unless demands are reasonably consistent, it is difficult to set appropriate periods for review.

SELF ASSESSMENT EXERCISE 1

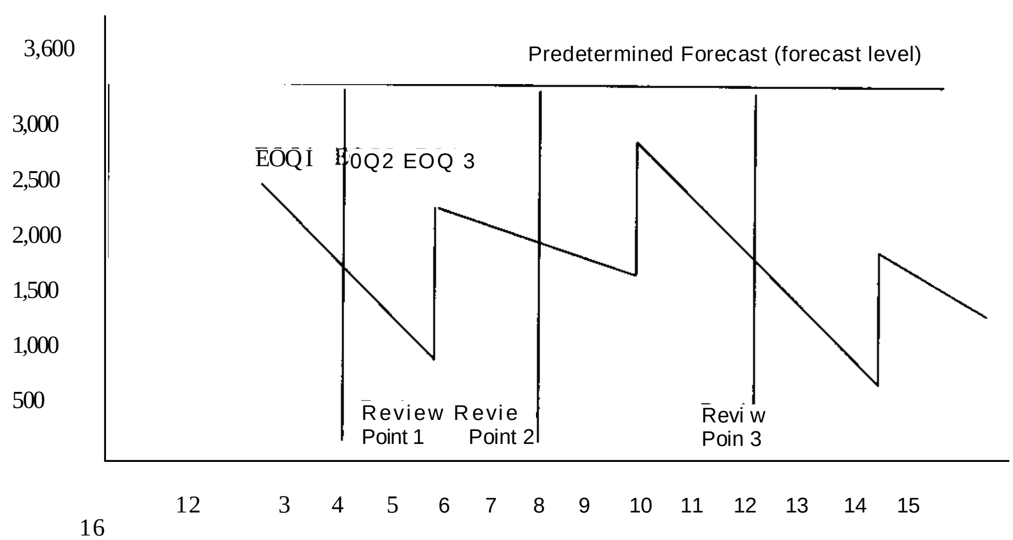
i What are the advantages and disadvantages of the Periodic review system of Inventory control?
How may inventory be classified?

Example 3

A simple manual periodic review system control of the 500 piece parts used in the assembly of the finished products by the periodic review system. The stock levels of al 500 parts are reviewed every 4 weeks and a replenishment order issued to bring the stock of each part up to a previously calculated level. This level is calculated at six-monthly intervals and is based on the anticipated demand for each part.

Based on the above system, the following graph shows the situation for one of the piece parts, parts No. 1101x, over a period of 16 weeks.

Stock Level



Time (weeks)
Stock levels of Part No. 1101 Figure 2

4.0 CONCLUSION

Inventory management techniques have been demonstrated as a tool for reducing cost and expressing inventory as an investment which is used to minimize cost and maximize profit.

5.0 SUMMARY

In this unit, stocks have been classified in raw material, work-in-progress, and finished goods.

- The overall objective of inventory control is to maintain stock at a level which minimizes total stock-out.
- There are two basic inventory control systems — the re-order level or two-bin system and the periodic review system.
- In the re-order level system, the usual replenishment order quantity is the EOQ.

6.0 TUTOR-MARKED ASSIGNMENTS

The following data relate to a given stock item:

Normal usage	1300 per day
Minimum usage	900 per day
Maximum usage	2000 per day
EOQ	30,000

Calculate the various controls levels.

7.0 REFERENCES / FURTHER READINGS

Lucey, T. (2002). *Quantitative Techniques 6th Edition,* (LLST) Educational Low-Priced Sponsored Texts.

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UNIT 5 INVENTORY CONTROL II

CONTENTS

1.0 INTRODUCTION

From Inventory Control I unit ... 3.3 (inventory management techniques), the importance of Economic Order Quantity in stock order arrangement was emphasized. In this unit, an attempt will be made to explain the different facet of EOQ in inventory control.

2.0 OBJECTIVES

After studying this unit, you should be able to:

- explain the basic Economic Order Quantity (EOQ);
- calculate order quantity when price discounts are possible

3.0 MAIN CONTENT

3.1 Economic Order Quantity (E0Q)

One of the major inventory management problems to be resolved is how much inventory should be added when inventory is to be replenished. If the firm is buying raw materials, it has to decide lots in which it has to be purchased on replenishment. If the firm is planning a production run, the issue is how much production is to schedule (or how much to make). These problems are called Economic Order Quantity problems the task of the firm is to determine the optimum or economic order quantity (or economic lot size). The determination of an optimum inventory level involves two types of costs — ordering cost and carrying cost. The economic order quantity is that inventory level that minimizes the total of ordering and carrying costs.

3.2 EOQ Assumption

To be able to calculate a basic EOQ, certain assumptions are necessary:

(a) that there is a known, constant stockholding cost;
(b) that there is a known, constant ordering cost;
(c) that rates of demand are known;
(d) that there is a known, constant price per unit;
(e) that replenishment is made instantaneously;
(f) no stock-outs allowed.

It is worthy of note that:

(i) it will be apparent that the above assumptions are somewhat sweeping and there are good reasons for treating any EOQ calculation with caution
(ii) Some of the above assumptions are relaxed later in the study (unit);
(iii) The rationale of EOQ ignores buffer stocks which are maintained to cater for variations in lead time and demand.

Example 1 – A graphical EOQ

The following data will be used to develop a graphical solution to the EOQ problem:

A company uses 50,000 gadgets per annum which are N10 each to purchase. The ordering and handling costs are N150 per order and carrying costs are 15% of purchase price per annum, i.e. it costs N1.50 per annum in carrying a gadget in stock (N10 x 15%).

To graph the various costs involved, the following: (pages 240 and 241)

3.2.1 The EOQ Formula

It is possible, and more usual, to calculate the EOQ using a formula. The formula method gives an exact answer, but do not be misled into placing undue reliance upon the precise figure. The calculation is based on estimates of costs, demands etc., which are, of course, subject to error. The EOQ formula is given below and you should learn it:

Basic EOQ formula

E C K) =

C_o

Where C_o = Ordering cost per order

D = Demand per annum

C, = Carrying cost per item per annum

Using the data from Example 1, the EOQ can be calculated:

= N150

50,000 gadgets

C_o = N10 x 15% or N1.50 per gadget

$$EOQ = \sqrt{\frac{2 . C_o . D}{C_o}}$$

$$= \sqrt{\frac{2 \times 150 \times 50{,}000}{1.5}}$$

10,000,000

3,162 gadgets

Notes:

(a) The closest value obtainable from the graph was approximately 3,200 which is very close to the exact figure.

(b) Always take care that demand and carrying costs are expressed for the same time period. A year is the usual period used.

(c) In some problems, the carrying cost is expressed as a percentage of the value whereas in others it is expressed directly as a cost per item. Both ways have been used in this example to provide a comparison.

The mathematical derivation is expressed below:

Derivation of basic EOQ formula:

Let D = annual demand

0 = order quantity

C_o = cost of ordering for one order

C, carrying cost for one item per annum

Average Stock = Q/2

Total annual stockholding , QC,/2

Number of orders per annum = D/Q

Annual ordering costs , DCo/Q

Total Cost $\frac{Qc_c}{2}+\frac{D.0_0}{Q}$

The order quantity which makes the total cost (Tc) at a minimum is obtained by differentiating with respect to Q and equating the derivative to zero.

$$\frac{dTc}{dQ} = \frac{\quad}{2} \;—\; \frac{\quad}{Q^2}$$

and when $\frac{dTc}{dQ}$ = 0 costs are at a minimum

i.e. $0 = \frac{C_c}{2} — \frac{D_c}{Q^2}$

and to find Q

$$\frac{DC_Q}{Q2} = \frac{Cc}{2}$$

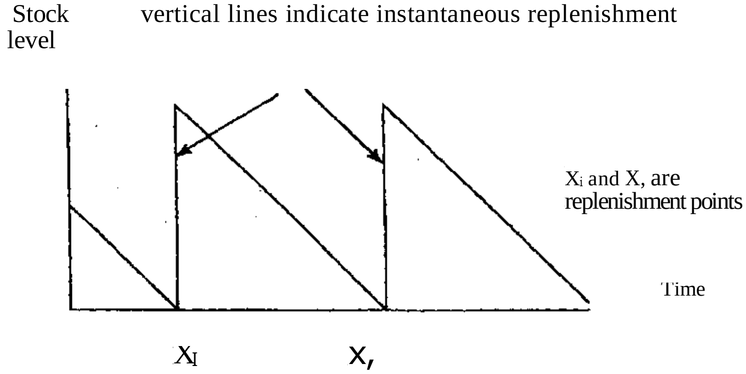
$$2DC_0 = Q^2Cc$$

$$\frac{2DC_{,2}}{Cc} = Q^2$$

$$Q \text{ (i.e. the EOQ)} = \sqrt{\frac{2\,.\,C_o\,D}{C_o}}$$

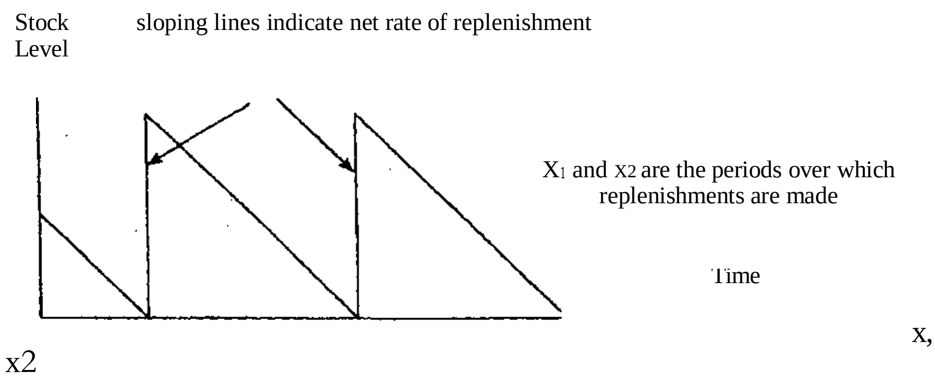
3.2.2 EOQ with Gradual Replenishment

In example earlier shown above, the assumption was that the widgets were ordered externally and that the order quantity was received as one batch, i.e. instantaneous replenishment as shown in the following diagram figure 10.1



Stock levels showing instantaneous replenishment
Figure 1

If however, the widgets were manufactured internally, they would probably be placed into stock over a period of time resulting in the following pattern:



Stock levels showing Non-instantaneous replenishment
Figure 2

The net rate of replenishment is determined by the rate of replenishment and the rate of usage during the replenishment period. To cope with such situations, the basic EOQ formula needs modification thus:

EOQ with gradual replenishment = $\sqrt{\frac{2 . C_2 . D}{Cc\,(1 — D)}}$

Where R = production rate per annum, i.e. the quantity that would be produced if production of the item was carried on the whole year.

All other elements in the formula have meanings as previously defined in paragraph 4. Note: The derivation of the above formula is given

Example of EOQ with Gradual Replenishment

Example 1

Assume that the firm described in Example 1 has decided to make the widgets in its own factory. The necessary machinery has been purchased which has a capacity of 250,000 widgets per annum. All other data are assumed to be the same.

EOQ with gradual replenishment = $\dfrac{2 \times 150 \times 50{,}000}{1.5\left(1 - \dfrac{50{,}000}{250{,}000}\right)}$

'1 $\dfrac{15{,}000{,}000}{1.5\,(0.8)}$

3,535 widgets

Notes:

(a) The value obtained above is larger than the basic EOQ because the usage during the replenishment period has the effect of lowering the average stock holding cost.

(b) As pointed out earlier on, the ordering costs for internal ordering usually include set-up and tooling costs as well as paper work and administration costs.

3.2.3 EOQ Where Stock-Outs Are Permitted

It will be recalled that the overall objective of stock control is to minimize the balance of the three main areas of cost, i.e. holding costs, ordering costs and stock-out costs.

Stock-out costs are difficult to quantify but nevertheless may be significant and the avoidance of these costs is the main reason why stocks are held in the first place. Where stock-out costs are known then they can be incorporated into the EOQ formula which thus becomes:

EOQ (where stock-outs are permitted and stock-out costs are known)

=

Cc Cs

Where Cs = stock-out costs per item and the other symbols have the meanings previously given.

Note: It will be seen that the formula is the basic EOQ formula multiplied by a new expression containing the stock-out cost. The derivation of the EOQ formula where stock-outs are permitted is given below.

Example

Assume the same data as in .... Except that stock-outs are now permitted. When a stock-out occurs and an order is received for widgets the firm has agreed to retain the order and, when replenishments are received, to use express courier service for the delivery at a cost of NO.75 per widget. Other administrative costs associated with stock-outs are estimated at NO.25 per unit. What is the EOQ?

C,, = N150

50,000

C_c= N1.50

C, = NO.75 + NO.25 NI.00

Thus, EOQ (with stockouts) Ai 2 x 150 x 50,000 i11.5 + 1

1.5 1

5,000

3.2.4 EOQ With Discounts

A particularly unrealistic assumption with the basic EOQ calculation is that the price per item remains constant. Usually some form of discount can be obtained by ordering increased quantities. Such price discounts can be incorporated into the EOQ formula, but it becomes much more complicated. A simpler approach is to consider the costs associated with the normal EOQ and compare these costs with the costs at each succeeding discount point and so find the best quantity to order.

3.2.5 Financial Consequences of Discounts

Price discounts for quantity purchases have three financial effects, two of which are beneficial and one adverse.

Beneficial Effects:

Savings come from:

(a) Lower price per item
(b) The larger order quantity means that fewer orders need to be placed so that total ordering costs are reduced.

Adverse Effect:

Increased costs arise from the extra stockholding costs caused by the average stock level being higher due to the larger order quantity.

Example 2 of EOQ with

A company uses a special bracket in the manufacture of its products which it orders from outside suppliers. The appropriate data are;

Demand	=	2,000 per annum
Order cost		N20 per order
Carrying cost =		20% of item price
Basic item price	=	NI 0 per bracket

The company is offered the following discounts on the basic price:

For order quantities	400 — 799	less 2%
	800 — 1,599	less 4%
	1,600 and over	less 5%

It is required to establish the most economical quantity to order.

This problem can be answered using the following procedure:

(a) Calculate the EOQ using the basic price.
(b) Compare the savings from the lower price and ordering costs and the extra stock-holding costs at each discount point (i.e. 400, 800 and 1,600) with the costs associated with the basic EOQ, thus:

$$\text{Basic EOQ} = \sqrt{\frac{2 \times 2{,}000 \times 20}{10 \times 0.2}} = 200 \text{ brackets}$$

Based on this EOQ, the various costs and savings comparisons are given in the following Table: 10.1

Order Quantity	200 (EOQ)	400	800	1,600	Line No.
Discount		2%	4%	5%	1
Average No. of Orders p.a.	10	5	21/2	11/2	2
Average No. of Orders saved p.a.		5	71/2	8%	3
Ordering cost savings P.a.		(5 x 20)=44100	(71/2 x 20)= N150	(8% x 20)=14175	4
Price saving per item per annum	-	20p (2000 x 20p) = 400	40p (2,000 x 40p)— 800	50p (2,000 x 50p) = 1,000	5
Total gains		N500	N950	141,175	6
Stockholding cost p. a.	(100 x 10 x 2) =N200	200 x 9.8 x 21= N392	400 x 9.6 x 2) = N768	(600 x 9.5 x 2) = 441,520	7
Additional costs incurred by increased order	-	(N392 - N200) = N192	(N768 - 14200) = 14568	(141,520-14200) 141,320	8
Net gain / (loss)		14308	N382	(N145)	9

Cost/Savings Comparisons EOQ to Discount Points

From the above table, it will be seen that the most economical order quantity is 800 brackets, thereby gaining the 4% discount.

Notes:

a) Line 2 is $\frac{\text{Demand of } 2{,}000}{\text{Order quantity}}$

b) Line 7 is the cost of carrying the average stock, i.e.

$$\frac{\text{Order quantity}}{2} \times \text{cost per item} \times \text{carrying cost percentage}$$

c) Line 9 is Line 6 minus Line 8.

