NATIONAL OPEN UNIVERSITY OF NIGERIA

ADVANCED MATHEMATICAL ECONOMICS
COURSE CODE: ECO 718

FACULTY OF SOCIAL SCIENCES
DEPARTMENT OF ECONOMICS

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INTRODUCTION
ECO718: ADVANCED MATHEMATICAL ECONOMICS (2 Units)

ECO718: Advanced Mathematical Economics is a two-credit and one-semester postgraduate course for Economics students. The course is made up of twelve units spread across sixteen lectures weeks. This course, Advanced Mathematical Economics, develops the students with the essential tools, skills and knowledge base necessary to operate as a practicing macroeconomist. In reality, the course also exposes the students to practical use of mathematical models in analyzing economic problems.

COURSE CONTENTS

The contents of ECO718: Advanced Mathematical Economics are such that it provides the concepts of calculus to the analysis of functions of several variables. Then it covers convex multivariate optimization. This is followed by analysis of constrained optimization and unconstrained optimization, as applied to problems of firm and consumer behavior. The course extends to matrix algebra in solving problems of production and allocation in mutually dependent economies. It also provides essential elements of dynamic optimization in discrete time. Other relevant topics covered in the course include Economic Models, Components of a Mathematical Model, Types of Functions, Functions of Two or More Independent Variables, Equilibrium Analysis in Economics, Constant Elasticity of Substitution Function, Integrals and Some Economic Applications, Differential Equations, Linear Programming, Input-Output Analysis and Linear Programming, Non-Linear Programming, and Game Theory.

COURSE AIMS

The aim of the course is to give the student comprehensive knowledge of the Mathematical economics with reference to:

1. Concept of Mathematical Economics,
2. Economic Models, Components of a Mathematical Model,
3. Types of Functions, Functions of Two or More Independent Variables
4. Equilibrium Analysis in Economics,
5. Matrix-Algebra,
6. Optimization: Constrained Optimization: Lagrange-Multiplier Method
7. Constant Elasticity of Substitution Function
8. Cobb Douglas Function as a Special Case of the CES Function
9. Differential Calculus/Derivatives and Some Economic Applications
10. Integrals Calculus and Some Economic Applications
11. Differential Equations
12. Simultaneous Equations Dynamic Models
13. Linear Programming: Simplex Method

14. Input-Output Analysis and Linear Programming
15. Non-Linear Programming,
16. Game Theory

COURSE OBJECTIVES

To achieve the aims of the course, there are general objectives which the course seeks to accomplish. These general objectives serve as a guide to the specific unit objectives and hence the course objectives are study guide for every student. The students at the completion of this course should be able to:

1. Explain the Concept of Mathematical Economics
2. Explain meaning of Economic Models, Components of a Mathematical Model
3. Discuss the Types of Functions, Functions of Two or More Independent Variables
4. Evaluate Matrix Algebra
5. Find the inverse of Matrix
6. Use the Cramer's Rule in solving simultaneous equations
7. Solve some simultaneous equations using matrix algebra
8. Solve mathematical problems on Constant Elasticity of Substitution Function.
9. Solve Optimization problems including Constrained Optimization using the Lagrange-Multiplier Method.
10. Solve Integral calculus
11. Solve Integration by Substitution
12. Solve Integration by Parts
13. Carry out Applications of Integral
14. Solve some first order Differential Equations
15. Solve some Simultaneous Equations Dynamic Models
16. Solve problems of Linear Programming
17. Use Matrix Algebra to evaluate input Output Models
18. Solve problems of Non-Linear Programming using substitution method
19. Non-Linear Programming,
20. Explain Game Theories and also evaluate games problems.

WORKING THROUGH THIS COURSE/SUCCESS GUIDE

This course consists of sixteen (16) units with study exercises called Student Assessment Exercises (SAE). Students would as a matter of necessity be required to submit home/class assignments for assessment drives. At the end of the course there is a final examination. This course is designed to be taught for a minimum of 16 weeks before examining students.

COURSE MATERIAL

The major component of the course, what you have to do and how you should allocate your time to each unit in order to complete the course successfully on time are listed follows:

1. Course guide
2. Study unit
3. Textbook
4. Assignment file
5. Presentation schedule

STUDY UNIT

This course consists of sixteen (16) units as earlier mentioned and these units are structured into four (4) modules as shown below:

MODULE ONE: OVERVIEW OF MATHEMATICAL ECONOMICS, MODELS, FUNCTIONS AND ECONOMIC EQUILIBRIUM ANALYSIS

UNIT 1 Overview of Mathematical Economics
UNIT 2 Economic Models, Components of a Mathematical Model
UNIT 3 Function, Types of Functions, Functions of Two or More Independent Variables
UNIT 4 Equilibrium Analysis in Economics

MODULE TWO: MATRIX ALGEBRA, SYSTEM OF LINEAR EQUATIONS AND MATRIX APPLICATION TO ECONOMICS: INPUT-OUTPUT ANALYSIS

UNIT 1 Matrix-Algebra
UNIT 2 System of Linear Equations and Cramer's Rule
UNIT 3 System of Linear Equations and Matrix Inversion
UNIT 4: Matrix Application to Economics: Input-Output Analysis

MODULE THREE: DIFFRENTIATIONS, INTEGRATION AND OPTIMIZATION TECHNIQUES

UNIT 1 Differential Calculus and Some Economic Applications
UNIT 2 Integral Calculus and Some Economic Applications
UNIT 3 Optimization Techniques
UNIT 4 Differential Equations

MODULE FOUR: LINEAR/NON-LINEAR PROGRAMMING, AND GAME THEORY

UNIT 1 Linear Programming: Simplex Method
UNIT 2 Non-Linear Programming,
UNIT 3Game Theory

Each study unit will take at least two hours, and it includes the introduction, objective, main content, self-assessment exercise, conclusion, summary and reference. Other areas border on the Tutor-Marked Assessment (TMA) questions. Some of the self-assessment exercise will necessitate discussion, brainstorming and argument with some of your colleges. You are advised to do so in order to understand and get acquainted with historical economic events as well as notable periods.

There are also textbooks under the reference and other (on-line and off-line) resources for further reading. They are meant to give you additional information if only you can lay your hands on any of them. You are required to study the materials; practice the self-assessment exercise and tutor-marked assignment (TMA) questions for greater and in-depth understanding of the course. By doing so, the stated learning objectives of the course would have been achieved.

TEXTBOOK AND REFERENCES RECOMMENDED

For further reading and more detailed information about the course, the students are advised to consult the following materials:

Adler, Ilan; Christos, Papadimitriou; & Rubinstein, Aviad (2014), "On Simplex Pivoting Rules and Complexity Theory", International Conference on Integer Programming and Combinatorial Optimization, *Lecture Notes in Computer Science* 17: 13 - 24,

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Testing, and Some New Empirical Values". *Journal of Political Economy*. 84 (5): 903 - 916.

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HOME WORK/ASSIGNMENT FILES

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your instructor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are four assignments in this course. The four course assignments will cover:
Assignment 1 – All TMAs' Questions in Units 1 – 4 (Module 1)
Assignment 2 – All TMAs' Questions in Units 5 – 8 (Module 2)
Assignment 3 – All TMAs' Questions in Units 9 – 12 (Module 3)
Assignment 4 – All TMAs' Questions in Units 13 – 16 (Module 4)

PRESENTATION SCHEDULE

The presentation schedule included in the course materials gives important dates in the year for the completion of tutor-marking assignments and attending tutorials. Remember, every student is required to submit all your assignments by due date. You should guide against failure to meet up with the deadline.