3.2.6 Marginal Costs and EOQ Calculations

It cannot be emphasized too strongly that the costs to be used in EOQ calculations must be true marginal costs, i.e. the costs that alter as a result of a further order or carrying another item in stock. It follows therefore that fixed costs should not be used in the calculations. In the examples used in this chapter, the costs have been clearly and simply stated. In examination questions, this is not always the case and considerable care is necessary to ensure that the appropriate costs are used.

4.0 CONCLUSION

In this unit, you have been inculcated with the different applications of inventory control through economic order quantity (E0Q).

5.0 SUMMARY

- The EOQ is the order quantity which minimizes the total costs involved which include holding and order costs.
- The basic EOQ calculation is based on constant ordering and holding costs, constant demand and instantaneous replacement.
- The basic ECQ formula is: $\sqrt{\frac{2 \cdot C_O \cdot D}{C_c}}$

where larger quantities are ordered to take advantage of price discounts, stock-holding costs increase, but savings are made in the price reductions and reduced ordering costs.

SELF ASSESSMENT EXERCISE 1

i What assumptions are usually made in EOQ calculations?

A customer has been ordering 5,000 units at the rate of 1,000unit per order during last year. The production cost is N12 per unit — N8 for materials and labour and N4 overhead cost. It costs N1,500 to set up for one run of 1,000 units and inventory carrying cost is 20% of the production cost. Since this customer may buy at least 5,000 units this year, the company would like to avoid making five different production runs. Determine the most economic production run.

6.0 TUTOR-MARKED ASSIGNMENTS

1) A company uses 100,000 units per year which cost N3 each. Carrying costs are 1% per month and ordering costs are N250 per order. What is the E0Q?
2) What would be the EOQ if the company made the items themselves on a machine with a potential capacity of 600,000 units per year?

7.0 REFERENCES / FURTHER READINGS

Lucey, T. (2002). *Quantitative Techniques 6th Edition,* (LLST) Educational Low-Priced Sponsored Texts.

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Pandey, I.M. (2005). *Financial Management,* UBS Publisher's Distributors PVT LTD.

MODULE 3

UNIT 1 INVENTORY CONTROL III

CONTENTS

1.0 INTRODUCTION

In this unit, a cursory study of the problem, how much to order, will be solved by determining the economic order quantity. The answer should be sought to the second problem, when to order. This is a problem of determining the re-order point.

2.0 OBJECTIVES

After studying this unit, you should be able to:

- plan safety or buffer stocks necessary to cope with variations in demand and/or lead time
- calculate sensitivity analysis as in inventory control

3.0 MAIN CONTENT

3.1 The Re-order Point

The re-order point is that inventory level at which an order should be placed to replenish the inventory to determine the re-order point under certainty, you should know:

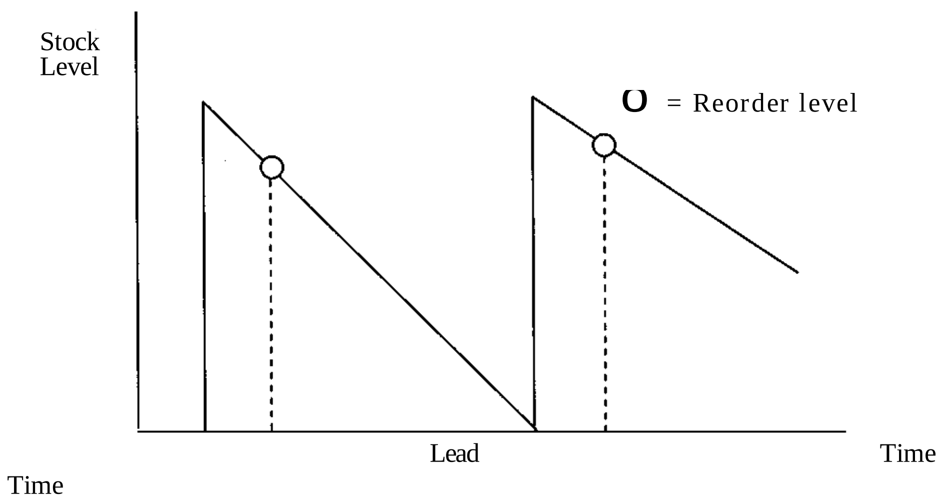
(a) lead time
(b) average usage, and
(c) economic order quantity (EOQ)

Lead time is the time normally taken in replenishing inventory after the order has been placed. By certainty, we mean that usage and lead time do not fluctuate. Under such situation, re-order point is simply that inventory level which will be maintained for consumption during the lead time. That is:

Re-order point Lead x Average usage

3.1.1 Reorder levels in conditions of certainty

Where the rate of demand and the lead time is known with certainty, the re-order level is the rate of demand times the lead time. This means that, regardless of the length of the lead time or of the rate of demand, no buffer stock is necessary. This results in a situation as follows:



Reorder level in conditions of certainty
Figure 1

3.1.2 Re-order Level and Safety Stock Relationship

It will be seen from the above figure that, in conditions of certainty, the re-order level can be set so that stock just reaches zero and is then replenished. When demand and/or lead time vary, the re-order level must be set so that, on average, some safety stock is available to absorb variations in demand and/or lead time. In such circumstances, the re-order level calculation can be conveniently considered in two parts:

a) the normal or average rate of usage times the normal or average lead time (i.e. as the re-order level calculation in conditions of certainty) plus
b) the safety stock.

3.2 Safety Stock

The re-order point in any typical example is computed under the assumption of certainty. It is always difficult to predict usage and lead time accurately. The demand for material may fluctuate from day-to-day or from week-to-week. Similarly, the actual delivery time may be difficult from the normal lead time. If the actual usage increases or the delivery of inventory is delayed, the firm can face a problem of stock-out which can prove to be costly for the firm. Therefore, in order to guard against stock-out, the firm may maintain a safety-stock — some minimum or buffer inventory as cushion against expected increased usage and/or delay in delivery time.

3.2.1 Safety Stock Calculation by Cost Tabulation

The amount of safety stock is the level where the total costs associated with safety stock are at a minimum. That is, where the safety stock holding cost plus the stock out cost is lowest. (it will be noted that this is a similar cost position to that described in the EOQ derivation described in earlier unit).

The appropriate calculations are given below based on the following illustration:

Example 1

An electrical company uses a particular type of thermostat which costs $5. The demand averages 800 p.a. and the EOQ has been calculated at 200. Holding costs are 20% p.a. and stock out costs have been estimated at $2 per item that is unavailable. Demand and lead times vary, but fortunately, the company has kept records of usage over 50 lead times as follows:

(a) Usage in lead time	(b) Number of times recorded	(c) Probability b/50
25 – 29 units	1	0.02
30 — 34 units	8	0.16
35 —39 units	10	0.20
40 — 44 units	12	0.24
45 — 49 units	9	0.18
50 — 54 units	5	0.10
55 — 59 units	5	0.10
Total	**50**	**1.00**

From the above the re-order level and safety stock should he

Solution:

Step 1 – Using the mid-point of each group, calculate the average usage in the lead time.

X	T	tx
27	1	27
32	8	256
37	10	370
42	12	504
47	9	423
52	5	260
57	5	285
Total	**50**	**2,125**

Table 2

$$\text{Average usage} = \frac{2,125}{50} \quad 42.5$$

Step 2 - Find the holding and stock out costs for various re-order levels.

A Re-order level	B Safety stock (A-42.5)	C Ilolding cost (B x N) N	D Possible shortages (mid-points) Table 2- A)	E Probability (from Table 1)	F No. of orders p.a. (8001 / 200)	C Shortage cost (D x E x I: x N 2) N	II Total cost (C + G)
45	2.5	2.5	2	0.18	4	2.88	
			7	0.10	4	5.60	
			12	0.10	4	9.60	20.58
50 •	7.5	7.5	2	0.10	4	1.60	
			7	0.10	4	5.60	14.70
55	12.5	12.5	2	0.10	4	1.60	14.10
60	17.5	17.5					17.50

From the Table (I) it will be seen that the most economical re-order level is 55 units. This re-order level, with the average demand in the lead time of 42.5, gives a safety stock of 12.5.

3.2.2 Safety Stock Calculation by Statistical Methods

The previous method of calculating safety stock was based on relative holding and stock out costs; but on occasions these costs, particularly the effects of stock outs, are not known. In such circumstances management may decided upon a particular risk level they are prepared to accept and the safety stock and re-order level are based upon this risk level. For example, management may have decided that they are prepared to accept a 5% possibility of a stock out.

Illustration of a safety stock calculation by statistical methods

Solution:

a) Safety stock given variable demand and constant lead time: From normal area tables, it will be found that 5% of the area lies above the mean +1.64a.

Safety stock = 1.64 x standard deviation of demand for 13.44 days

= 1.64 x (0.4 x ·4 13.44)

= 2.40

Note: The standard deviation of daily demand, 0.4 is multiplied by \)13.44 because standard deviations are not added.

b) Safety stock given variable lead time and constant demand:

Safety stock = 1.64 x standard deviation of lead time for a demand of 3,162

$$= 1.64 \times (0.75)$$
$$= \underline{1.23}$$

Example 2

Using the data from example 1, it was found that the average demand during the lead time was 42% units. The company has carried out further analysis and has found that this average lead time demand is made up of an average-demand (D) of 3,162 units per day over an average lead time (L) of 13.44 days. Both demand and lead time may vary and the company has estimated that the standard deviation of demand (o.D) is 0.4 units and the standard deviation of lead time (aL) is 0.75 days. The company is prepared to accept a 5% risk of a stock out and wish to know the safety stock required in the following three circumstances:

(i) Where demand varies and lead time is constant;
(ii) Where the lead time varies and demand is constant;
(iii) Where both demand and lead time vary.

c) [illegible] of the separate safety stocks already calculated:

$$= 2.40 + 1.23$$
$$= \underline{3.63}$$

Note:

(a) Safety stock calculations based on risk levels are commonly used.

(b) Where lead time and demand vary it can be expensive to maintain sufficient stocks for low stock out risks.

The above example uses for properties of the Normal Distribution and values obtained from Normal Area tables. It is but one further application of the statistical principles covered earlier in the book using continuous probability distributions. Discrete probabilities and expected values can also be used for incorporate variability.

3.3 Inventory Control and Sensitivity Analysis

When an inventory control value has been calculated, for example, the EOQ, the re-order level, the total stockholding cost etc. management stay wish to know how sensitive the value is to changes in the factors used to calculate it.

For example, is the EOQ greatly affected when ordering cost changes? The process by which this is done is known as *sensitivity analysis.* The following example shows a typical analysis:

Example: 3
Henderson Ltd. made the following estimates for a component they use:
Annual usage 1,125
Ordering costs N50 per order
Carrying costs per year N5 per component
Based on these estimates, an EOQ of 150 and expected total stock costs of N750 p.a. were calculated, as follows:

$$\text{EOQ} = \frac{42 \times 50 \times 1,125}{5} = 150$$

Expected total stock costs Ordering cost p.a. + Holding Cost p.a.

$$\frac{(1\ 125)}{150} \times N50 + \frac{145 \times 150}{2} = \underline{N750}$$

During a year, the EOQ of 150 was used for re-ordering, but the actual usage of components turned out to be 20% higher at 1,350.

(a) Calculate the actual total stock costs.
(b) Calculate what the total stock costs would have been if the correct EOQ had been used.
(c) Find out how sensitive total costs are to errors in the usage estimates.

Solution:

a) Actual total stock costs = $\frac{(1,350)}{150} \times N50 + \frac{5 \times 150}{2} = \underline{N825}$

b) Correct EOQ = if $\frac{2 \times 50 \times 1,350}{3} = 164$

Total costs if an EOQ of 164 was used

$$= \frac{(1,350)}{164} \times N50 + \frac{145 \times 164}{2} = \underline{N822}$$

c) Thus N3 (i.e. N825 — 822) extra costs were incurred by using the 150 EOQ based on the incorrect usage.

This means that a 20% increase in demand has caused only approximately $\{^1/2\%\}$ change in total costs. We conclude that total costs are insensitive to errors in demand estimates.

From management's viewpoint, this is good news. It means that even if errors are made in the estimates, which are all too likely, it doesn't make much difference to the final result.

SELF-ASSESSED EXERCISES

(i) How is the re-order level set when demand and lead times are known with certainty?

(ii) Why is the establishment of safety stocks essentially a cost balancing exercise?

4.0 CONCLUSION

At the end of this unit, you have been taught that safety or buffer stocks are needed to cope with variations in demand and/or lead time. This is essential in the skill of managing inventory in an organisation.

5.0 SUMMARY

That safety stocks are necessary because of demand and/or lead time variation. Re-order level is the average demand over the average lead time plus safety stock.

The safety stock level can be established by comparing the safety stock holding cost and the stock out cost at various re-order levels.

6.0 TUTOR MARKED ASSIGNMENT

The demand for an item is normally distributed with a mean of 50 per day and a standard deviation of 5 units. Given a lead time of 20 days, what is the re-order level and safety stock to meet 90% of demands.

7.0 REFERENCES AND FURTHER READINGS

Lucey, T. (2002). *Quantitative Techniques 6th Edition,* (LLST) Educational Low-Priced Sponsored Texts.

Owen, F. and Jones, R. (1975). *Modern Analytical Techniques,* Polytech Publishers Limited, Stockport.

Pandey, I.M. (2005). *Financial Management,* UBS Publisher's Distributors PVT LTD.

UNIT 2 TIME SERIES ANALYSIS

CONTENTS

1.0 INTRODUCTION

Time series analysis is one of the most powerful methods in use, especially for short-term forecasting purposes. From the historical data, one attempts to obtain the underlying pattern so that a suitable model of the process can be developed, which is then used for purposes of forecasting or studying the internal structure of the process as a whole. Time series analysis looks for the dependence between values in a time series (a set of values recorded at equal time intervals) with a view to accurately identify the underlying paths of the data.

In the case of quantitative methods of forecasting, each technique makes explicit assumptions about the underlying pattern. For instance, in using regression models you had first to make a guess on whether a linear model should be chosen and only then could you proceed with the estimation of parameters and model development. We could rely on mere visual inspection of the data or its graphical plot to make the best choice of the underlying model. However, such guess work though not uncommon, is unlikely to yield very accurate or reliable results. In time series analysis, a systematic attempt is made to identify and isolate different kinds of patterns in the data. The four kinds of patterns that are most frequently encountered are horizontal, non-stationary, seasonal and cyclical. Generally, a random component is also superimposed.

The question of the choice of a forecasting method is taken up. The characteristics of various methods are summarized along with likely situations where these may be applied. Of course, considerations of cost and accuracy desired in the forecast play a very important role in the choice.

2.0 OBJECTIVES

After reading this unit, you should be able to:

- discuss the role of time series analysis in short-term forecasting;
- decompose time series into its various components;
- identify the underlying patterns of a time series;

3.0 MAIN CONTENT

3.1 Time Series

The time series takes a variable and examines the way in which its magnitude fluctuates over a period of time. For example, consider the way in which output of a particular firm has varied year by year since 1985. Why should you do this?

Whenever we are interested in planning, you must make forecasts of what is likely to happen in the future. Despite complicated techniques applied, the inference will depend, to a large extent, on what has happened in the past. Thus, a reliable analysis of what has happened in relevant fields in recent years is the first step in obtaining a reliable estimate of what is likely to happen in the future.

Any housewife will tell you that egg prices are likely to rise during particular months of the year, and to fall during other months. How does she arrive at this?

On the basis of past experience, you can undertake a simple analysis of more complex problems to predict the future more reliably and more precisely.

3.1.1 The Trend

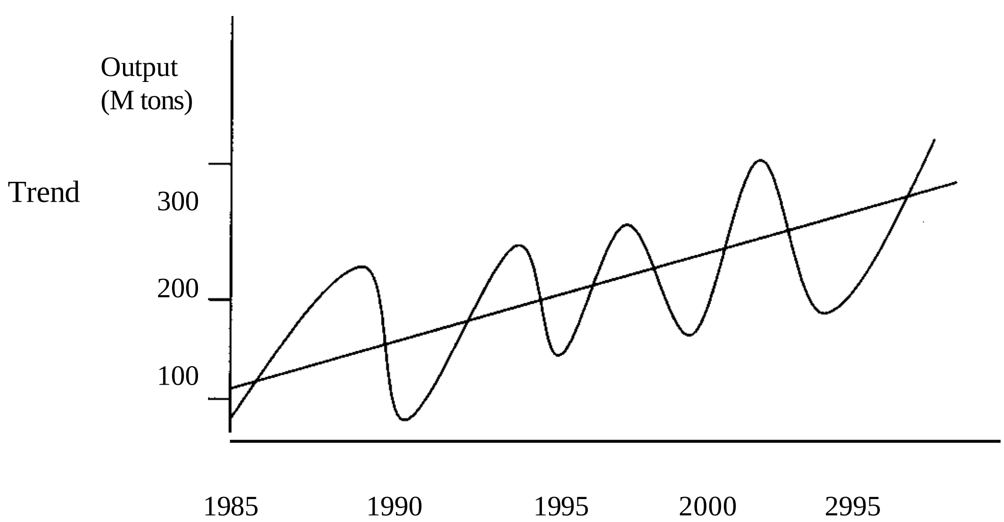
Let's consider the following series:

Table 1 Output of ABC Limited 1985 — 2002 Million Tons

Year	Output
1985	68
1986	100
1987	120
1988	177
1989	100
1990	80
1991	140
1992	200
1993	230
1994	180
1995	280
1996	200
1997	250
1998	170
1999	320
2000	230
2001	210
2002	330

Seeing the figures presented, information reveals two things — figures of output show that although there have been very marked fluctuations year by year, there is a general upward rise in the figures. Moreover, certainly from 1993 to 2002, the figures tend to show a peak output every second year, followed by a fall in the intervening years.

This becomes much more apparent if you look at the graph below of this time series in Figure 1:



The general rise in output can be clearly seen. This general tendency of figures to move in a given direction is known as TREND. In figure 1 above, the trend has been shown as a linear or straight line trend. Other series to be examined later in the trend will vary its direction, at first rising and then falling or vice versa. Eg: curvi-linear. An important characteristic of such trends is that they change direction slowly over time, and so, baring catastrophes, the continuation can be sketched in from a careful examination of the existing trend line.

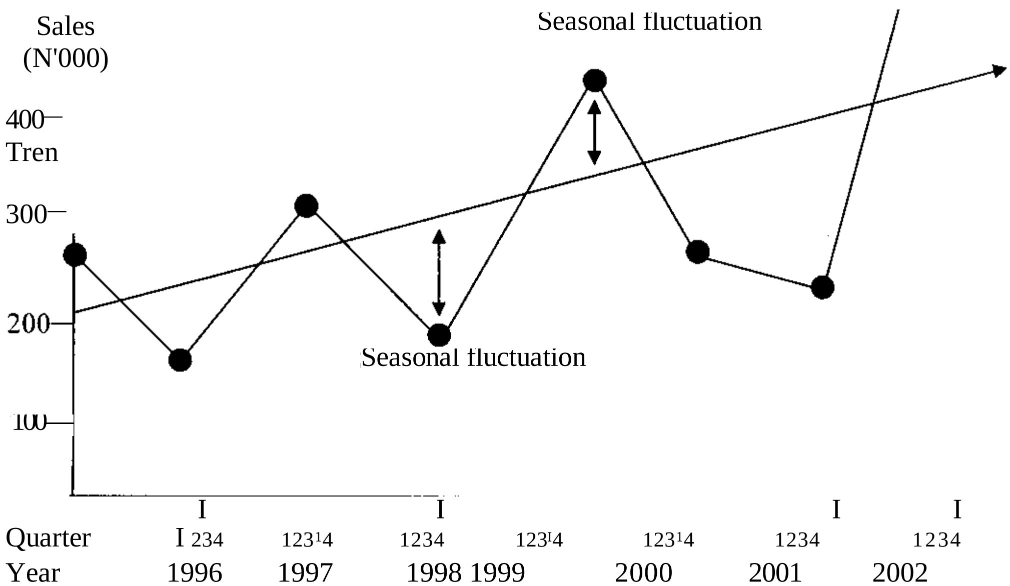
Much more obvious in figure 1 is the succession of rises and falls in output occurring at regular intervals. Looking at this graph suggests a certain pattern of behaviour in output in 2003. What is it?

Formal analysis of such a series can reveal much more useful data about what may happen in the future.

3.1.2 Seasonal Variation

Many series are of such a nature that figures become of much significant when they are analysed on a monthly or quarterly basis rather than annually. Most economic series are like this — sales, for example, show marked variations throughout the year.

Sales tend to rise at Christmas. But if such fluctuations occur, are they sufficiently regular to be predictable? If they are, then you must try to ascertain the direction and probable magnitude of the saving. Figure 2 below shows hypothetical sales data plotted quarterly over a period of years.



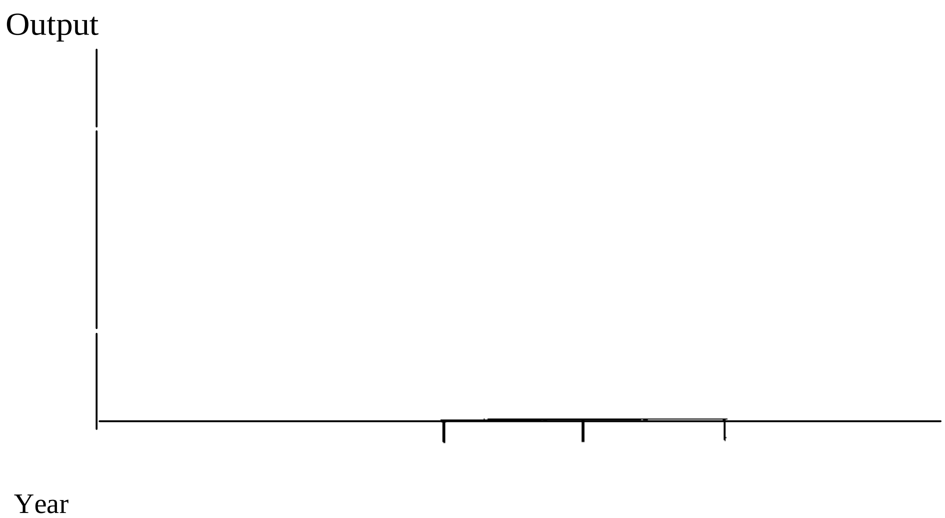
Seasonal Variation
Figure 2

The data show a remarkable regularity — every year, there are two sales peaks, in the second and fourth quarter, while every year, sales slump in the third quarter. The pattern is regular enough for us to predict that the same thing will happen in future years.

If y take the trend as the norm, such regular fluctuations around t trend are given the name of SEASONAL VARIATIONS. It should now be apparent that if you can obtain a reliable estimate of the trend, and of the magnitude of the seasonal variation from the trend, you have gone some way towards obtaining a reliable basis for forecasting the future sales figures.

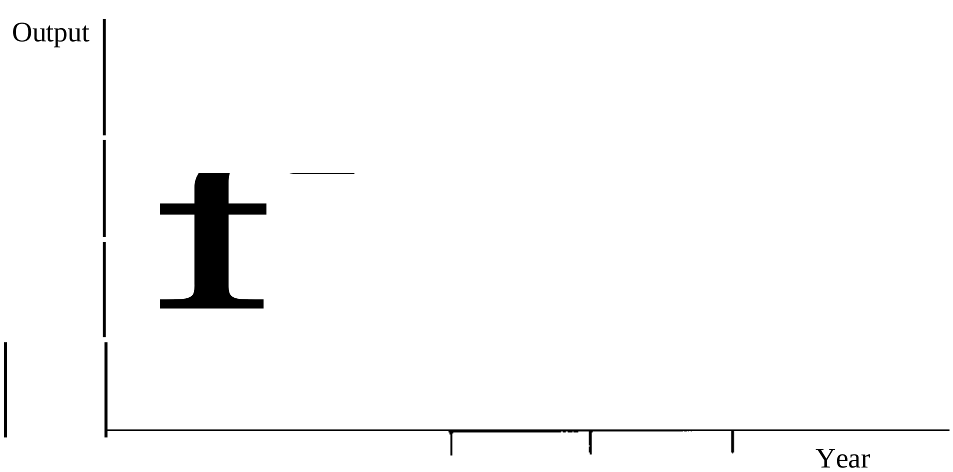
Figure 3

(a)



Linear Trend

(b)



Curvilinear Trend

3.1.3 Residuals

Unfortunately for the planner, the data he has is a compound of many influences, most of which cannot be foreseen and may not occur again in a similar form. Exports will affect the dock strike, production by a shortage of some vital raw material, sales by a sudden change in taxation. Thus, the figures gotten can be broken down into TREND AND SEASONAL VARIATION AND RESIDUALS.

By their very nature residual influences are chance happenings and are not amenable to analysis. This does not mean that residual influences can be ignored by any planner of repute for it could make nonsense of his forecast.

The importance of the residuals lies on this fact. If over the years residual influences have had a minimal effect on the figures, one can use ones forecasts with some degree of confidence that will not be unduly affected by external events. But if the data has been affected regularly and, to a large extent, by residual factors, then we must use our forecasts with caution.

3.2 Time Series Analysis

Taking a look at some simple time series and seeing how the trend could be obtained and if seasonal variation can be calculated, you will begin as statisticians have done by considering the trend and its nature and scope.

Firstly, take the case of a series with obvious linear trend i.e. which shows a constant tendency, in spite of fluctuations, for all figures to move in a direction — to rise or to fall, but not both.

Economic or business oriented time series are made up of four components — trend, seasonality, cycle and randomness. Further, it is usually assumed that the relationship between these four components is multiplicative as shown below:

$$X_t \qquad T_t\, S_t\, C_t\, R_t \qquad \ldots\ldots\ldots\ldots \qquad (1)$$

where Xt = the observed value of the time series

T = trend

seasonality

C, = cycle

and R = randomness

Alternatively, one could assume an additive relationship of the form X, = T, +S_L + Rt

But additive models are not commonly encountered in practice. Focus will be to work with a model of the form (1) and shall systematically try to identify the individual components.

If the time series represents a seasonal pattern of L period of L periods, then by taking a moving average of L periods, we would get the mean value for the year. Such a value will obviously be free of seasonal effects, since high months will be offset by low ones. If 1\4, denotes the moving average of equation (1), it will be free of seasonality and will contain little randomness (owing to the averaging effect) which is expressed thus:

$$Tt\ C_t \qquad (2)$$

The trend and cycle components in equation (2) can be further decomposed by assuming some form of trend. One could assume different kinds of trends such as:

- Linear trend — implies a constant rate of change (Figure I)
- Parabolic trend — implies a varying rate of change (Figure II)
- Exponential or Logarithmic trend — implies a constant percentage rate of change (Figure III)
- Curvi-linear trend (an S curve) — implies slow initial growth, with increasing rate of growth followed by a declining growth rate and eventually saturated (Figure IV).

Figure I — Linear Trend

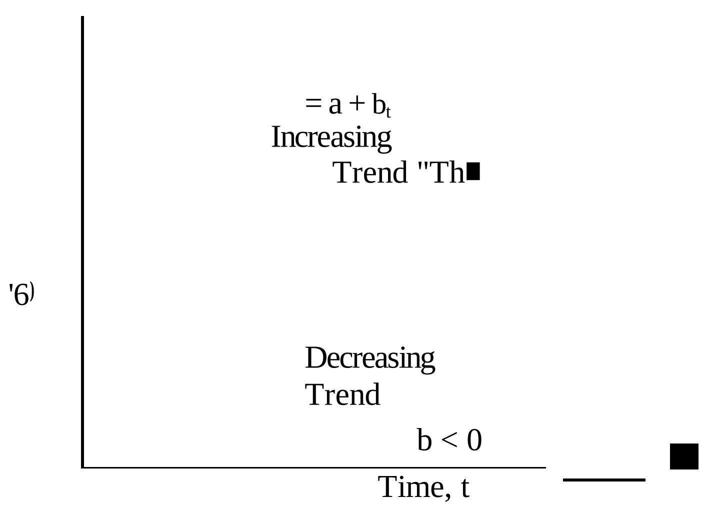


Figure II — Parabolic Trend

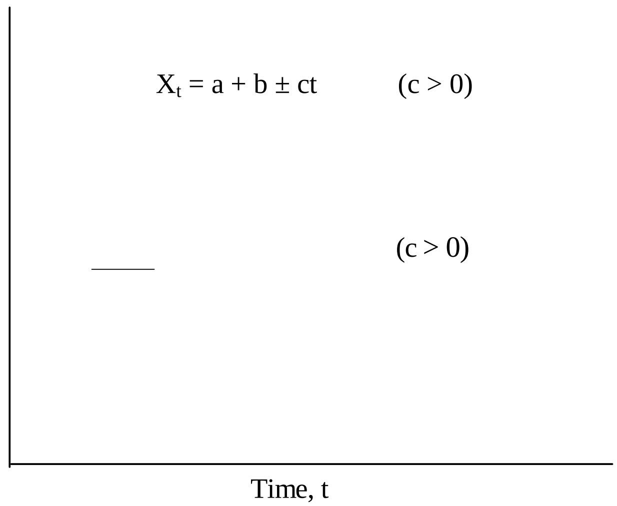
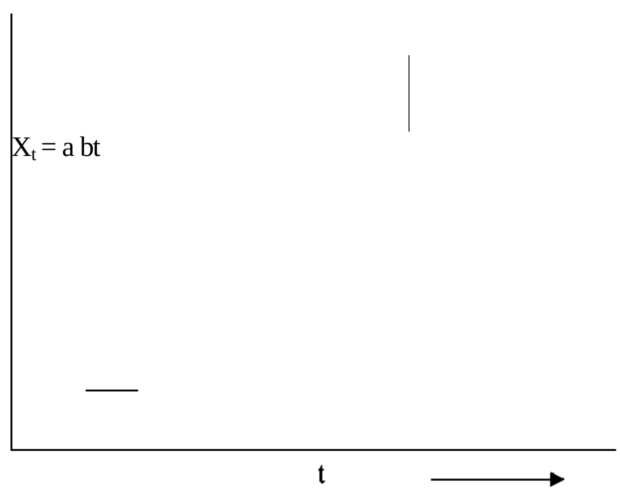
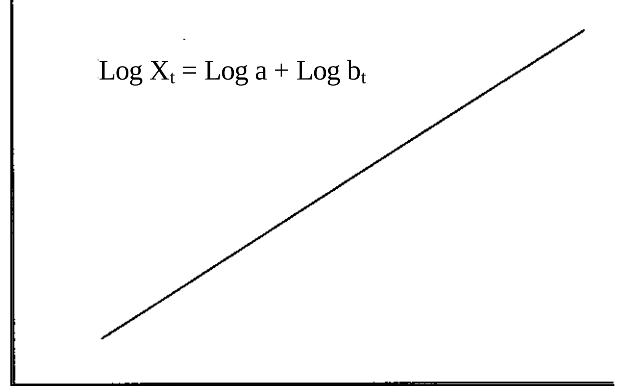


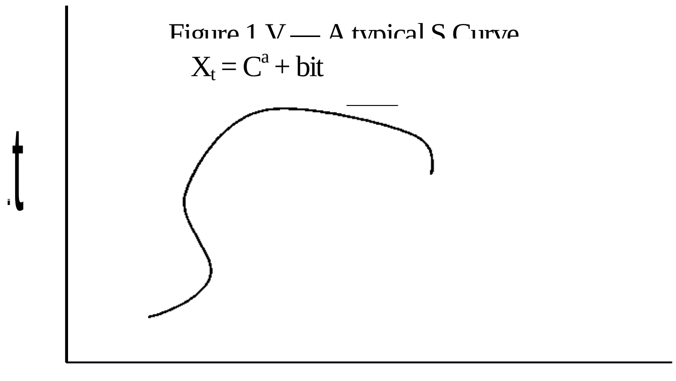
Figure III — Exponential Trend



(a)



(b)



Knowing the pattern of the trend, the appropriate mathematical function could be determined from data by using the methods of regression. This would be established by the values of parameters of the chosen trend model. Assuming a linear trend gives:

$$T_t = a + b, \qquad (3)$$

The cycle component C, can now be isolated from the trend T, in equation ... (2) by the use of equation .... (3) as follows:

$$T_t \qquad a + b, \qquad (4)$$

As already indicated, if a linear trend is not adequate, one may wish to specify a non-linear one. Any pattern for the trend can be used to separate it from the cycle. In practice, however, it is often difficult to separate the two; one may prefer to work with the trend cycle figures of equation .... (2)

The isolation of the trend will ad little to the overall ability to forecast. This will become clear when we take up an example problem for solution.