ASSESSMENT

The assessment for this course is a combination of continuous assessment and final examinations. The continuous assessment is 30% of the final grade and it is made up of all home works/assignments given in the course of lectures as contained in the units of the four modules. The home works and/or assignments must be submitted to your instructor for formal Assessment in compliance with the deadlines. Classroom tests will also be given to facilitate learning of the more challenging areas of the course. A final examination will be written at the end of the course and this will cover 70%.

TUTOR-MARKED ASSIGNMENTS (TMA)

There are four homework/assignments to be done by each student this course. Students are required to submit all the assignments as the TMAs constitute 30% of the total score.Assignment questions for the units in this course are contained in the Home Work/Assignment File. Students are advised to eagerly demonstrate that researched knowledge outside the course material. When you have completed each assignment, send it, together with a TMA form, to your instructor. Make sure that each assignment reaches your instructor on or before the deadline given in the Presentation File. If for any reason, you cannot complete your work on time, contact your instructor before the assignment is due to discuss the possibility of an extension. Extensions will not be granted after the due date unless there are exceptional circumstances.

FINAL EXAMINATION AND GRADING

The final examination will be of three hours' duration and have a maximum score of 70% of the total course grade. The examination will consist of questions which reflect the types of study/practice exercises and tutor-marked problems students have previously encountered. All topics of the course will be assessed.

Revise the entire course material using the time between finishing the last unit in the module and that of sitting for the final examination. You might find it useful to review your self-assessment exercises, tutor-marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

COURSE MARKING SCHEME

In this course, the total marks of 100% is to be earned by students. The allocation of total score would include the best three assignments out of four that are marked 10% each making a total of

30% while the final Examination would be graded over 70%. This is explained in the table below:

Table 1: Scores Allocation

Assignment	**Marks**
Assignments (Best three assignments out of four that is marked)	30%
Final Examination	70%
Total	**100%**

COURSE OVERVIEW

The Table presented below indicates the units, number of weeks and assignments to be taken by the students in order to successfully complete the course.

Table 2: Course Overview

Units	**Title of Work**	**Week's Activities**	**Assessment (end of unit)**
	Course Guide		
MODULE 1: OVERVIEW OF MATHEMATICAL ECONOMICS, MODELS, FUNCTIONS AND ECONOMIC EQUILIBRIUM ANALYSIS			
1	Overview of Mathematical Economics	Week 1	Assignment 1
2	Economic Models, Components of a Mathematical Economic Models	Week 2	Assignment 2
3	Functions/Ordered Pairs, Types of Functions, and Continuity of Functions	Week 3	Assignment 3
4	Equilibrium Analysis in Economics	Week 4	Assignment 4
MODULE 2: MATRIX ALGEBRA, SYSTEM OF LINEAR EQUATIONS AND MATRIX APPLICATION TO ECONOMICS: INPUT-OUTPUT ANALYSIS			
1	Matrix-Algebra	Week 5	Assignment 1
2	System of Linear Equations and Cramer's Rule	Week 6	Assignment 2
3	System of Linear Equations and	Week 7	Assignment 3

	Matrix Inversion		
4	Matrix Application to Economics: Input-Output Analysis	Week 8	Assignment 4
MODULE 3:DIFFRENTIATIONS, INTEGRATION AND OPTIMIZATION TECHNIQUES			
1	Derivatives and Some Economic Applications	Week 9	Assignment 1
2	Integrals and Some Economic Applications	Week 10	Assignment 2
3	Optimization Techniques	Week 11	Assignment 3
4	Differential Equations	Week 12	Assignment 4
MODULE 4: LINEAR/NON-LINEAR PROGRAMMING, AND GAME THEORY			
1	Linear Programming	Week 13	Assignment 1
2	Non-Linear Programming	Week 14	Assignment 2
3	Game Theory	Week 15 & 16	Assignment 3
	Examination	Week 17, 18 & 19	

HOW TO GET THE MOST FROM THIS COURSE

In distance learning the study units replace the university lecturer. This is one of the great advantages of distance learning; you can read and work through specially designed study materials at your own pace and at a time and place that suit you best. Think of it as reading the lecture instead of listening to a lecturer. In the same way that a lecturer might set you some reading to do, the study units tell you when to read your books or other material, and when to embark on discussion with your colleagues. Just as a lecturer might give you an in-class exercise, your study units provide exercises for you to do at appropriate points.

Each of the study units follows a common format. The first item is an introduction to the subject matter of the unit and how a particular unit is integrated with the other units and the course as a whole. Next is a set of learning objectives. These objectives let you know what you should be able to do by the time you have completed the unit.

You should use these objectives to guide your study. When you have finished the unit, you must go back and check whether you have achieved the objectives. If you make a habit of doing this you will significantly improve your chances of passing the course and getting the best grade.

The main body of the unit guides you through the required reading from other sources. This will usually be either from your set books or from a readings section. Some units require you to undertake practical overview of historical events. You will be directed when you need to embark on discussion and guided through the tasks you must do.

The purpose of the practical overview of some certain historical economic issues are in twofold. First, it will enhance your understanding of the material in the unit. Second, it will give you practical experience and skills to evaluate economic arguments, and understand the roles of history in guiding current economic policies and debates outside your studies. In any event, most of the critical thinking skills you will develop during studying are applicable in normal working practice, so it is important that you encounter them during your studies.

Self-assessments are interspersed throughout the units, and answers are given at the end~~s~~ of the units. Working through these tests will help you to achieve the objectives of the units and prepare you for the assignments and the examination. You should do each self-assessment exercises as you come to it in the study unit. Also, ensure to master some major historical dates and events during the course of studying the material.
The following is a practical strategy for working through the course. If you run into any trouble, consult your tutor. Remember that your tutor's job is to help you. When you need help, don't hesitate to call and ask your tutor to provide it.

1. Read this Course Guide thoroughly.
2. Organize a study schedule. Refer to the `Course overview' for more details. Note the time you are expected to spend on each unit and how the assignments relate to the units. Important information, e.g. details of your tutorials, and the date of the first day of the semester is available from study centre. You need to gather together all this information in one place, such as your dairy or a wall calendar. Whatever method you choose to use, you should decide on and write in your own dates for working breach unit.
3. Once you have created your own study schedule, do everything you can to stick to it. The major reason that students fail is that they get behind with their course work. If you get into difficulties with your schedule, please let your tutor know before it is too late for help.
4. Turn to Unit 1 and read the introduction and the objectives for the unit.
5. Assemble the study materials. Information about what you need for a unit is given in the `Overview' at the beginning of each unit. You will also need both the study unit you are working on and one of your set books on your desk at the same time.
6. Work through the unit. The content of the unit itself has been arranged to provide a sequence for you to follow. As you work through the unit you will be instructed to read sections from your set books or other articles. Use the unit to guide your reading.
7. Up-to-date course information will be continuously delivered to you at the study centre.