To isolate seasonality, one could simply divide the original series {equation ... (I)) by the moving average {equation ... (2)} to obtain:

$$\frac{X_t}{M_t} = \frac{T_t C_t R_t}{TC_t} = S_1 R_t \ldots\ldots\ldots\ldots\ldots (5)$$

Finally, randomness can be eliminated by averaging the different values of equation (....5). The average is done on the same months or seasons of different years (for example, the average of all Januaries, all Februaries .... all Decembers). The result is a set of seasonal values free of randomness, called seasonal indices, which are widely used in practiced.

In order to forecast, one must reconstruct each of the components of equation ... (1). The seasonality is known through averaging the values in equation ... (5) and the trend through equation .... (3). the cycle of equation ... (4) must be estimated by the users and the randomness cannot be predicted.

To illustrate the application of this procedure to a dual forecasting of a time series, below is an example to consider using decomposition (adapted by Srivastava et. al, 1978).

A firm producing farm equipment wants to predict future sales based on the analysis of its past sales pattern. The sales of the company for the last five years arc given in the table below:

Table 12.2 Ouarterlv sales of a firm during 2003 — 2007

Year	Quarter			
	I	II	III	IV
2003	5.5	5.4	7.2	6.0
2004	4.8	5.6	- 6.3	5.6
2005	4.0	6.3	7.0	6.5
2006	5.2	6.5	7.5	7.7
2007	6.0	7.0	8.4	7.7

The procedure involved in the study consists of:

(a) de-scasonalising the times series which is done by constructing a moving average M and taking the ratio Xt/M_t which we know from equation ... (5) represents the seasonality and randomness.

(b) fitting a trend line of the type T_t = a -I- N to the de-seasonalisecl time series.

(c) identifying the cyclical variation around the trend line.

(d) using the above information for forecasting sales for the next year.

3.2.1 De Scasonalising The Time Series

The moving averages and the ratios of the original variable to the moving average have first to be computed — as in table 2 below:

Table 3 Computation of Moving Averages M and the ratios X /M

Year	Quarter	Actual sales	4 Quarter Moving Total	Centre Moving Total	Centre Moving Average Total (Mil	$\frac{X_I}{M_I}$
2003	I	5.5				
	II	5.4				
	III	7.2		23.8	6.0	1.200
	IV	6.0	24.1	23.5	5.9	1.017
2004	I	4.8	23.4	23.2	5.8	0.828
	II	5.6	23.6	22.5	5.6	1.000
	III	6.3	22.7	21.9	5.5	1.145
	IV	5.6	22.3	21.9	5.5	1.018
2005	I	[illegible]	21.5	22.6	5.7	0.702
	II	[illegible]	22.2	23.4	5.9	1.068
	III	[illegible]	22.9	24.4	6.1	1.148
	IV	[illegible]	23.8	25.1	6.3	1.032
2006	I	5.2	25.0	25.5	6.4	0.813
	II	6.5	25.2	26.1	6.5	1.000
	111	7.5	25.7	26.8	6.7	1.119
	IV	7.2	26.4	27.5	6.9	1.043
2007	I	6.0	27.2	28.2	7.1	0.845
	II	7.0	27.7	28.9	7.2	0.972
	III	8.4	28.6			
	IV	7.7	29.1			

It should be noted that the 4 Quarter Moving Totals pertain to the middle of two successive periods. Thus, the value (1) computed at the end of Quarter IV, 1983 refers to middle of Quarter II, III, 1988 and the next moving total of 23.4 refers to the middle of Quarter III and IV, 1983. Thus, by taking their average, we obtain the centre moving total of

$$\frac{(23.1 \text{ -I- } 23.4)}{2} = 23.75 = 23.8$$

to be placed for Quarter III, 1983.

Similarly, for the other values in case the number of periods in the moving totals or average is odd, centering will not be required.

The seasonal indices for the quarterly sales data can now be computed by taking averages of the X_t/M_t ratios of the respective quarters for different years as shown in Table 3 below:

Table 4 Computation of Seasonal Indices

Year	Quarters			
	I	H	III	IV
2003	-	-	1.200	1.017
2004	0.828	1.000	1.145	1.018
2005	0.702	1.068	1.148	1.032
2006	0.813	1.000	1.119	1.043
2007	0.845	0.972		-
Mean	0.797	1.010	1.153	1.028
Seasonal Index	0.799	1.013	1.156	1.032

The seasonal indices are computed from the quarter means by adjusting these values of means so that the average over the year is unity. Thus, the sum of means in Table 3 above is 3.988 and since there are four quarters, each mean is adjusted by multiplying it with the constant figure of 4/3.988 to obtain the indicated seasonal indices. These seasonal indices can now be used to obtain the de-seasonalised sales of the firm by dividing the actual sales by the corresponding index as shown in Table 4 below:

Table 5 De-Scasonalised Sales

Year	Quarter	Actual sales (Y)	Seasonal Index	Deceseasonalised sales
2003	1	5.5	0.799	6.9
	II	5.4	1.013	5.3
	III	7.2	1.156	6.2
	IV	6.0	1.032	5.8
2004	I	4.8	0.799	6.0
	11	5.6	1.013	5.5
	III	6.3	1.156	5.4
	IV	5.6	1.032	5.4
2005	I	4.0	0.799	5.0
	II	6.3	1.013	6.2
	III	7.0	1.156	6.0
	IV	6.5	1.032	6.3
2006	1	5.2	0.799	6.5
	11	6.5	1.013	6.4
	III	7.5	1.156	6.5
	IV	7.2	1.032	7.0
2007	I	6.0	0.799	7.5
	11	7.0	1.013	6.9
	III	8.4	1.156	7.3
	IV	7.7	1.032	7.5

3.2.2 Fitting a Trend Line

The next step after de-seasonalising the data is to develop the trend line. We shall here use the method of least squares (trend line). Choice of the origin in the middle of the data with a suitable scaling simplifies computations considerably.

To fit a straight line of the form Y = a + bX to the de-easonalised sales, we proceed as shown in Table 5 below. Example:

Table 6 Computation of Trend

Year	Quarter	Deseasonalised sales (Y)	X	X^2	XY
2003	I	6.9	-19	361	-131.1
	II	5.3	-17	289	-90.1
	III	6.9	-15	225	-93.0
	IV	5.8	-13	169	-75.4
2004	I	6.0	[illegible]	121	-66.0
	II	5.5	[illegible]	81	-49.5
	III	5.4	[illegible]	49	-37.8
	IV	5.4	[illegible]	25	-27.0
2005	I	5.0	[illegible]	9	-15.0
	II	6.2	[illegible]	1	-6.2
	III	6.0	[illegible]	1	6.0
	IV	6.3	[illegible]	9	18.9
2006	I	6.5	[illegible]	25	32.5
	II	6.4	[illegible]	49	44.8
	III	6.5	[illegible]	81	58.5
	IV	7.0	[illegible]	121	77.0
2007	I	7.5	13	169	97.5
	II	6.9	15	225	103.5
	111	7.3	17	289	124.1
	IV	7.5	19	361	142.5
TOTAL		125.6	0	2660	114.2

the trend line is a = SY = 125.6 6.3
20

• **114.2 = 0.04X**

EX2 **2660**

• **6.3 + 0.04Xb 3.2.3**

Identifying Cyclical Variation

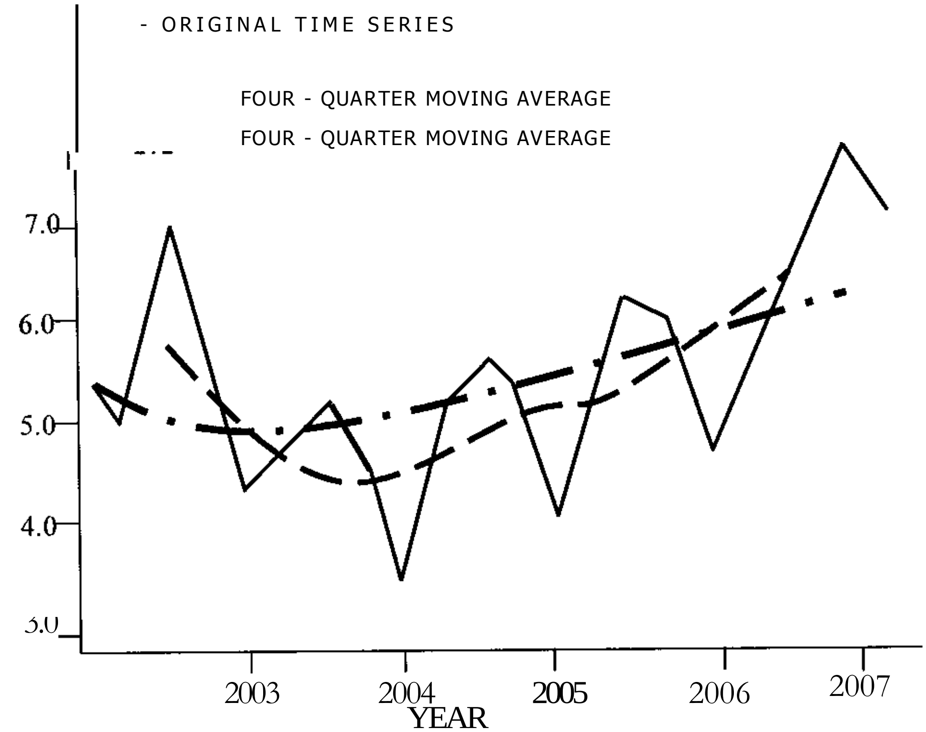
The cyclical component is identified by measuring de-seasonalised variation around the trend line, as the ratio of the actual de-seasonalised sales to the value predicted by the trend line. The computations are shown in Table 6 below:

Table 7 Computation of Cyclical Variation

Year	Quarter	De-scasonalised Sales (Y)	Trend a + bX	Y / a + bX
2003	I	6.9	5.54	1.245
	II	5.3	5.62	0.943
	III	6.2	5.70	1.088
	IV	5.8	5.78	1.003
2004	**I**	6.0	5.86	1.024
	11	5.5	5.94	0.926
	111	5.4	6.02	0.897
	IV	5.4	6.10	0.885
2005	I	5.0	6.18	0.809
	II	6.2	6.26	0.990
	III	6.0	6.34	0.946
	IV	6.3	6.42	0.981
2006	I	6.5	6.50	1.000
	II	6.4	6.58	0.973
	III	6.5	6.66	0.976
	IV	7.0	6.74	1.039
2007	I	7.5	6.82	1.110
	II	6.9	6.90	1.000
	III	7.3	6.98	1.046
	IV	7.5	7.06	1.062

The random or ·rregular variation is assumed to be relatively insignificant. Attempts have been made to describe the time series in this problem using the trend, cyclical and seasonal components. Figure V represents the original time series, its four quarter moving average (containing the trend and cycle components) and the trend line.

Figure 5 — Time Series with Trend and Moving Averages



3.3 Forecasting with the Decomposed Components of the Time Series

Suppose that the management of the firm is interested in estimating the sales for the second and third quarters of 2008. The estimates of the deseasonalised sales can be obtained by using the trend line:

$$6.3 + 0.04\,(23)$$

$$7.22 \text{ (2nd Quarter 2008)}$$

$$\text{and } Y = 6.3 + 0.04\,(25)$$

$$7.30 \text{ (3rd Quarter 2008)}$$

These estimates will now have to be seasonalised for the second and third quarters respectively. This can be done as follows:

For 2008 2nd quarter scasonalised sales estimate = 7.22 x 1.013
= 7.31

For 2008 3rd quarter seasonalised sales estimate = 7.30 x 1.56
= 8.44

Thus, on the basis of the above analysis, the sales estimates of the firm for the second and third quarters of 1988 are N7.31k and N8.44k respectively.

These estimates have been obtained by taking the trend and seasonal variations into account. Cyclical and irregular components have not been taken into account. The procedure for cyclical variations only helps to study past behaviour and does not help in predicting the future behaviour.

Moreover, random or irregular variations are difficult to quantify.

SELF ASSESSMENT EXERCISES

(i) What do you understand by time series analysis? How would you go about conducting such an analysis for forecasting the sales of a product in your firm?

(ii) Find the 4 — quarter moving average of the following time series representing the quarterly production of Tea in Adamawa State.

	Quarters			
Year	I	II	III	IV
2003	5	1	10	17
2004	7	1	10	16
2005	9	3	8	18
2006	5	2	15	19
2007	8	4	14	20

4.0 CONCLUSION

In this unit, you have been shown the role of time series in business forecasting and how to apply it in the different scope in decision making.

5.0 SUMMARY

Some procedures for time series analysis have been described in this unit with a view to making more accurate and reliable forecasts of the future.

In this unit, time series models of historical data on demand and variable interest were discussed, expressing that one is dealing with projecting into the future from the past. It is mainly short-term forecasting models.

The decomposition method has been discussed — with time series broken up into seasonal, trend, cycle and random components from the given data and reconstructed for forecasting purposes — with detailed example to illustrate the procedure.

6.0 TUTOR MARKED ASSIGNMENT

Choice of the origin in the data with suitable scaling simplifies computations considerably. Find the least squares line of best fit from the following variables:

X	1	2	3	4	5	6	7
	1	2	6	7	10	16	21

7.0 REFERENCES AND FURTHER READINGS

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Owen, F. and Jones (1975). *Modern Analytical Techniques.* Polytech Publishers Limited, Stockport.

UNIT 3 CORRELATION

CONTENTS

1.0 INTRODUCTION

We could quote cases where it has been useful to establish a relationship between two sets of variables. In fact, the establishment of such relationship has, in some cases, changed our whole way of life. A good example of this is the relationship between the numbers of cigarettes smoked and the incidence of lung cancer. Another example is the relationship between the probabilities of infection and the degree of exposure to a certain disease. However, it can be just as useful to establish that a relationship does not exist — the statistician has been the scourge of the purveyors of quack medicine! A further use of establishing relationships is to make predictions about a variable that is difficult or costly to measure directly. For example, if we can establish a relationship between the destructive power of a bomb and the amount of explosive it contains, we can now examine the form that such relationships take.

2.0 OBJECTIVES

After completion of this unit, you should be able to:

- explain the meaning of correlation and its role in decision making;
- compute the correlation coefficient between two variables from sample observations;
- test for the significance of the correlation coefficient in business decision;

3.0 MAIN CONTENT

3.1 The Correlation Co-Efficient (R)

Let us suppose that we take two variables, x and y. we suspect that there is a relationship between the variables and we plot the results on a graph. The two variables which we suspect are related form a *bi-variate distribution,* and the graph drawn is called a *scatter diagram.* The scatter diagram might look like this

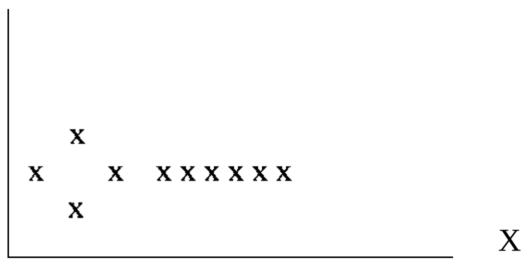


Figure 13.1

We notice that large values of y are associated with large values of x, and small values of y are associated with small values of x. we would say that in this case the variables are positively correlated. In other words, positive correlation exists when both variables increase together. Were we to plot the number of motor vehicles registered against the number of road accidents we would expect there to be a positive correlation. Can you think of any other examples yourself'?