8. Work before the relevant due date (about 4 weeks before due dates), get the Assignment File for the next required assignment. Keep in mind that you will learn a lot by doing the assignments carefully. They have been designed to help you meet the objectives of the course and, therefore, will help you pass the exam. Submit all assignments no later than the due date.
9. Review the objectives for each study unit to confirm that you have achieved them. If you feel unsure about any of the objectives, review the study material or consult your tutor.
10. When you are confident that you have achieved a unit's objectives, you can then start on the next unit. Proceed unit by unit through the course and try to pace your study so that you keep yourself on schedule.
11. When you have submitted an assignment to your tutor for marking do not wait for its return `before starting on the next units. Keep to your schedule. When the assignment is returned, pay particular attention to your tutor's comments, both on the tutor-marked assignment form and also written on the assignment. Consult your tutor as soon as possible if you have any questions or problems.
12. After completing the last unit, review the course and prepare yourself for the final examination. Check that you have achieved the unit objectives (listed at the beginning of each unit) and the course objectives (listed in this Course Guide).

TUTORS AND TUTORIALS

There are some hours of tutorials (2-hours sessions) provided in support of this course. You will be notified of the dates, times and location of these tutorials together with the name and phone number of your tutor, as soon as you are allocated a tutorial group.

Your tutor will mark and comment on your assignments, keep a close watch on your progress and on any difficulties you might encounter, and provide assistance to you during the course. You must mail your tutor-marked assignments to your tutor well before the due date (at least two working days are required). They will be marked by your tutor and returned to you as soon as possible.

Do not hesitate to contact your tutor by telephone, e-mail, or discussion board if you need help. The following might be circumstances in which you would find help necessary. Contact your tutor if.

- You do not understand any part of the study units or the assigned readings
- You have difficulty with the self-assessment exercises
- You have a question or problem with an assignment, with your tutor's comments on an assignment or with the grading of an assignment.
- You should try your best to attend the tutorials. This is the only chance to have face to face contact with your tutor and to ask questions which are answered instantly. You can raise any problem encountered in the course of your study. To gain the maximum benefit

from course tutorials, prepare a question list before attending them. You will learn a lot from participating in discussions actively.

CONCLUSION/SUMMARY

The course~~,~~ gives the student comprehensive knowledge of the mathematical economics with reference to: economic/mathematical model, types of functions, functions of two or more independent variables, equilibrium analysis in economics, matrix-algebra, theory optimization including free and constrained simultaneous equations dynamic models, linear programming: simplex method, input-output analysis and linear programming and non-linear programming.

This course also gives you an insight into differential calculus/derivatives and some economic applications as well as integrals calculus and some economic applications. Finally, Game theory is also examined to enhance the student techniques of logical reasoning. After the successful study of this course, the student would have developed acute mathematical skills with the techniques required for analytical execution of economic problems with high levels of accuracy within an economic setting. We wish you success with the course and hope that you will find it fascinating and handy.

MODULE 1: OVERVIEW OF MATHEMATICAL ECONOMICS, MODELS, FUNCTIONS AND ECONOMIC EQUILIBRIUM ANALYSIS

UNIT 1 **Overview of Mathematical Economics**
UNIT 2 **Economic Models, Components of a Mathematical Model**
UNIT 3 **Types of Functions, Functions of Two or More Independent Variables**
UNIT 4 **Equilibrium Analysis in Economics**

UNIT 1: OVERVIEW OF MATHEMATICAL ECONOMICS

CONTENTS

1.0 Introduction
2.0 Objectives
3.0 Main Content
 3.1 Concept of Mathematical Economics
 3.2 Why Study Mathematical Economics?
 3.3 Economic Applications of Mathematics

4.0 Conclusion
4.0 Summary
6.0 Tutor-Marked Assignment
7.0 References/Further Readings

1.0 INTRODUCTION

This unit focuses on concept of mathematical economics, rationale for studying mathematical economics and the various areas where mathematics can be applied in economics.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Describe what mathematical economics is all about
- Discuss the rationale for mathematical economics
- Formulate economic problems in mathematical expressions

3.0 MAIN CONTENT

3.1 Concept of Mathematical Economics

Mathematical economics is an aspect of economics that utilizes mathematical methods to resolve economic problems in order to arrive at results. Such mathematical methods refer to differential and integral calculus, differential equation, matrix algebra, mathematical programming etc. By convention, the subject thus relies on numerical observations to predict economic behavior.

SELF ASSESSMENT EXERCISE

Describe in your own words what you understand as Mathematics for Economists

3.2 Why Study Mathematical Economics?

1. Mathematical economicsaids mathematical formulation/economic modelling of theoretical and economic interactions.
2. Mathematics aid economists in carrying out quantifiable assessmentsin order toforecasttrend/forthcoming economic activity.
3. Mathematics aid economists in the act of making positive claims about historical economic arguments.
4. It is particularly useful in solving optimization problems where a policymaker, for example, is considering the best policy option out of a given menu of economic policies.

SELF ASSESSMENT EXERCISE

Explain major advantages for studying mathematical economics

3.3 Economic Applications of Mathematics

The economic applications include:

- Policy maker, household or firm optimization
- Input-output modeling & analysis
- Equilibrium analysis application in which economic unit such as a household or firm or economic system such as a market or overall economy is modeled as not changing

- Comparative statics application in which a variation from one equilibrium to another equilibrium is induced by a variation in one or more factors
- Dynamic analysis application which entails tracing changes in an economic system over time, for example from the growth rate of national income.

SELF ASSESSMENT EXERCISE

Describe the various areas where mathematics can be applied to economics

4.0 CONCLUSION

Advanced mathematical economics is an important aspect of economic analysis as it provides the skills for application of mathematical knowledge to economic problems and also to analyze and simulate real economic situations. Its fundamentals embrace understanding of the one variable calculus as well as models of simultaneous equations.

5.0 SUMMARY

In this unit, we have been able to articulate the concept of mathematical economics, rationale for mathematical economics and also the various areas of economics where mathematics can be applied for purpose of decision making.

6.0 TUTORED MARKED ASSIGNMENTS

1. Briefly give an overview definition of mathematical economics.
2. In what ways can it be deduced that the study of mathematical economics is justifiable?
3. Describe the various areas where mathematics can be applied to economics

7.0 REFERENCES/FURTHER READINGS

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UNIT 2: ECONOMIC MODELS, COMPONENTS AND TYPES OF MATHEMATICAL/ECONOMIC MODEL

CONTENTS

1.0 INTRODUCTION

This unit discusses the meaning of economic model, components of a mathematical economic model, what makes good mathematical economic models and the various types of models. Also, our focus of discussion is on meaning of Cobb-Douglass Function and constant elasticity of substitution function (CES), properties of Cobb-Douglass Function and CES function while also providing solutions to some numerical problems on Cobb-Douglass Function and CES.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Define an economic model
- Discuss the various components of a mathematical model
- Explain what makes good mathematical models
- Discuss types of models
- Explain what the Cobb-Douglass function (CES) is all about
- Solving numerical problems on Cobb-Douglass function
- Explain what the constant elasticity of substitution function (CES) is all about
- Solving numerical problems on constant elasticity of substitution function (CES)

3.0 MAIN CONTENT

3.1. Meaning of Economic Model

An economic model is a depiction of a system using mathematical concepts. In effect, an economic model is a formal exemplification of an economic theory based on a set of mathematical equations. The process of developing an economic model is called mathematical modeling. An economic model is used to study the effects of different components, and to make predictions about economic behaviour.

3.2 Components of Mathematical Economic Model

Mathematical Component: This entails a discussion of the number of equations, linear or non-linear form of these equations and the type of equation in addition to the type of variables say endogenous, or exogenous variables.

Stochastic Component: Every economic model consists of a stochastic disturbance often known as the error term. The error term is a random variable that measures all the factors affecting the endogenous variable under study but not integrated in the model hence the incompleteness property of a model.