Now it is possible that the scatter diagram might look like this:

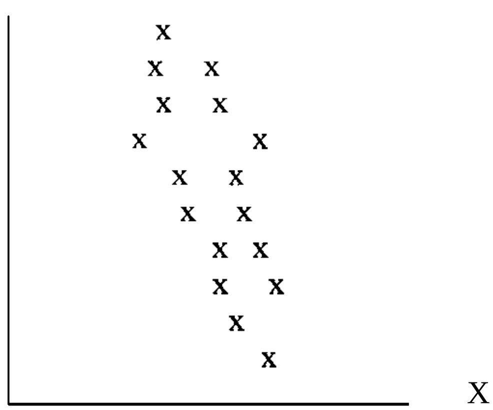


Figure 13.2

In this case small values of y are associated with large values of x and large values of y arc associated with small values of x. here, the variables are negatively correlated, which exists when an increase in one variable is associated with a decrease in the other. An example of negative correlation is the association between the number of T.V. licenses issued and cinema admissions. You should try to compile a list of such examples.

A third case is where the scatter diagram looks like this:

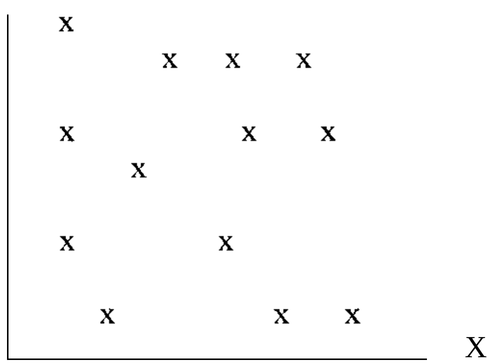


Figure 13.3

These points are scattered at random and here we can say that there is an absence of correlation between the variables. This of course will be the most common of the three cases if we randomly select bi-variate distributions.

Consider now the three diagrams below:

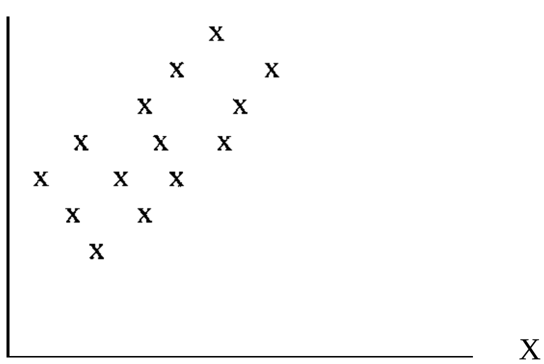


Figure 13.4(a)

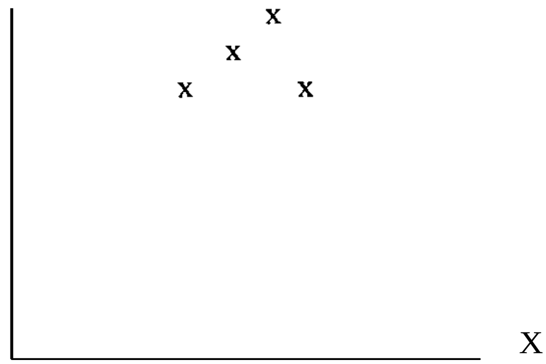


Figure 13.4(b)

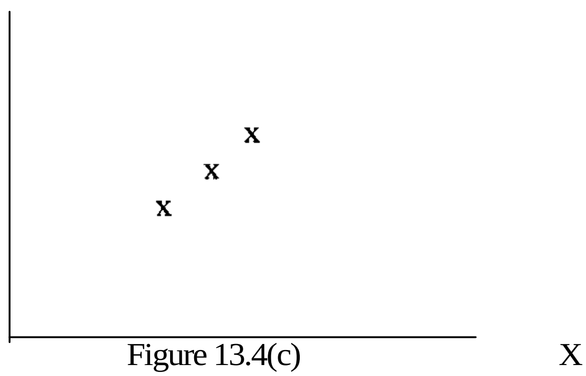


Figure 13.4(c)

In each case we see that the variables are positively correlated. The diagrams clearly show that association is strongest in case (c) and weakest in case (a). in case (c) all the points are on a straight line and for any value of one variable we could predict precisely the corresponding value of the other variable. Case (c) shows *perfect positive correlation.* Now in practice we will not meet perfect positive correlation, but we will meet examples that approach it quite closely. With some justification we can regard such examples as showing perfect positive correlation and any deviations of points from a straight line as resulting from experimental error.

Clearly, we need some measure of correlation, and our measure must exist between positive and negative correlation. How can we derive such a measure? Well, you are probably thinking back to the chapter on the Time Series and releasing that some connection must exist between correlation and the 'least squares' method used in that chapter. We saw that if a number of points were plotted on a graph the least squares line of best fit could be given as y = $mx + c$, where,

$$Exy \quad \underline{aEy}$$

$$Ex2 — \underline{(D)^2}$$

and

$$1\ (Ey — Ma)$$

If this is not clear in your mind you should read again the appropriate section.

Examine carefully how we obtained this line. We selected a point, say F,, with the coordinates $(X_n Y_n)$ and said that the value of y on this line when $X = X_n$ *was* $X_n M + C$; and the deviation of the point (X_n l',,) from the point on the line was

$$Y_n — (X_n M + C) = Y_n — X_n M\ C$$

Now what are the implications of this? We are calculating the value of y on the line for a given value of *x* — which implies that we are using the values of *x* to predict the values of y. we are supposing that all deviations from the line occurs as a result of errors in the value *y,* and so we have been measuring the *vertical* deviations of the points from the line. Now must we find the regression line this way? We could have considered instead the horizontal deviations of the points from the line.

To do this we would have to consider the line

$$X = M_l y + C$$

And for any value y,, the value of x on the line would be

$$Y_n M_l + C$$

Hence the deviation of the point $(x_n y_n)$ from this line is

$$x_n — (y_n m_l + c = x_n — y_n m_l — ci$$

You can see this in the diagram below:

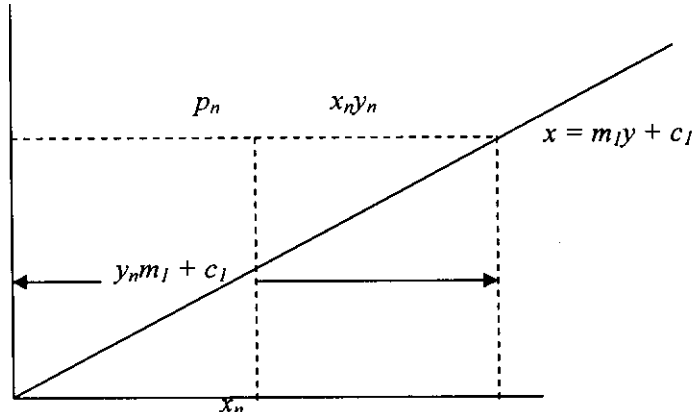


Figure 13.5

(You were asked to consider this idea in the tutorial following the analysis of least squares.)

We can use the same method to find m_l and c_l that we used to find *m* and *c.* this would give:

Exy — ExEy

Ex2 — (£,4²

and

C_i= 1 (Ex — M_lEy)

To find the regression line y = *mx* **+c,** we took the value of x as given and calculated the deviations from y this line is called the regression of *y* on x, and is used to estimated a value of y when we are given a value of x. to the regression line of x = $m_l y + c_l$, we took the value of y as given and calculated the deviations from x. this is the regression line of x on y and is used to estimate a value of x given a value of y. in the first case we assume that all deviations from a linear relationship are caused by differences between the observed values of y, and the values of y calculated from the regression equation

Example 1

When we considered the least squares method we found that for the bi-variate distribution

1	7	3	4	5	6	7
1	2	6	7	10	16	21

the least squares line of best fit was

y = 3.29x — 4.16

This is the regression line of y on x. let us now find the regression line of x on y.

X		y^2	xy
1	1	1	1
2	9	4	4
3	6	36	18
4	7	49	28
5	10	100	50
6	16	256	96
7	21	441	147
78	63	887	344

a =28 Ey = 63 Ey^2 = 887 Exy = 344

E,xy — ExEy

Ex2 — $(Ex)^2$

$2y^2$ — $(E1)^2$

344 — 28 x 7

7

887— $(63)^2$

7

0.288

= 1 (Ex — *M₁Ey)*

C 1 (28 — *0.28 x 63)*

7

1.43

The equation is x = 0.288y + 1.43

3.2 The Significant of the Correlation Coefficient

Now let us see if we can obtain a measure of correlation, and all this the correlation coefficient *(r)*. We will define *r* as the product of the coefficients *m* and m_i in the regression equations, i.e.

MMI

We shall consider first the case of perfect, positive correlation: the points would all lie on a straight line with a positive slope. The regressive lines would coincide and the two equations y = *mx* + *c* and x = $m_1y + c$, would be identical. Suppose we re-arrange the second equation:

mly ci

So, y = x - c_i

m_i

The gradient of this line is 1. As the regression lines coincide, the gradient of this

m

is also *m*. Hence,

1

and the product of the coefficients will be

$1 \times mi = 1$

m_l

Thus, with perfect, positive correlation, *r* = 1: a sensible value. Now let us consider the case where no correlation exists. The regression lines will be parallel to the axes and look like the diagram on the facing page.

You may be wondering why the regression lines are parallel to the axes. If no correlation exists, then y does not depend upon x, and the regression equation is y = *c.* you will remember from the introductory section that the line y = *c* is parallel to the *x* axis. Likewise, the regression

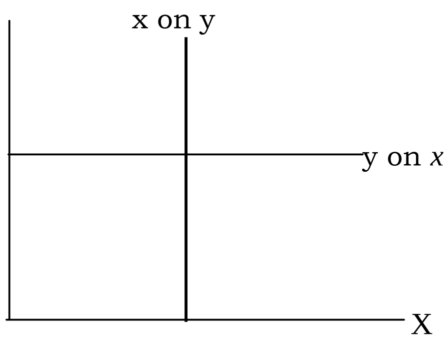


Figure 13.6

Equation x on y must be x = c_l, which is parallel to the y axis, in this case $m = 0$ and $m_l = 0$, so the correlation coefficient is 0 x 0 = 0

When correlation is absent the value of the correlation coefficient is zero, again a sensible value.

Finally let us consider perfect negative correlation. Again the regressive lines will coincide, but this time they will both have a negative slope. The regression lines will have the equations

$$mx + c \text{ and} x = \qquad + c_i$$

Earlier we found that m =' so in this case — m = -1 and the product
m_i m_i

of the coefficients will be — m_l x — 1 = 1. Hence if perfect negative correlation exists the m$_i$

value of our correlation coefficient will be one — the same as for perfect positive correlation. Now this will just not do! Our correlation coefficient must be able to distinguish between positive and negative correlation, but our coefficient $r = mm$, will not do this! What shall we do? Let us try making r the squares root of the product of the coefficient s, i.e.

$7'^2$

Now when correlation is absent is $r^2 = 0$ and $r = 0$ which is quite satisfactory. When correlation is perfect $r^2 = 1$ and $r = \pm 1$. This is much better.

The positive root can signify perfect positive correlation and the negative root perfect negative correlation. But how will we know which root to take? Well, if correlation is positive, the gradient of the regression lines will be positive. Likewise if correlation is negative both m and m, will be negative. So if m and *m,* are positive take the positive root of their product and if m and m_1 are negative take the negative root. measure of correlation, then

Let us summaries what we have learned so far. If we take ? = mm, as a

—1<r^{2}Cl

If 0 correlation is absent
1 correlation is perfect and positive
—1 correlation is perfect and positive

so it follows that if 0 < *r* Cl, correlation is positive but not perfect and if — I < r^2 <1, correlation is negative but not perfect The closer *r* between the two variables.

approaches to its limits (-1 and +*1)* the stronger is the association

Now let us consider the correlation coefficient for the example we

3.29 x 0.288 and *r r* +0.97

considered earlier. We found that m = 3.29 and that ml = 0.288 so r^2= A high degree of positive correlation exists. This leads us to expect that this.

both regression lines would almost coincide. The diagram clearly shows

Now we have defined r^2 as *mmi,* and we have seen that it is a

satisfactory measure of correlation. It must follow that:

7'2 Sty - E_g"

E$_x$2 E I D!

E y 2

Now suppose that we divide the top and bottom of each fraction by *n* the number of pairs in the bi-variate distribution. Then

I T.ay -ixLy

1 lxy

$\frac{1}{}$ $\Sigma xy - \underline{\Sigma x \Sigma y}$

1 a y

You should recognize that the denomination of the first fraction is the variation of x and that the denomination of the second fraction is the variance of y. so we can write

L FY-xY

n n [L

var (x)

var (5')

LSfY xLL ExY - D-4:Y9

var (x) var (5')

We call the numerator of this last fraction the Covariance so now we can

say

R = COI__Igg-(1Ce?-VIC

Gx csr

Writing the correlation coefficient in this way we have overcome the problem of the sign of the coefficient: it is the same as the sign of the covariance.

Example 2

Index of Earnings and prices *(1996 – 100)*

	1999		2000				2001			
Quarter	3	4	1	2	3	4	1	2	3	4
Earnings	121	125	130	135	138	145	150	154	158	162
Prices	113	115	118	120	121	124	127	131	133	135

We wish to find the correlation coefficient between earnings and prices. It will simplify the arithmetic if we take deviations from 140 on the earnings index and deviations from 125 on the prices index.

x	Y	x^2	Y^2	Xy
-19	-12	361	144	228
-15	-10	225	100	150
-10	-7	100	49	70
-5	-5	25	25	25
-2	-4	4	16	8
5	-1	25	1	-5
10	2	100	4	20
14	6	196	36	84
18	8	324	64	144
22	10	484	100	220
18	**-13**	**1844**	**539**	**944**

Table 1

Covariance (xy) = 1 [Exy — Laid

= 1 [944-18x-13]
10 10

= 1 [944 + 23.4] =96.74
10

10 844 — 18^2]
10

=J 1 [1844 — 32.4] = '181.16 13.46

10

σ_x = 1[— (Ev)^2] = \I 1 [539 — (-13)^2]
10 10

= \I 1 [539 — 16. = \I 52.21

7 23

10

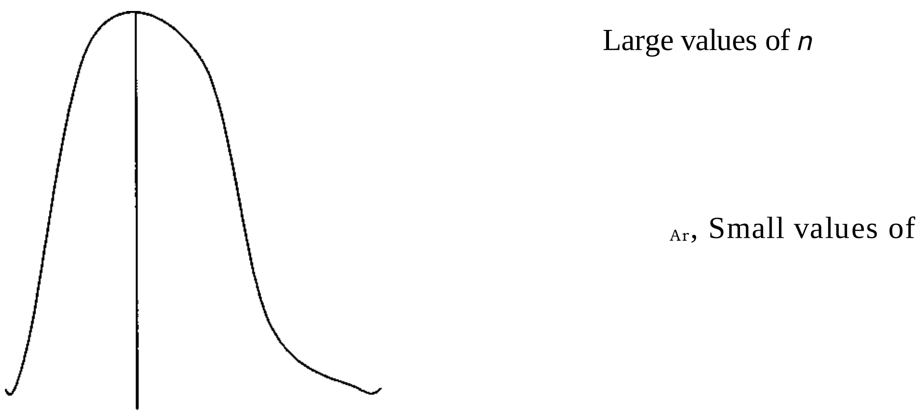
r = Covariance (xv) = 96.74 = 0.994

σx ay = 13.46 X 7.23

3.3 The Significance of the Correlation

Calculating the correlation coefficient has little meaning unless we can test its significance. Suppose we have a bi-variate population with a zero correlation coefficient, and we draw samples of *n* pairs from this population. Of course, it would be unreasonable to expect all our samples to have a zero correlation coefficient: some will be greater than zero and some less, but we would expect the average value of correlation coefficients to be zero. This give us some idea as to how we can assess the significance of *r.* suppose a sample of *n* pairs gives a correlation coefficient of *r,* and suppose we adopt the Null Hypothesis that the sample is drawn from an uncorrelated population then if we can find the standard error we should be able to calculate the probability of obtaining our observed value of *r.*

Now we have already stated that correlation coefficients of samples all drawn from the same population are liable to sampling fluctuations. If the sample size is fairly large, then the distribution of *r* will be Normal. However, unrepresentative pairs will have a significant effect on the correlation coefficient of the sample, and the range of the correlation coefficient will be greater (the effect of an unrepresentative pair will be lost in a large sample). The distribution of the correlation coefficient for small values of n will not be Normal. This is illustrated in the diagram below



The distribution of the correlation coefficient
Figure 7

When n is small, the distribution of the correlation coefficient forms the so-called 'T' distribution. The deviation and use of this distribution is beyond the scope of this book, and unfortunately most calculations of the correlation coefficient are from small samples. However, if you turn to the appendix, you will find that values of r at the 5% and I% levels of significance are tabulated against differing sizes of n. consulting this table for n = 10 (the number of pairs in the last example). We can state that if no correlation exists there is a 5% chance of r exceeding 0.63 and a 1% chance of r exceeding 0.76. Now the value of r we obtained in the last example was 0.994. What can we conclude? The probability that correlation is absent is certainly very much less than 1% and we must state that positive correlation seems highly likely.