SELF ASSESSMENT EXERCISE

Describe the various components of economic mathematical model

3.3 What Makes Good Mathematical Economic Models?

- Variable Representation Property: In mathematical models, parameters are most often represented by variables and once there is a change in the values of any variable, it mathematically brings a change upon the model.
- Simplification: Every mathematical model simplifies essential structure of the economy being described.

- Incompleteness Property: Mathematical models are necessarily incomplete in specification. The reason being that a mathematical model is a representation of real life situation and there is therefore no mathematical model that can include every aspect of real life.

SELF ASSESSMENT EXERCISE
Describe the qualities of Mathematical Models

3.4 Types of Models

3.4.1 Linear Model: This is a model that has its variable raised to the power of one. Examples are demand and supply models given mathematically as; $q = a - bp$ and $q = -d + cp$ respectively. For example:

$$y = 28 - 0.5x \quad (1)$$

$$y = 500 + x \quad (2)$$

From the above example, we can see that the equations have a power of one (1) and this is the reason why they are linear models. In addition, these models are linear because they can be plotted on a graph as a straight lines with the equationwhere -0.5 and 1 are the slope coefficientsof the respective linesor equations (1) and (2) and 28 and 500 are the intercept on $y - axis$ for equations (1) and (2) respectively.

3.4.2 Polynomial Models: A polynomial is an equation of the following form

$$y = d_0 + d_1x + d_2x^2 + d_3x^3 + \ldots + d_nx^n$$

where y is the dependent variable, x is the independent variable, d_0 is a constant and $x_i \ (i = 1, 2, \ldots n)$ are the underlying parameters of the equation. The specified equation is a polynomial of degree *n* because *n* is the highest power of x. Other relevant eexamples of polynomial models include the following:

i. Polynomial of degree one, $y = a + d_1x$. This is a linear function and it is of degree one because 1 is the highest power of x in the equation.

ii. Polynomial of degree two, $y = a + d_1x + d_2x^2$. This is a quadratic equation and it is of degree two because 2 is the highest power of x in the equation.

iii. Polynomial of degree three, $y = a + d_1x + d_2x^2 + d_3x^3$. This a cubic equation and it is of degree three because 3 is the highest power of x in the equation. Economically speaking, an example of a polynomial equation of degree three is the total cost function.

3.4.3 Reciprocal Model:*(Please, define the model before giving example)* Examples of reciprocal models include the following:

$$Q = a + bZ^{-1}$$

Where Q is dependent variable,
Z is explanatory variable,
a is intercept coefficient,
b is slope coefficient.

3.4.4 Translog Indirect Utility Model: The translog equation is given by:

$$-Log_e Q = \delta + \sum_{p=1}^{P} d_p Log_e[w_p / M] + \sum_{p=1}^{P}\sum_{l=1}^{P} \phi_{pl} Log_e[w_p / Z][w_l / Z]$$

Where Q is indirect utility,
P is price level for the k^{th} commodity,
Z is income level [see Christensen, Jorgenson & Lau (1975)]

3.4.5. Dynamic Model vs. Static Model: A dynamic model is a model that accounts for dependent changes in the state of the system, thus, dynamic models are constructed to determine the time-path of variables such as GDP, employment, consumption, investment, price, sale, income etc. Example of dynamic model are shown below:

$$Y = 2000 + 4Z_{t-1} + 60Z^2$$
$$P = -0.8 - 90Q_{t-1}$$

As shown above, the "t" indicates time trend and this makes the mathematical model dynamic model. Dynamic models are useful in the following regards:
i. deciding the speed of adjustment,
ii. deciding whether or not there is going to be an adjustment to equilibrium in event of disequilibrium.

The static mathematical model is often referred to as the Steady state model because it is a mathematical model that is time invariant as it analyzes the system in equilibrium. In effect, static models simply reflect changes from one point in time to another without respect to speed of arriving at the new position. Below are some examples of static model.

$$Y = 1000 + 6Z + 40Z^2$$
$$P = -0.8 - 90Q$$

As seen above, there is no time which is denoted by letter **"t"** in the equation. Dynamic models are of two types; namely: difference and differential equations.

TABLE 1: DIFFERENCE BETWEEN STATIC AND DYNAMIC MODELS

Dynamic Models	**Static Models**
Dynamic models keep changing with reference to time	Static models are at equilibrium of in a steady state

Dynamic modeling comprises of series of operations, state changes, activities and interface	Static modeling comprises class and object diagrams used in depicting static elements
Dynamic model is a behavioral representation of static components of the system	Static model is structural representation of the system
Dynamic modeling is highly elastic as it varies with time, that is, time dependent observation of the system.	Static modeling is rigid as it is time independent observation of a system

SELF-ASSESSMENT EXERCISE

1. What is the relationship between static and dynamic models?
2. What is the relationship between Static and Dynamic models?

3.4.6. Non-linear Models: Cobb-Douglass Model: These models can be classified into two (2) types and these include mathematical models that are nonlinear in the variables but still linear in terms of the unknown parameter. This class of non-linear models includes model which are made linear in the parameters through transformation. Example is the Cobb Douglas function that relates output (Y) to labour (L) and capital (K) as shown below:

$$Y = (aL^{q}K^{g})$$

This function is nonlinear in the variable while it is linear in the parameters a, q and g. There is also the nonlinearity in parameters. These category of models cannot be made linear following transformation. The CD function is a functional form of the production function transmits the technological relationship between two inputs and output.

The Mathematical Formulation of Cobb-Douglas Function with two factors is as follows:

$$Y = WK^{q}L^{(1-q)}$$

where Y = total output

L = units of labour input (number of person-hours)

K = units of capital input (machinery, equipment, and buildings etc.)

W = total factor productivity

$q, 1-q$ are the coefficients of output elasticity of capital and labor, separately.

The CD function can be mathematically formulated in a general multi-factors case as a linear relation thus:

$$\ln(Y) = b_0 + \sum b_i \ln(I_i)$$

where Y = total output
I = units of input
b_i = input coefficients

In the two-factor case the CD function form can be mathematically formulated as a linear relation as follows:

$$\ln(Y) = \ln(W) + q\ln(K) + (1-q)\ln(L)$$

The Cobb-Douglas Function can be specified as a Special Case of CES Functionas:

$$Y = W(qL^{o} + (1-q)K^{o}]^{-\frac{\varepsilon}{o}} \quad (1)$$

Taking the log of the CES function in equation (1), we have equation (2) as follows:

$$\ln(Y) = \ln(W) - \frac{\varepsilon}{o}\ln[qL^{o} + (1-q)K^{o}] \qquad (2)$$

By applying l'Hôpital's rule, such that $o = 0$ we derive a limiting case that corresponds to CD function as follows: As can be seen in equation 2, if $o \to 0$ $\frac{\varepsilon}{0}$,and hence $\frac{\varepsilon}{o}\ln\left(qL^{o} + (1-q)K^{o}\right)$ will tend to ∞).

The features of Cobb-Douglas Function are as follows:

a. The CD function is characterized by constant returns to scale such that $q + (1-q) = 1$,
b. The CD function is characterized by decreasing returns to scale such that $[q + (1-q)] < 1$
c. The CD function is characterized by increasing returns to scale such that $[q + (1-q)] > 1$
d. The constant returns to scaleassumption holds under a perfect competition

SELF ASSESSEMENT QUESTION

Discuss the properties of CD function
Discuss the meaning and properties of CD function
Mathematically formulate CD function

Numerical Example 1: Consider the following Cobb-Douglas function which is to be optimized

$$Q = 81z^{\frac{1}{4}}y^{\frac{3}{4}}$$
$$s.t.\ 3z + 6y = 90$$

(a) Optimize the CD function

(b) Determine the optimal value of the function at z = 124 & y = 54 respectively

(c) Use the results to explain fully the concept of MRTS between the factor inputs z & y.