This section on the correlation coefficient will end with a word of warning. When dealing with correlation coefficient one must never forget common sense. A significant correlation coefficient does not necessarily imply cause and effect. Many highly significant correlations are quite non-sensical. Statisticians in the past pointed to the highly significant correlation between the annual issues of broadcasting licenses and the annual rate of admissions into mental institutions. The only logical conclusion you can draw from this is that quite by chance both statistics were increasing at the same rate. Even when a significant correlation seems reasonable you should not assume that cause and effect is established, as a third variable may be influencing both the others and thus explain the correlation. The high degree of positive correlation between overcrowding causes a high infant mortality rate; through in fact both are indicative of income levels.

3.4 Rank Correlation

Example 3

The correlation coefficient r is rather awkward to calculate and you may prefer to use instead the Rank Correlation coefficient. This is not as accurate as the correlation coefficient r but does still give a fairly reliable indication as to the degree of correlation between two variables. Let us examine the following vicariate distribution.

2001 Months	Production of Private Cars (thousands)	Value of Exports (N million)
February	134	627
March	142	771
April	138	787
May	156	755
June	176	799
August	119	782
September	159	834
October	151	810
November	145	759
December	180	832

Table 2

If we wish to calculate the rank correlation coefficient we must rank the values in each column. Let us consider the production of private cars over the ten months. The highest output was in December so we give this month a rank of one. The second greater output was in June and we give this month a rank of two. We continue in this way ranking each month according to its output of private cars. When we have done this we rank each month according to the value of exports. Our bi-viate distribution would now look like this:

Months	**Output (x)**	**Rank of Output**	**Exports** (y)	**Rank of Export** (y)
February	134	9	627	10
March	142	7	771	7
April	138	8	787	
May	156	4	755	
June	176	2	799	4
Au:1.1st	119	10	782	6
September	159	3	834	1
October	151	5	810	3
November	145	6	759	8
December	180	1	832	2

Table 3

We can now calculate the correlation coefficient using the ranks instead of the actual values. This will considerably simplify the arithmetic. Suppose we write p for the rank correlation coefficient. Then

$$\frac{\text{Co-variance } (XY)}{(Tx\ cry}$$

before we actually calculate p for this bi-variate distribution let us consider some. If preliminaries. The ranks (X) are identical to the ranks (Y) although they will not necessarily appear in the same order. It follows from this that Ex = Ey and that a, = a_y. If you cannot see that this must be so verify these facts for yourself using the ranks in the table above. Now we can write

$$\frac{\text{Co-variance } (XY)}{a^2\,(x)}$$

so we need calculate only one variance that two standard deviations. Let us now consider the bi-variate distribution with n pairs. If we calculate p for this distribution we may well obtain an expression for p which is easier to manage than the one above. The ranks (A') in this distribution will be

1, 2, 3 ...., n

though not necessarily in that order. We will need to calculate the variance of the ranks *(X)*.

Variance *(X)* = [Ex2 (a)2}

Now E x = 1 + 2 + 3 + ...+*n* i.e. the sum of the first n integers. When you worked the tutorial following geometric progression (it was in the chapter on Discounted Cash Flow) you found the sum of this series and you were asked to keep the result. Find this result and check that it is $\underline{n^2 + n}$. We can now write EX = $\frac{n^2 + n}{2}$ in the same

$$Ex^2 = 12 + 22 + 32 + \ldots + n^2$$

Now it can be shown that 12+22 + 32 + ...+ $n^2 = \frac{n(n+1)(2n+1)}{6}$

If you wish to verify this; an exercise in the next tutorial will help you to do so.

We can now say

$$\text{Variance (X)} = \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6} - \left(\frac{n^2+n}{2}\right)^2 \text{X } \frac{1}{n}\right]$$

$$- \quad 1\left[\frac{n(n+1)\,2n+n)}{6} - \frac{n^4 + 2n^3 + n^2}{4n}\right]$$

$$- \quad 1\left[\frac{n(n+1)(2n+1)}{6} - \frac{n^3 + 2n^2 + n}{4}\right]$$

$$- \quad \frac{1(n+1)(2n+1)}{6} - \frac{n^2 + 2n + 11}{4}$$

$$\bullet \quad \frac{1(n+1)(2+1)}{6} - \frac{(n+1)^2]}{4}$$

Well, this does not seem to be much of a simplification, but if we now consider the covariance the simplification will eventually occur.

Covariance *(XY)* . 1 [EXY — $\underline{\text{EXEY}}$]

$$- \quad 1\,[\text{EXY} - \underline{(\text{EX})2}\,]$$

Now we have already found that the last term in the expression can be written as

$$\frac{(n+1)2.}{4}$$

We shall now see if we can find another way of writing EXY. To do this we shall use the fact that

0( )02 , 2xy +y2

So $$E(X — Y)^2 = EX^2 - 2EXY + EY^2$$

The quantity $(X — Y)^2$ could be found by taking the difference between the ranks, squaring them and summing the squares. If we write $D^2$for this quantity

$$ED^2 = X^2 — 2XY + Y2$$

So $$\frac{EXY = EX^2 + EY^2 — ED^2}{2}$$

As $EX^2 = \quad EY^2$,

$$EXY = \frac{2EX^2 — DE\ 2\ Ex2}{2} \frac{ED2}{2}$$

We have already found an expression for EX^2, so

$$EXY = \frac{n(n+1)(2n+1)}{6} — \frac{ED^2}{2}$$

We can now write the Covariance like this:

Covariance (XY) $$\frac{n(n+1)(2n+1)}{6} — \frac{ED^2}{2n} — \frac{(n+1)^2}{4}$$

$$\frac{(n+1)(2n+1)}{6} \frac{ED^2}{2n} — \frac{(n+1)^2}{4}$$

and the rank correlation coefficient like this:

$$\frac{\frac{(n\ 1)\ (2n+1)}{6} - \frac{ED^2}{2n} — \frac{(n+1)^2}{4}}{\frac{(n+1)\ (2n+1)}{6}\ \frac{(n+1)^2}{4}}$$

$$= 1 — \frac{\frac{ED^2}{2n}}{\text{Or } \frac{4\text{-}1)\ (2n\ 1)}{6}\ \_\ \frac{(n+1)^2}{4}}$$

$$= 1 — \frac{6ED^2}{2n\ (n\ \text{-1-}\ 1)\ (2n\ \text{-1-}\ I)\ \_\ 3n(n+1)^2}$$

$$= 1 — \frac{6ED^2}{n(n^2 — 1)}$$

So the attempt at simplification has proved to be worthwhile as the rank correlation coefficient will now be simple to calculate.

X	*Y*	*D*	D^2
9	10	-1	1
7	7	0	0
8	5	3	9
4	9	-5	25
,	4	-2	4

Table 13.4a

X	*Y*	*D*	D^2
10	6	4	16
3	1	2	4
5	3	2	4
6	8	-2	4
1	2	-1	1
		0	68

Table 13.4b

Notice that ED = 0. This must be so (why?) and is a useful check on the calculations performed.

$$1 — \frac{6 \text{ X } 68}{1000^2\ \text{-}} \quad = 0.588$$

4.0 CONCLUSION

In this unit, you have been taken through the concept of correlation coefficient, its computation between two variables from sample observations and how to apply correlation in decision-making.

5.0 SUMMARY

In this unit, you have been taken through to-

- understand the meaning of correlation;
- compute the correlation coefficient between two variables from sample observations;
- test for the significance of the correlation coefficient in business decision;
- compute the rank correlation coefficient when rankings rather than actual values for variables are known;

6.0 TUTOR MARKED ASSIGNMENT

1) What do you understand by the term correlation?

2) Explain how the study of correlation helps in forecasting demand of a product.

7.0 REFERENCES AND FURTHER READINGS

Owen, F. and Jones (1975). *Modern Analytical Techniques.* Polytech Publishers Limited, Stockport.

Pandey, I.M. (2005). *Financial Management,* 9th Edition Vikas Publishing House PVT LTD. New Delhi, India

UNIT 4 REGRESSION ANALYSIS

CONTENTS

1.0 INTRODUCTION

In industry and business today, large amounts of data are continuously being generated. This data may be pertaining, for instance, to a company's annual sales, capacity utilisation, turnover, profits, manpower levels, absenteeism or some other variable of direct interest to management.

Or there might be a technical data regarding a process such as temperature or pressure at certain crucial points, concentration of a certain chemical in the product or the breaking strength of the sample produced or one of a large quality attributes.

The accumulated data may be used to get information about the system (as for instance, what happens to the output of the plant when temperature is reduced by half) or to visually depict the past pattern of behaviour (as often happens in company's annual meetings where records of company progress are projected) as simply used for control purposes to check if the process or system is operating as designed (as for instance of quality control).

The interest in regression is primarily for the first purpose, mainly to extract the main features of the relationships hidden in or implied by the mass of data.

2.0 OBJECTIVES

After completion of this unit, you should be able to:

- explain the role of regression in establishing mathematical relationships between dependent and independent variables from given data
- use the least square criterion to estimate the model parameters;
- determine the stand or the errors of estimate of the forecast and estimated parameters
- make meaninLi ul forecasts from given data by fitting any function, linear in unknown parameters.

3.0 MAIN CONTENT

3.1 The Need for Statistical Analysis

For the system understudy there may be variables and it is of interest to examine the effects that some variables exert (or appear to exert) on others. The exact functional relationship between variables may be so complex but we may wish to approximate to this functional relationship by some simple mathematical function such as straight line or a polynomial which approximates to the true function over certain limited ranges of the variables involved.

Broadly speaking, one would have to undergo the following sequence of steps in determining the relationship between variables, assuming we have data points already:

(1) Identify the independent and response variables;
(2) Make a guess form of the relation (linear, quadratic, cycle etc.) between the dependent and independent variables (graphically plot systematic tabulation to suggest trends or patterns);
(3) Estimate the parameters of the tentative entertained model in (2) above. E.g. if a straight line was to be fitted — what is the slope and intercept of this line?
(4) Having obtained the mathematical model, conduct an error analysis to see how good the model fits into the actual data;
(5) Stop, if satisfied with model otherwise repeat (2) to (4) for another choice of the model form in (2).

What is Regression?

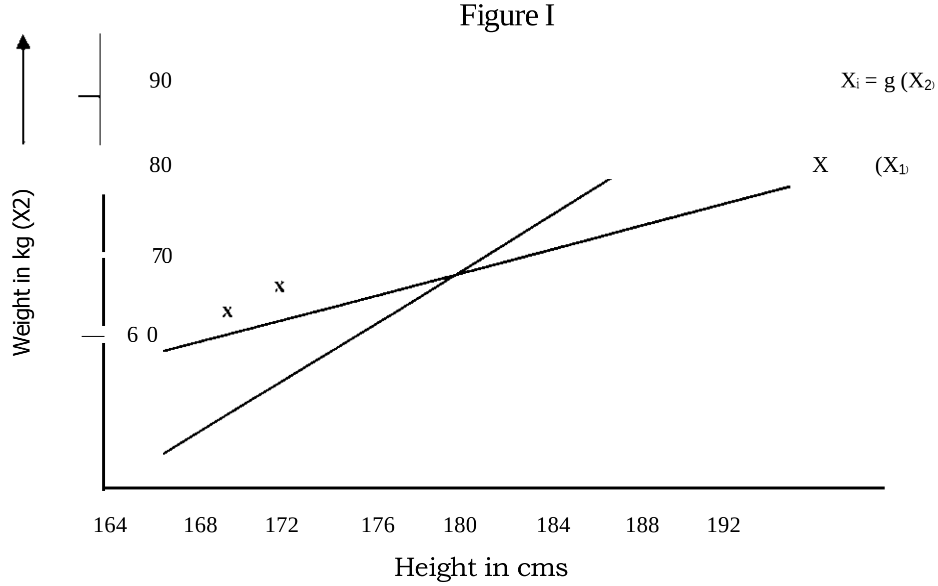
Suppose one considers the height and weight of adult males for some given population. If one plots the pair (X_1, X_2) = (height, weight). diagram like figure 1 will result. This is conventional scattered diagram.

Note that for any given height, there is a range of observed weights and vice versa. The variations will be partially due to variation errors but primarily due to variations between individuals. Thus, no unique relationship between height and weight can be expected. But one can note that average observed weight for a given observed height increases as height increases.

The locus of average of observed weight for given observed height (as height varies) is called the Regression Curve of weight on height. Let us denote it by $X_2 = f(x_i)$. There also exists a regression curve of height on weight similarly defined which we can denote by $X_1 = g(X_2)$.

Let us assume that these two "curves" are both straight lines (which in general may not be). In general, these two curves are not the same as indicated by the two lines in figure 14.1 below:

Figure I



A pair of random variables such as (height, weight) follows some sort of bi-variate probability distribution. When we are concerned with the dependence of a random variable but not a random variable, an equation that relates Y to X is usually called a regression equation. Similarly, when more than one independent variable is involved, you may examine the way in which a response Y depends on variables X_I, $X_2 \dots X_n$. The determination of a regression equation from data which cover certain areas of the X — space as $Y = f(X_i, X_2 \dots X_n)$.

3.1.1 Linear Regression

The simplest and most commonly used relationship between two variables is that of a straight line — written in first order model as Y = pa + p, x + c .... (1). (That is, for a given X, a corresponding observation Y consist of the value of 13_0 + p, X plus an amount €, the increment by which an individual Y may fall off the regression line. Equation (1) is the model of what is believed. pa, 130are called the parameters of the model whose values have been obtained from the actual data. When a model is linear or non-linear, it is referring to linearity or non-linearity in the parameters. The value of the highest power of independent variable in the model is called the order of the model.

For example:

$$= 13o \pm \quad x + 132\,X^2\,T\,€$$

is a second order (in X) linear (in the P_s) regression model. Now in the model of equation (1) $130,$ 13, and € are unknown and in fact would be difficult to discover from observation. However, pa, p, remain fixed, and although we cannot find them exactly without examining all possible occurrences of Y and X, we can use the information provided by the actual data to give us estimate b_0 and 13_1 of 13_0 and P_I.

Thus, we can write:

$$7(\qquad b_0 + b_i X \qquad (2)$$

where Sr (Y hat) denotes the predicted value of Y for a given X, when b_0 and b_1 are determined. Equation (2) could then be used as a predictive equation; substitution of a value of X would provide a prediction of the true mean value of Y for that of X.

3.2 Fitting a Straight Line

Least Squares Criterion

In fitting a straight line (or any other function) to a set of data points, we would expect some points to fall above or below the line resulting in both positive and negative error terms as shown in figure 14.2 below:

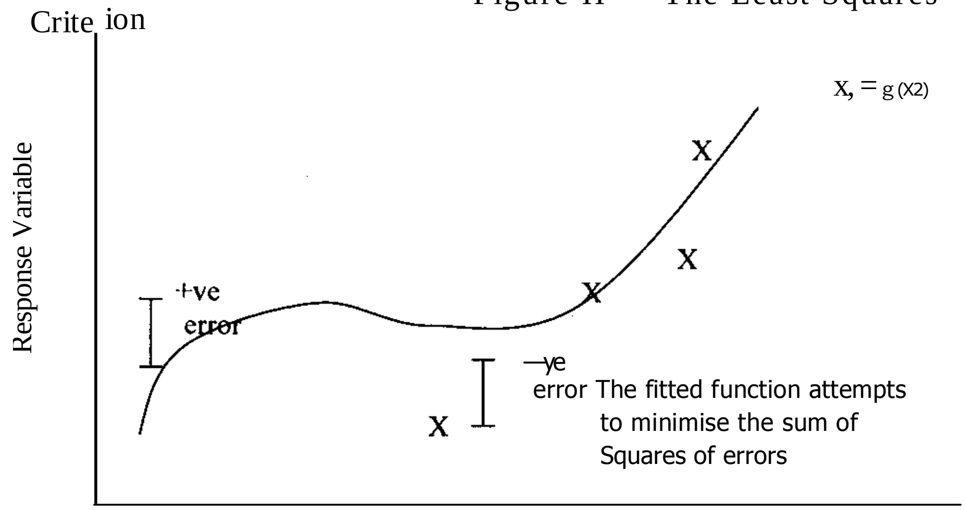


Figure II — The Least Squares

Independent Variable
Figure 14.2

It is true that we would like the overall error to be as small as possible. The most common criterion in the determination of model parameters is to minimise the sum of squares of errors or residuals as they are often called. This is known as the least square criterion, and is the one most commonly used in Regression analysis (as well as Time Series Analysis).

This is, however, not the only criterion available. One may, for instance, minimise the sum of absolute deviations, which is equivalent to minimising the mean absolute deviation (M.A.D).

The least squares criterion, however, has the following main advantages:

(i) It is simple and intuitively appealing.
(ii) It results in linear equations (normal equations) for solution of parameters which are easy to solve.
(iii) It results in estimates of quality of fit and intervals of confidence of predicted values rather easily.

For other discussions and equation development refer to Time Series Unit on least square criterion.

An Example:

Data on the annual sales of a company in Naira over the past eleven years is shown in the table below. Determine a suitable straight line regression model, Y = (3$_0$ -I- x C for the data in the table:

Year	**Annual Sales in Naira**
1988	1
1989	5
1990	4
1991	7
1992	10
1993	8
1994	9
1995	13
1996	14
1997	13
1998	18

Solution:

The independent variable in this problem is the year whereas the response variable is the annual sales. Although you could take the actual year as the independent variable itself, a judicious choice of the origin at the middle year 1993 with the corresponding X values for other years as —5, —4, —3, —2, —1, 0, 1, 2, 3, 4, 5 would simplify calculations.

(From b $\quad \dfrac{EX_iY_i — (YX_1/'Y_1)/n}{EX_i^2 — (EX)^2 In}$ .... (3)

(Ref: Least Squares Criterion in Time Series Unit)

we see that to estimate the parameter b, we require the four summations:

$EX_i, Ey, E x_i^2, EX_iY_i$

Thus, calculations can be shown below where the totals of the four columns yield the four desired summations:

S/N	X1	Y_i	$X,^2$	X,Y,
1	—5	1	25	—5
2	—4	5	16	—20
3	—3	4	9	—12
4	—2	7	4	—14
5	—1	10	1	—10
6	0	8	0	0
7	1	9	1	9
8	2	13	4	26
9	3	14	9	42
10	4	13	16	52
11	5	18	25	90
	EX, = 0	EY, = 102	EX_1^2= 110	'CY; = 158

Table14.1

We find that n = 11

Xxi = 0 0/11 = 0

EXY = 102 Y = 102/11

X^2 = 110 = = 158

$$b, = \frac{EXLEY,}{EN^2 — (EXY /n}$$

$$\frac{158}{110} = 1.44$$

The fitted equation is:

Y + b, (X —7)

Or

9.27 + 1.44X

Thus, the parameters 130 and 13_1 of the model Y = 130 + 131x + E are estimated b0 and bl which in this case are 9.27 and 1.44 respectively. Now that the model is completely specified, we can obtain the predicted values cr, and the errors as residuals Y Y

- — - I respectively to the eleven observations enough, the ANOVA table 14.2 below:

S/N	X,	Yi		Y_1—Y_1
1	—5	1	2.07	—1.07
2	—4	5	3.51	1.49
3	—3	4	4.95	—0.95
4	—2	7	6.39	—0.61
5	—1	10	7.83	2.17
6	0	8	9.27	—1.27
7	1	9	10.71	—1.71
8	2	13	12.15	0.85
9	3	14	13.59	0.41
10	4	13	15.03	—2.0
11	5	18	16.47	1.53

Table 14.2

To determine whether the fit is good enough, the Analysis of Variance (ANOVA) table can be constructed:

(Sum of squares - SS due to regression

= b, ,

(Associated degrees of freedom = 1)

= [I,X,Y, —(y,x,zy,) / n 1

Ex,2 (Exi)2in

$= \frac{(158)^2}{110} = 226.95$

The total (corrected) SS (Associated degrees of freedom = 11 —1 = 10)

I x’ (y) (1 · 1

1194 — $(102)^2/11$

– 1194 — 945.82

• 248.18

The value R^2 $= \frac{\text{SS due to regression}}{\text{SS about mean}}$

$= \frac{226.05}{248.18} = 0.9145$

indicating that regression line explains 91.45% of the total variation about the mean.

Note:

sum of squares about the mean		sum of squares about regression	+	sum of squares due to regression

Any sum of squares is associated with it a number called its degree of freedom. This number indicates how many independent pieces of information involving the n independent numbers Y Y -1, - 2, ..., Yn are needed to compile the sum of squares.

3 2 1 Examining the fitted Straight Line

In fitting the linear model $Y = h_o + p, x + E$ using the least square criterion as indicated earlier in 3.2, no assumption was made about probability distribution. The method of estimating the parameters 13_0 and P_i tried only to minimise the sum of squares of the errors or residuals, and that simply involved the solution of simultaneous linear equations (Refer to Time Series Chart). However, in order to be able to evaluate the precision of the estimated parameters to provide confidence intervals for forecast values, it is necessary to make the following assumptions in the basic model $Y, = 13_0 + Jb_iX_i + x + E$, $i = 1,2 \ldots n$.

With some mathematical assumptions, the following could be determined — (Ref. Time Series and Correlation units).

(1) Standard error of the slope b, and confidence interval of 13_1.
(2) Standard error of the intercept b_0 and confidence interval for 13_0.
(3) Standard error or Y, the predicted value.
(4) Significance of regression.
(5) Percentage variation explained.

Note:

Errors that occur in many real life situations tend to be normally distributed due to the Central Limit Theorem. In practice, an error term such as is a sum of errors from several sources. Then no matter what the probability distribution of the separate errors may be, their sum will have a distribution that will tend more and more to the normal distribution as the number of components increases, by the Central Limit Theorem.

The studies in Time Series and correlation have established these facts. You are encouraged to refer to them.

An Example of the Calculations:

The various computations stated for a straight line regression situation in Section 3.2.1 will now be illustrated for the example of annual sales data for a company that was considered earlier in Section 3.2. Recall that the fitted regression equation was

$$t = 9.27 + 1.44X \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

3.3 Variety of Regression Models

The methods of regression analysis have been illustrated in this unit for the case of fitting a straight line to a given set of data points. However, the same principles are applicable to the fitting of a variety of other functions which may relevant in certain situations like seasonal, which leads to cyclical forecast and Trend, Polynominal of various order, Multiplicative Models, Linear and Non-linear Regression (as found in the Time Series and Correlation Units).

4.0 CONCLUSION

In this unit, fundamentals of linear regression have been highlighted. Broadly, the fitting of any chosen mathematical function to given data is known as regression analysis. It has been established that the estimation of the parameters of this model is accomplished by the least squares criterion which tries to minimise the sum of squares of the errors for all the data points.

5.0 SUMMARY

We have touched on the following salient issues:

- How the parameters of a fitted straight line model are estimated, has been illustrated.
- After the model is fitted to the data, the next logical question is to find out how good the quality of fit is?
- Making of qualitative statements regarding confidence limits for estimates of the parameters as well as the forecast values.
- Comparison of alternative regression models in hypothesis.
- Emphasis that the method of least squares used in linear regression is applicable to a wide class of models.
- Regression is a potent device for establishing relationships between variables from the given data and the discovered relationship can be used for predictive purposes. Some of the models used in forecasting of demand rely heavily on regression analysis.

6.0 TUTOR-MARKED ASSIGNMENTS

1) Explain the need for linear regression in statistical analysis.

2) In establishing a relationship between variables from a given data, discuss briefly highlighting the fundamental steps to be considered.

7.0 REFERENCES AND FURTHER READING

Indira Gandhi National Open University (2000). Young Printing Press, Delhi.

Owen, F. and Jones (1975). *Modern Analytical Techniques.* Polytech Publishers Limited, Stockport.

UNIT 5 PROJECT PLANNING

CONTENTS

1.0 INTRODUCTION

Planning for a project is expedient if it is to be undertaken efficiently; moreover, if it involves a number of people. All too often, people work in isolation, quite ignorant of what other people working on the same project are doing. It is the task of management to coordinate efficiently the efforts of such people. In the past, planning has not been so important as it is today as projects were less complex, the rule of thumb method would work quite well. However, as projects have become more complex, management (banking, finance etc.) researchers have turned their attention increasingly to systematic planning. This unit will consider the complexity of this day products (services) and you will soon recognise the need for systematic planning. Shortly after the Second World War (1945), researchers evolved a method called Network Analysis. The impact of the method has been quite dramatic, largely because it is applicable to such a wide variety of projects. It has been used to plan production projects, servicing projects, research projects, sales projects, operational projects and military projects etc.

2.0 OBJECTIVES

At the end of this unit, you should be able to:

- discuss project planning;
- explain network in project development with deterministic and probabilistic times
- construct simple network diagrams

3.0 MAIN CONTENT

3.1 Nature of Projects

Projects are found in series of stages, a life cycle, which include:

- Planning
- Execution
- Phase — out

At each stage of this life cycle, a variety of skillful requirements are involved. In effect, project unit human resources and with diverse knowledge and skills, most of whom remain together for less than full life of the project. Some go from project to project as their contributions are needed (full or part-time basis) — consulting firms — architects, writers, publishers, Accountants, Financial Analysts etc. tend to be involved with project on a regular basis.

3.1.1 Project Planning / Management

The length, size and scope of projects varies widely according to the nature and purpose of the project. Nevertheless, all projects have something in common. They go through a life cycle, which typically consists of five phases:

(1) Concept — recognizing the need for a project.
(2) Feasibility Analysis — examines the expected costs, benefits and risk of undertaking the project.
(3) Planning — spells out the details of work and provides estimates of the necessary resources — human, time and cost.
(4) Execution — ensure project is done.
(5) Termination — closure is achieved.

3.2 The Network – Symbols, Diagram / Convention

How does the method work? Well the first stage of analysis is to divide the project into a number of different activities. An activity is merely a particular piece of work identifiable as an entity within the project. If for example, the project under consideration is the servicing of a motor car, then one of the activities would be check the brakes for wear.

Now an activity within a network is represented by an arrow, with the description of the activity written on it viz:

Check the brakes for war

Note: In addition to activities, events are identified too.

Events mark the point in time when activity is completed and the next activity can be started. Events are normally represented by circles:

Check the brakes wear  Figure I

The event 111 represents the point in time when the car is ready to have its brakes tested.

Network ?

A network is a convenient method of showing the logical sequence of activities in a project.

Suppose that in a certain project, there are two activities A and B, and activity B cannot be started until activity A is completed. Using the network symbols, these activities can be represented like this:

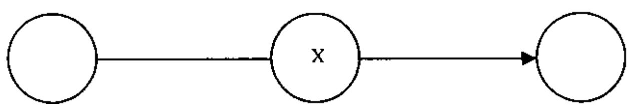
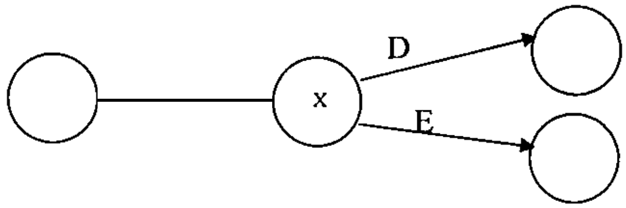


Figure 2

The event represents the point in time when activity A is completed, but it also represents the point of time when activity B is can begin. This represents the situation when activity B depends upon activity A; when two or more activities are dependent upon the same. activity. The situation when neither activity D nor E can start until activity C is complete would be represented like this:

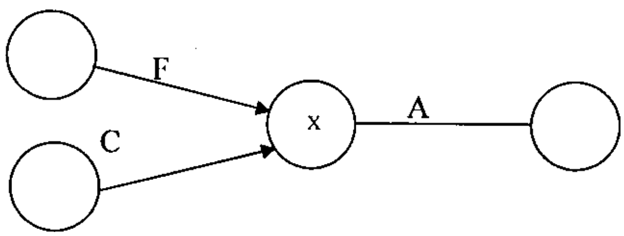
Figure 3



The event x represents the point of time when activity C is completed, and also the point of time when activities D and E can start, so the diagram clearly shows that D and E depend on C.

Also, it is likely that an activity depends upon more than one other activity. If activity A cannot start until activities G and F are both complete — represent as follows:

Figure 4



Dependence Tables

From the introduction, the first task of network analysis is to sort out the logical sequence of activities. This is done by constructing a dependency table. List all the activities, next is the list of activities that they depend upon. E.g. there is one way in which the Financial Manager can affect the volume of credit sales and collection period and consequently investment in accounts receivable. This is through a dependable change in credit policy.

Credit policy is used to refer to the combination of three decision variables (assuming) logically:

(o)	Introduction	Preceding activity
(i)	Credit Standard	(a)
(ii)	Credit Terms	(b)
(iii)	Collection Efforts, financial manager has	(c) on which the

influence. Following the preceding activity will enhance proper decision.

Constructing a dependence table (trend) is often the most difficult part of project analysis. Obviously, the construction of dependence tables of any action (activity) like that of establishing a credit arrangement is in essence, a team activity. All the experts engaged on the project have to be consulted.

Network Diagram

A network diagram below is composed of a number of arrows and nodes. The arrows represent the flow of project activities. A network diagram is generally the preferred approach for usual portrayal of project activities as follows:

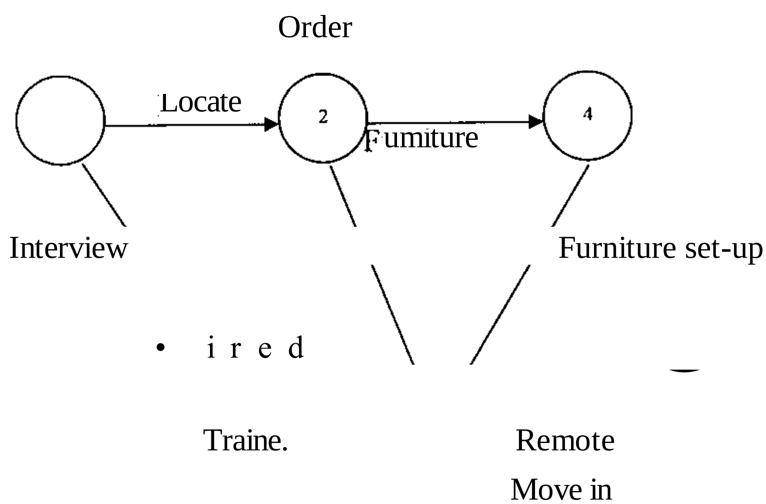


Figure 5

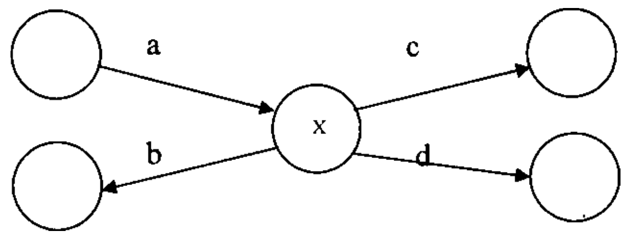
The network diagram describes sequential relationship among major activities on a project. For example, activity 2 —> 4 cannot be started; according to the network until activity 1 —> 2 has been completed (figure 4 above).