(d) State and derive with regards to CD function, Euler's theorem

Solution to Numerical Example 1: Form the Langragian function as follows:

$$L = 81z^{\frac{1}{4}}y^{\frac{3}{4}} + \lambda[90 - 3z - 6y]$$

$$L_z = \frac{\partial L}{\partial z} = 20.25z^{-\frac{3}{4}}y^{\frac{3}{4}} - 3\lambda = 0 \qquad (1)$$

$$L_y = \frac{\partial L}{\partial y} = 60.75z^{\frac{1}{4}}y^{-\frac{1}{4}} - 6\lambda = 0 \qquad (2)$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = 90 - 3z - 6y = 0 \qquad (3)$$

Diving equation (1) by equation (2), we have:

$$\frac{20.25z^{-\frac{3}{4}}y^{\frac{3}{4}}}{60.75z^{\frac{1}{4}}y^{-\frac{1}{4}}} = \frac{3\lambda}{6\lambda}$$

$$\frac{20.25y}{60.75z} = \frac{3}{6}$$

$$121.5y = 182.25z$$

$$y = \frac{182.25z}{121.5}$$

$$y = 1.5z$$

Substituting for y in equation (3), we have that:

$$90 - 3z - 6(1.5z) = 0$$

$$90 - 3z - 9z = 0$$

$$12z = 90$$

$$z = \frac{90}{12}$$

$$z = 7.5$$

$$y = 1.5(7.5)$$

$$= 11.25$$

For the S. O. C. shows that $\begin{vmatrix} L_{zz} & L_{zy} \\ L_{yz} & L_{yy} \end{vmatrix} < 0$, therefore the solution set is a maximum

3.4.7. Models of Constant Elasticity:

Models of constant elasticity are most often given as: just as the name suggest, it is a model with a constant value of elasticity. The Constant Elasticity of Substitution Function (CES) function is a neoclassical function that displays constant elasticity of substitution. In other words, it is a technological function that has a constant percentage change in factor proportions due to a percentage change in marginal rate of technical substitution.

It can be given in its non-logarithmic form.

$$Q = A\left[\alpha L^{-\rho} + (1-\alpha)K^{-\rho}\right]^{-\frac{k}{\rho}}$$

The logarithmic representation of the model is given as:

$$Ln_e Q = Ln_e d - (c/a)Ln_e[bL^{-a} + (1-b)K^{-a}]$$

Where Q is the output of production,

L is labour man-hours,

K is capital stock [units],

d is production intercept,

b is constant labour share,

(1-b) is constant capital share,

e is natural logarithmic base.

The mathematical formulation of CES function is given as follows:

$$Y = W(qL^{-o} + (1-q)K^{-o}]^{-\frac{\varepsilon}{-o}}$$

Where, Y = quantity of output

- W = factor productivity
- q = share parameter
- L = units of labour inputs
- K = units if capital inputs
- $o = \dfrac{\sigma - 1}{\sigma}$ is substitution parameter
- = $\sigma = \dfrac{1}{1-\delta}$ is elasticity of substitution

- ε = homogeneity degree.
- Where ε = 1 **(**constant return to scale),
- ε < 1 (decreasing return to scale),
- ε > 1 (increasing return to scale).

The CES function exhibits constant elasticity of substitution between factors. Leontief and Cobb Douglas (CD) functions are special cases of the CES function. That is,

1. Suppose that substitution coefficient is equal to 1, that is, σ goes to 1, we have a linear or perfect substitutes function;
2. Suppose that the substitution coefficient is equal to zero, that is, σ goes to0 in the limit, we get the CD function; and
3. Suppose that the substitution coefficient approaches negative infinity we get the Leontief or perfect complements function.
4. The CES is homogenous of degree 1.

SELF ASSESSMENT EXERCISE

Explain what you understand by elasticity of substitution

Numerical Example 2: For the following CES function,

$$Q = 20\left[0.5z^{-(-0.5)} + 0.5y^{-(-0.5)}\right]^{-\frac{1}{(-0.5)}},$$

Cost constraint, $150 = 5z + 2y$

(a) Optimize the CES Production Technology subject to the constraint.

(b) Estimate the elasticity of substitution

(c) Interpret the results economically.

Solution to Numerical Example 1: Form the Langrangian function as follows:

$$L = 20\left[0.5z^{-(-0.5)} + 0.5y^{-(-0.5)}\right]^{-\frac{1}{(-0.5)}} + \lambda[150 - 5z - 2y]$$

$$L_z = \frac{\partial L}{\partial z} = 40\left[0.5z^{0.5} + 0.5y^{0.5}\right]\left(0.25z^{-0.5}\right) - 5\lambda = 0$$

$$= 10z^{-0.5}\left[0.5z^{0.5} + 0.5y^{0.5}\right] - 5\lambda = 0 \qquad (1)$$

$$L_y = \frac{\partial L}{\partial y} = 40\left[0.5z^{0.5} + 0.5y^{0.5}\right]\left(0.25y^{-0.5}\right) - 2\lambda = 0$$

$$= 10y^{-0.5}\left[0.5z^{0.5} + 0.5y^{0.5}\right] - 2\lambda = 0 \qquad (2)$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = 150 - 5z - 2y = 0 \qquad (3)$$

Diving equation (1) by equation (2), we have as follows:

$$\frac{10z^{-0.5}\left[0.5z^{0.5}+0.5y^{0.5}\right]}{10y^{-0.5}\left[0.5z^{0.5}+0.5y^{0.5}\right]}=\frac{-5\lambda}{-2\lambda}$$

$$\frac{z^{-0.5}}{y^{-0.5}}=2.5$$

$$\frac{y^{0.5}}{z^{0.5}}=2.5$$

$$y^{0.5}=2.5z^{0.5}$$

$$y=2.5^{2}z$$

$$y=6.25z$$

Substituting for y in equation (3), we have that:

$$\textit{Substituting for } y \textit{ in equation } (3), \textit{ we have}$$

$$150=5z+2(6.25z)$$

$$150=17.5z$$

$$z=\frac{150}{17.5}$$

$$z=8.57$$

$$y=6.25(8.57)$$

$$=53.57$$

For the S. O. C. show that $\begin{vmatrix} L_{zz} & L_{zy} & L_{z\lambda} \\ L_{yz} & L_{yy} & L_{y\lambda} \\ L_{\lambda z} & L_{\lambda y} & L_{\lambda\lambda} \end{vmatrix} > 0$

b. For elasticity of substitution, evaluate $\sigma=\frac{d\ln(z/y)}{d\ln(f_y/f_z)}=\frac{d(z/y)}{d(f_y/f_z)}\frac{f_y/f_z}{z/y}$

or simply compute, $\sigma=\frac{1}{1-\delta}$

$$\frac{1}{1+\rho}=\frac{1}{1-0.5}=2$$

c. Given that $\sigma>1$, substitution among factor inputs is said to be elastic. Suppose that the substitution coefficient is equal to zero, that is, σ goes to0 in the limit, we get the CD function; and suppose that the substitution coefficient approaches negative infinity we get the Leontief or perfect complements function.

SELF ASSESSMENT EXERCISE

Describe the properties of CES function
Explain the concept of elasticity of substitution

4.0 CONCLUSION

A mathematical model is an applied or empirical representation of the economic reality. Mathematical Models are therefore used for the sole purpose of testing economic theories. Economists use economic theories and models as primary tools of economic analysis. The treatment of the Cobb-Douglass function under this unit was carried out focussing on the Cobb-Douglas production technology and not the Cobb-Douglass utility function.

5.0 SUMMARY

In this unit, we have discussed economic model, components of a mathematical model, qualities of good mathematical models and types of models. Also, in this unit, we have discussed the meaning of CD function, mathematically formulated CD function, mathematically derived the relationship between CD function and CES function and also solved some numerical problems of CD function. The CES arises as a specific type of aggregator function which combines two types of factor inputs into an aggregate magnitude. This aggregator function exhibits constant elasticity of substitution.

6.0 TUTOR-MARKED ASSIGNMENT

1. Discuss the various components of a mathematical economic model.
2. What are the differences between static and dynamic models
3. Distinguish between an economic model and a mathematical model
4. Briefly discuss 6 different types of mathematical economic models.
5. How would you define models in economics?

6. When do we need to apply models in economic analysis?
7. How would you define a Cobb-Douglas function?
8. Given the CD equation:

$$Z = WL^{\alpha}K^{1-\alpha},\ [0 \le \alpha \le 1]$$

(a) Prove that $MP_{LK} = MP_{KL}$

(b) Prove that $Z_L L + Z_K K = Z$

(c) Prove that the elasticity of substitution $\sigma = 1$

9. Consider a firm annual production function is of the form given below:

$$G(z, y) = 4z^{0.25}y^{0.75}$$

Estimate the minimum cost of the firm producing 200 units of output given that the unit annual costs for z and y are given as ₦13 and ₦15 respectively.

10. Given the CD equation:

$$Z = L^{\alpha}K^{1-\alpha},\ [0 \le \alpha \le 1]$$

Suppose L *and* K both grow at constant, though different rates, that is,

$$L = L_0 e^{nt},\ K = K_0 e^{mt},$$

(a) Find $\frac{dz}{dt}$

(i) Using direct substitution

(ii) Using chain rule (t is "time" factor)

(b) Prove that the degree of homogeneity is 1

11. Consider that a firm's annual production function is of the form given below:

$$D(z, y) = 30z^{0.4}y^{0.6}$$

Estimate the maximum output (D) of the firm if it supplies total output of 460 while the unit price of input z is ₦24 and that of y is ₦60.

12. Given the CES function

$$Y = W(qL^{-o} + (1-q)K^{-o}]^{-\frac{\varepsilon}{-o}}$$

(a) Prove that the CES function is homogeneous of degree 1

(b) What is the economic interpretation of linear homogeneity

13. Given the CES function

$$Y = W(qL^{-o} + (1-q)K^{-o}]^{-\frac{\varepsilon}{-o}}$$

(a) Derive the elasticity of substitution for the CES function

(b) Define the elasticity of substitution. What is the economic implication of it?

14. For the following CES function,

$$Q = 550\left[0.6z^{-0.4} + 0.4y^{-0.4}\right]^{-\frac{1}{0.4}},$$

Cost constraint, $300 = 10K + 4L$

(a) Estimate the elasticity of substitution

(b) Interpret the results economically.

(c) Optimize the CES function subject to the constraint.

(d) Use the optimal capital-labour ratio to examine the optimization solution.

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UNIT 3: FUNCTIONS, TYPES OF FUNCTIONS AND CONTINUITY OF FUNCTIONS

CONTENTS

1.0 INTRODUCTION

This unit provides both theoretical and mathematical discussion of the concept of functions, types of functions and continuity of functions.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Explain the meaning of a function
- Discuss the various types of functions
- Explain the difference between functions and ordered pairs as well as functions of two or more independent variables

3.0 MAIN CONTENT

3.1. Function and Ordered Pair

Function:A function is a distinct relation which charts each entry of set G with a parallel entry of set H on the condition that both the sets G and H are non-empty. Having total regard to the previous definition, it can be said mathematically that a function is a binary relation over two arrays or circles of values that links to each entry or element of the first set to just one element of the second set. Typical examples are functions from integers to integers or from the real numbers to real numbers.Accordingly, f: G → H is a function such that for g∈G there is a unique entry h∈ H whereby (g, h) ∈ f. In effect, a function relates each element of a set with accurately one element of another set which are most often the same set. This can be demonstrated as follows:

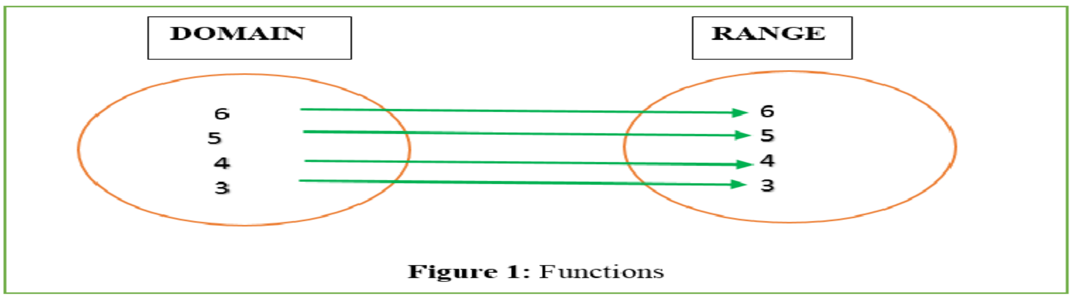

Figure 1: Graph of a Function
Source: Mathematics-Wikipedia

Ordered Pair: An ordered pair is a pair of elements a, b having the property that (g, h) = (s, m) if and only if g = s, and h = m. The relation is such that each value from the set of the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.

Consider the following relation, $[(-4,0)\ \ (1,6)\ \ (-3,2)\ (2,9)]$

From these ordered pairs, we can generate the sets of first components and second components as follows:

$$[(-4,0)\ \ (1,-6)\ \ (-3,2)\ (2,9)]$$
$$1st\ \ components\ \ [-4,1,-3,2]$$
$$2nd\ \ components\ [0,-6,2,9]$$

Ltaking 1 from the set of first components, it can be seen that there is exactly one ordered pair with 1 as a first component, $(1,-6)$. Therefore, the list of of values from the set of second components associated with 1 is exactly one number, -6. In effect, there should not more than one ordered pair with 1 as a first component. The relation is a function.

Consider the following relation, $[(5,0)\ \ (2,6)\ \ (7,-3)\ (5,4)]$

From these ordered pairs, it can be seen that there are two ordered pairs with 5 as a first component, $(5,0)$ & $(5,4)$. Therefore, the list of of values from the set of second components associated with 5 are two, namely 0 and 4. This negates the principle of ordered pair as there are more than one ordered pair with 5 as a first component. Thus, the relation is not a funtion.

3.2. Relationship between Function and Ordered Pairs: The relationship between a function and an ordered pair is intuitive in the sense that, a function is a process or a sequence that associates to each element of a set Z a single element of a set Y.