A path is a sequence of activities that leads from the starting node to the finishing node. Thus, the sequence 1 2 —▪ 4 —> 5 —> 6 is a path. There are two other paths in the network namely: 1 —> 2 —> 5 6 and 1 —▪ 3 5 6. The path with the longest time is of particular interest because it governs project completion time. The project life cycle equals the expected time of the longest path; the longest path is the critical path, and its activities are referred to as critical activities. The allowable slippage for any path is called slack, and it reflects the difference between the length of a given path and the length of the critical path.

Network Conventions

Developing and interpreting network diagrams requires some familiarity with networking conventions. Discussion centres only on some of the most basic and most common feature of network diagram. As above, figure 1, 2, 3 and 4 provides background for understanding the basic concepts associated with precedence diagrams and permits us to solve typical problems. See some more examples like the following:

Figure 6



When multiple activities enter a node, this implies that all those activities must be completed before any activity that is to begin at that node can start. Hence figure 5 — activities "a" and "b" must both be finished before either activity "c" or activity "d" can start.

When two activities both have the same beginning and ending nodes, a dummy note and activity is used to preserve the separate identify of each activity. In the diagram below, activities "a" and "b" must be completed before activity "c" can commence. See figure t below:

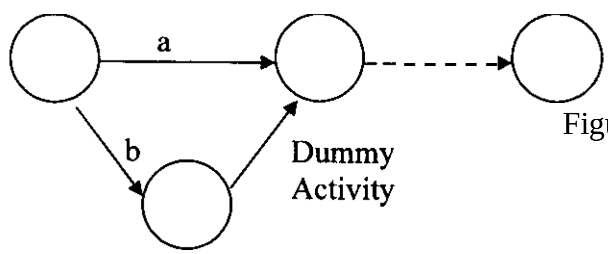


Figure 15.7

Note: The 'if' Event Numbering Rule

If the network above consisted of real rather than imaginary activities, then description of each activity would be written above each arrow. It is convenient to use a coding system to describe a particular activity, and we do this by numbering the events according to the 'if' rule. The rule states that the event at the end of an activity must be assigned a greater number than the event at the beginning of an activity. There is no single way of numbering the events, for the 'if rule' allows considerable latitude. One numbering system that obeys the rule is shown in figure 7 below but of course there are many others.

We can now describe activity A by its 'if numbers 1 —2.

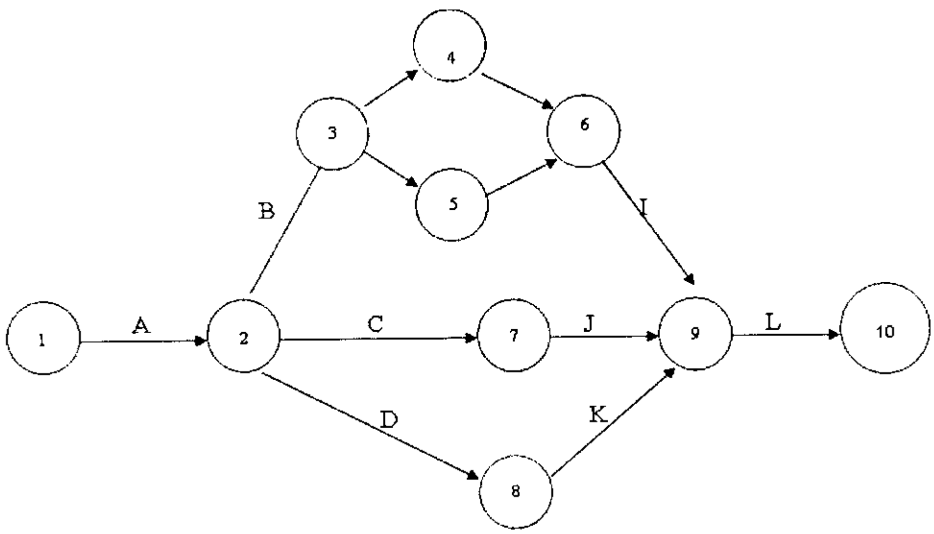
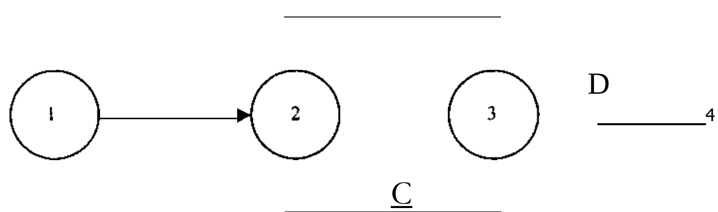


Figure 8

Example:

Activity	Preceding
B and C	A
	B and C

Figure 9



This network does show a logical sequence of activities, but we cannot accept it, as activities B and C both have the same 'if number. This problem is overcome by introducing a dummy activity — represented by a broken line.

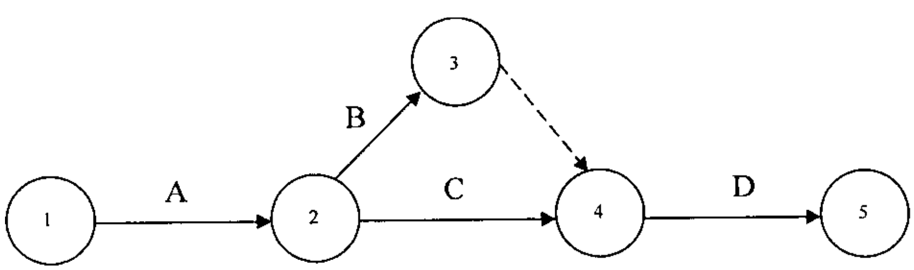


Figure 10

The dummy activity does not represent any activity as such — it is inserted to preserve the sequential numbering system of events. Such a dummy is called an identity dumni) . Sometimes it is necessary to insert a dummy to preserve not the sequential numbering system, but the logical sequence of events.

3.3 Total Project Time: Critical Path Method (CPM)

The main determinant of the PERT and CPM networks are analysed and interpreted is whether activity time estimates are probabilistic or deterministic. If time estimates can be made with a high degree of confidence, the actual times will not differ significantly, we say the estimates are deterministic.

If estimated times are subject to variation, we say the estimates are probabilistic. Probabilistic time estimates must include an indication of the extent of probable variation.

Rules for when and where logical dummies would be needed for critical path method. The rules are most convenient to use as stated below:

(1) If an activity occurs in the right-hand column of a dependence table but not in the left-hand column, it cannot depend upon another activity having been completed. Hence, it must be a 'starting activity'. If there is more than one start activity, you may need dummies,

(2) If any activity occurs more than once in the right-hand column then you will need to introduce dummies. If the activity occurs n times then (n — 1) dummies will have to be drawn from its end event.

Example:

The dependence table of a certain project looks like the following:

Activity	Preceding Activity
B, C and D	A
E and F	
	G and H

G, H, C and D

I, J and K

Firstly, note that there is only one starting activity. Both G and H occur twice, so the dummy will be needed from the end event of both these activities. The network will look like the following figure 10:

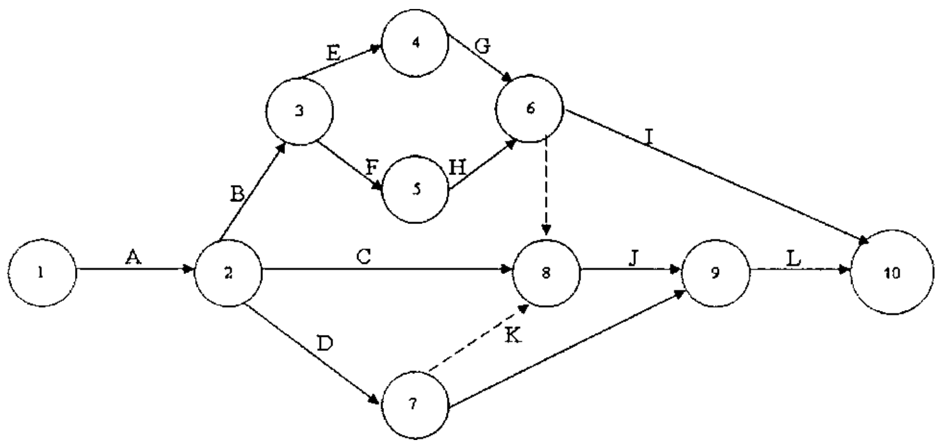


Figure II

Let us suppose that the network describes the activities in a manufacturing process. The network clearly shows that activities B, C and D can all be started together as long as we have sufficient resources to do so. Let us assume that we can easily obtain all the resources that we need.

Assume the project is such that the only resource needed is labour, and that all the labour available is equally capable of performing any activity. The time taken to complete each activity is known thus.

Activity
A B C D E F G H I J K L

Time (hrs)
34 5 6 2 1 7 4 35 6 2

How long will the project take?

Answer: Examine all routes through the network. See there are seven possible routes! See the list and find the time taken.

Route	Time		Hours
A B E G I L	3+4+2-FE7+3+2	=	21
A B E G Dummy J L	314+2+7+0+5+2	=	23
A B F H I L	3+4+1+4+3+2	=	17
A B E H Dummy J L	3+4+1+4+0+5+2	=	19
A C J L	3+5+5+2	=	15
A D Dummy I L	3+6+0+5+2	=	16
A D K L	3+6+6+2	=	17

The project cannot be completed in less than 23 hours. This is determined by the longest route through the network — called the critical path. Activities on this route must be completed on time otherwise the total project time will lengthen: (i.e. the activities have critical times). You should realise that it will be very difficult to calculate the total project time without first drawing the network. Try to find the total project time just using the dependence table, and you will confirm how true it is?

Note·
Finding the earliest event times has certainly helped to determine the total project time, but it has helped in isolating the critical path. To do this, use latest event times i.e. the latest time that each event can occur if the network is to be completed on time. Think back to critical activities — they must be started on time, otherwise the total project time will lengthen. What does this imply?

Each event on the critical path must have the same earliest and latest times. Using this fact, one can easily identify the critical path in the network.

SELF ASSESSMENT EXERCISE

Draw the network of the following 'dependable table'.

Activity	Preceding Activity
B and C	A
	C and D
	E and F

Remember that there must be one start event and one end event.

3.4 Gantt Chart

This is a popular tool for planning and scheduling simple projects. It enables a manager to initially schedule project activities and then to monitor progress over time by comparing planned progress to actual progress. A Gantt chart for a bank's plan to establish a new direct marketing department is shown below:

Activity start	0	2	4	6	8	10	12	14	16	18	20
Local new facilities											
Interview sros I ective staff											
Flire and train staff											
Select and order furniture											
Remodel and install •hones											
Furniture received and set up											
More in start up											

To prepare the chart, the personne in charge of the project identifies the major activities first that would be required. Next, time estimates for each activity is made, and the sequence of activities will occur, their planned duration and when they will take place. Then, as the project progresses and the manager would be able to see which activities were ahead of schedule and which were delaying project. This enables the manager to direct attention where it was needed most to hasten the project in order to finish on schedule.

The advantage of a Gantt chart is its simplicity, and this accounts for its popularity. However, Gantt charts fail to reveal certain relationships among activities that can be crucial to effective project management.

For instance, if one of the early activities in a project suffers a delay, it would be important for the manager to be able to easily determine which later activities would result in a delay.

It assists in allocating resources to the project known as 'loading the network'. On a Gantt chart, the activities are represented by lines having lengths proportionate to the duration of each activity.

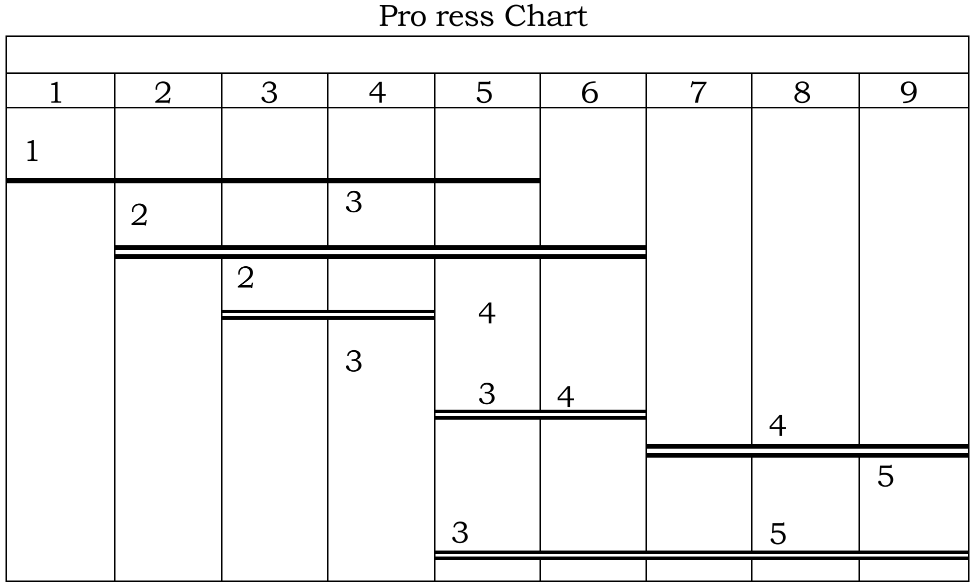
Each activity on the Gantt chart is identified by its number. It helps to dictate the project that is overloaded or under-loaded. The Gantt chart can be used effectively to show how a project is actually progressing and this is done by drawing a line above the activity symbol which is proportional to the amount of work completed. Gantt chart is an important element of project control.

SELF ASSESSMENT EXERCISE

(1) Suppose that the labour available to work on a project was as follows:
2 men up to the end of the 11th hour
3 men from the 1 1 th hour to the 23" hour

Draw the simple Gantt chart to take account of the supply of labour.

Pro ress Chart



The progress chart shows the state of a project at the end

4.0 CONCLUSION

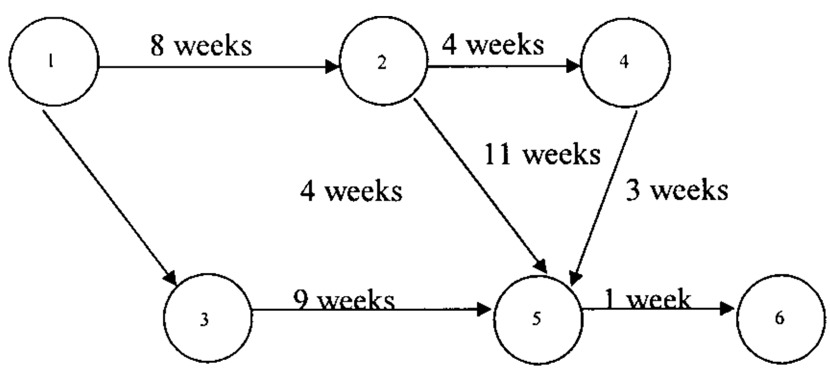
You have been able to see the need of a manager to be vested with creativity and imagination in the smooth management of business through proper project planning with aid of network studies. This helps in evaluating the job/duty and giving it proper direction.

5.0 SUMMARY

Projects consist of activities with set objectives. From the study, you realise that projects are to be properly planned, the analysis understood through network symbols, convention and diagrams. With simple network presentation, a manager can identify problems of the project on time and determine the probable time needed to achieve the objective of the project and how the resources are allocated and managed.

6.0 TUTOR-MARKED ASSIGNMENT

Given the following information:



Determine.

(a) The length of each path;
(b) The critical path;
(c) The expected length of the project;
(d) Amount of slack time for each path.

7.0 REFERENCES AND FURTHER READINGS

Owen, F. and Jones, R. (1975). *Modern Analytical Techniques,* Polytech Publishers Limited, Stockport.

National Open University of Nigeria (2006). MBA 701: *Production and Operations Management.*