Mathematically therefore, a function *f* from a set Z to a set Y is defined by a set F of ordered pairs (z, y) such that $z \in Z$, $y \in Y$, and every element of Z is the first component of correctly one ordered pair in F.By implication, it so follows that for every z in Z, there is just one element y such that the ordered pair (z, y) belongs to the set of pairs outlining the function *f*. The set F is called the graph of the function as shown in figure 1 above.

Relatively, a relationship may not serve as a function because of the following:

i. A given value in set A has no relation in set B

ii. A given value in set A is related to more than one value in set B

The common notation of a "function" is f(z) ($f(x)$)called f of z or g(z) called g of z. Some Examples of functions include the following:

(1) "f of z equals z squared", that is, $F(z) = z^2$

(2) "f of z equals x cubed plus one", that is, $F(z) = z^3+1$

(3) $f(z) = 2 + z + z^3$

Function 3 can as well be written as:

- $f(s) = 2 + s + s^3$
- $g(z) = 2 + z + z^3$
- $h(w) = 2 + w + w^3$

The variable (s, z, x) all take the same position as x. So also are g and h taking same position as f. Consequently,

$$f(4) = 2 + 4 + 64 = 70$$

SELF ASSESSMENT EXERCISE

Explain the meaning of a function.

Explain the conditions that makes a "relation" to serve as a function.

What is a linear equation?

3.3Types of Functions

3.3.1. Explicit Function: Explicit function is the classic$y = f(x)$ type of function which shows how to go directly from x to y. this is specified as follows:

$$y = x^8 + 6x + 10$$

In this case, once x is known, y can be evaluated.

3.3.2. Implicit Function: Implicit function is the function that is not given directly. This can be demonstrated as follows:

$$x^5 – 5xy + y^2 = 0$$

Consequently, even when *x* is known, it requires an indirect approach to find y.

3.3.3. Constant Function: This is the function f*: R→R* whereby *f(x) = h = k*, for *z ∈R* and k is a constant in R. The domain of the function f is R and its range is a constant, k.

3.3.4. Signum Function: This is the sign function whereby *f: R→R* defined by *f(x) = {1, if x> 0; 0,* if *x = 0; -1, if x< 0.*

3.3.5. Quadratic Function: This is the degree two polynomial function given as $f(x) = dx^2 + ax + k$, where $d \neq 0$ and *d, a, k* are constant and *x* is a variable & R is both domain and range of the function.

3.3.6. Cubic Function: This a degree three polynomial function given as $f(x) = dx^3 + ax^2 + kx$ +g, where d ≠ 0 and d, a, k, and g are constant &x is a variable & R is both domain and range of the function.

3.3.7. Rational Function: This is a rational polynomial function where f: R→R is defined as = f(x)/g(y) in which g(y) ≠ 0.

3.3.8. Modulus Function: This is an absolute value function denoted | | such that *f: R→ R* is defined by *f(x)* = |*x*| and for every non-negative value of x, *f*(*x*) = z and for every negative value of *x*, *f(x)* = -*x*. accordingly therefore, *f(x)= {x, if x ≥ 0; – x, if x< 0.*

3.3.9. Exponential Functions

Exponential models are growth equations which when transformed becomes semi-logarithmic.

$$Q = de^{d_1 Z}$$

On transformation $\ln\left(\frac{Q}{d}\right) = d_1 Z$

Transformed $LnQ = \ln d + d_1 Z$

Where Q is dependent variable,
Z is explanatory variable,
d is intercept coefficient,
d_1 is slope coefficient.

3.3.10. Logarithmic Functions: Logarithmic function is the inverse of an exponential function. That means the logarithm of a given number x is the exponent to which the base *b*, must be raised, to have that number z.The logarithm of z to base b is denoted as $\log_b(z)$. Unambiguously, the defining relation between exponentiation and logarithm is:

$$\log_b(z) = y$$

$$where\ b^y = z,\ z > 0, b > 0, b \neq 1, (b = e = 2,718)$$

For purpose of pedagogy, it can be shown that the logarithm base 2 of 256 is 8, or $\log_2(256) = 8$. That is,

$$\log_b(z) = y \Leftrightarrow \log_2(256) = 8$$

$$as\ 2^8 = 256$$

Therefore, the logarithmic of 256 to base 2 is 8. The inverse of the exponential function $y = b^z\ is\ z = b^y$. So, the logarithmic function $\log_b(z)$ is equivalent to the exponential equation $z = b^y$.

3.3.11. Trigonometric or Sinusoidal function: Trigonometric functions are real functions which transmit an angle of a right-angled triangle to ratios of two side lengths. For example, given an acute angle A = θ of a right-angled triangle, the hypotenuse h is the side that connects the two acute angles. This is shown in figure 1 below:

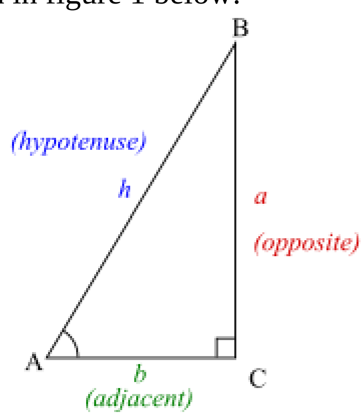

Figure 1: Right-Angled Triangle
Source: Wikipedia

If the angle θ is given, the ratio of any two side lengths is determined on the basis of θ. These ratios define different trigonometric functions of θ, which includesine, cosine, tangent, cotangent, secant and cosecant. These are shown in the table below:

Table 1: Relationships between Trigonometric Functions

Function	Radians Measurement
$\sin\theta = \dfrac{opp}{hyp}$	$\sin\theta = \dfrac{1}{\csc\theta}$
$\cos\theta = \dfrac{adj}{hyp}$	$\cos\theta = \dfrac{1}{\sec\theta}$
$\tan\theta = \dfrac{opp}{adj}$	$\tan\theta = \dfrac{1}{\cot\theta}$
$\csc\theta = \dfrac{hyp}{opp}$	$\csc\theta = \dfrac{1}{\sin\theta}$
$\sec\theta = \dfrac{hyp}{adj}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\cot\theta = \dfrac{adj}{opp}$	$\cot\theta = \dfrac{1}{\tan\theta}$

Trigonometric functions can be evaluated as coordinate values of points on the Euclidean plane that are related to the unit circleradius one centered at the origin of the coordinate system. Thus, whereas, the right-angled triangle explanationscertifies the classification of the trigonometric functions for angles between 0 and 90°, the unit circle classificationsprovides an extension of the domain of the trigonometric functions to all positive and negative real numbers.

In figure 2 below, the six trigonometric functions of angle θ are represented as Cartesian coordinates of points in relation to the unit circle. The ordinates of A = sin θ, B = tan θ and D = csc respectively, while the abscissas of A = cos θ , C = cotθ and E = sec θ correspondingly.

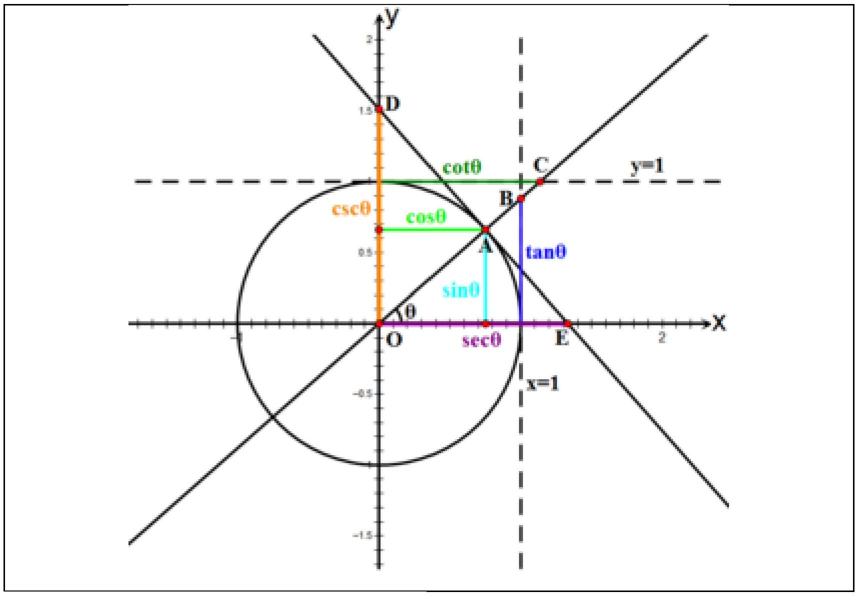

Figure 2: Unit Circle and Cartesian coordinates
Source: Wikipedia

Circling a ray from the direction of the positive half of the x-axis by an angle θ for >0, and for <0yields intersection with the unit circle at point A . Also, by extending the ray to the line *x*=1, we have a coordinate values of B and to the line of *y*=1, we have another coordinate points of C.

The tangent line to the unit circle at point A, intersects the *x*and *y*-axis at point E and D respectively. The coordinate values of these points give all the existing values of the trigonometric functions for real values of θ as the *x*- and *y*-coordinate values of point A such thatfulfills the Pythagorean identity in which: $\sin^2\theta + \cos^2\theta = 1$

Therefore, with the application of the Pythagorean identity, we have that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \cot\theta = \frac{\cos\theta}{\sin\theta}, \sec\theta = \frac{1}{\cos\theta}, \csc\theta = \frac{1}{\sin\theta}$$

SELF ASSESSMENT EXERCISE
Explain the meaning of a trigonometry function.
Explain the relationship between logarithmic and exponential functions

3.3 Continuity of Functions

A function *g(x)* is continuous at a point *z* = *b*, in its domain if the following three conditions are satisfied:

3. f(b) exists. In other words, the value of f(b) is finite
4. $\lim_{x\to b}$ f(x) exists and it is finite

5. $\lim_{x \to b} f(x) = f(b)$

In effect, the function *f(x)* is continuous in the interval $I = [x_1, x_2]$ if the conditions above are satisfied for each point in the interval I.

Examples of typical continuous functions include, trigonometric functions, polynomial functions and exponential functions as well as logarithmic functions in their domain.

SELF ASSESSMENT EXERCISE

Explain the different types of functions. Define a relation?

Under what conditions would a "relation" not served as a function?

3.3.1 Solving Numerical Problems on Continuity of Functions

Numerical Example 1:

Consider the function

$$g(x) = \frac{x-9}{x^2-81}$$

Determine whether or not the function is continuous (a) @ x =9, (b) @ x = 8.

Solution:

Applying the above conditions for continuity, we have as follows:

(a) @ x = 9,

$$g(9) = \frac{9-9}{9^2-81} = \frac{0}{0}$$. Thus, g(x) does not exist.

$$\ell im_{x \to 9} \left(\frac{x-9}{x^2-81} \right)$$

$$= \ell im_{x \to 9} \left(\frac{x-9}{(x+9)(x-9)} \right)$$

$$= \ell im_{x \to 9} \left(\frac{x-9}{x+9} \right)$$

$$= \frac{1}{9+9} = \frac{1}{18}$$

$$g(x) @ x = 9 \neq \ell\, im_{x \to 9} \left(\frac{x-9}{x^2-81} \right)$$

$$i.e.\ 0 \neq \frac{1}{18}$$

In sum, g(x) does not exist, *and* $\lim_{x \to b} g(x)$ = g(9)

Thus, the function is not continuous at $x = 9$.

(b) @ $x = 8$

$g(8) = \frac{8-9}{8^2-81} = \frac{1}{17}$. This indicates that $g(x)$ exists @$x=8$

$$\lim_{x \to 8} = \left(\frac{x-9}{x^2-81} \right) = \frac{8-9}{8^2-81} = \frac{1}{17}$$

$$g(x) @ x = 8 = \lim_{x \to 8} = \left(\frac{x-9}{x^2-81} \right)$$

Accordingly, the function is continuous at $x= 8$.

SELF ASSESSMENT EXERCISE

Describe the relationship between an explicit and an implicit function.

4.0 CONCLUSION

Operationally, a function is characterized in such a way that an input produces a corresponding output. This brings into focus the concept of ordered pair. The intuition is that a function is a rule that defines a relationship between dependent and independent variables. By swapping the dependent and independent variables in a given function, one obtains an inverse function and this in turn changes the roles of the variables in question.Functions are ever-present in mathematical analysis because they are essential for formulatinginteractions in the social and physical sciences.Non-algebraic functions, such as exponential and trigonometric functions, are transcendental functions.

5.0 SUMMARY

In this unit, we have discussed meaning of function, types of functions as well as continuity of functions and also we solved some numerical problems on continuity of functions.

6.0 TUTOR-MARKED ASSIGNMENT

1. Discuss with relevant equations, the relationship between function and ordered pairs
2. What are the differences between functions of two or more independent variables?
3. Using a pictographic analogy, distinguish between a relationship and a function
4. Is the following function continuous at the indicated points

$$f(d) = \frac{d^2 + 18d + 2}{d - 14} \quad @ \ d = 2, 6$$

5. Are the following functions continuous at the indicated point?

$$g(d) = 28d^2 - 10d + 9 \ @ \ d = 9$$

$$g(d) = \frac{d-8}{d^2 - 64} \ @ \ d = 0$$

6. Are the following functions continuous at the indicated point

$$g(d) = \frac{d^2 - 2d - 24}{d - 6} \ @ \ d = 4$$

$$g(d) = \frac{d^2 + 6d + 12}{d - 6} \ @ \ d = 8$$

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UNIT 4: EQUILIBRIUM ANALYSIS IN ECONOMICS

CONTENTS

1.0 INTRODUCTION

This unit focuses on concept of equilibrium in economics, partial equilibrium analysis, general equilibrium analysis and importance of equilibrium in economic analysis.

2.0 OBJECTIVES

After successful study of this unit, students should be able to do the following:

- Explain the meaning of equilibrium in economics
- Discuss the types of equilibrium in economics
- Explain why equilibrium is important in economic analysis

3.0 MAIN CONTENT

3.1 Concept of Equilibrium in Economics

Economic equilibrium is a state in which economic forces are balanced such that economic variables remain unchanged from their stability values in the absence of external stimuli. Economic equilibrium is also known as market equilibrium. The point of equilibrium represents a theoretical state of stability where all economic transactions that ought to take place with respect to the initial state of all relevant economic variables have taken place.

3.2 Types of Equilibrium in Economics

3.2.1 Partial Equilibrium/Marshallian Analysis: This is the type of equilibrium analysis that determinesthe price of one good with the assumption that prices of all other goods remain constant. A good example of Partial Equilibrium is Marshallian theory of demand and supply.

3.2.3 General Equilibrium/Walrasian equilibrium Analysis: This is Walrasian (Léon Walras, 1874) type of equilibrium analysiswhere determination of price of a good is considered within the context of several interacting markets. In effect, general equilibrium analyzes numerous markets in the quest of ascertaining interaction of demand and supply in an all-inclusive equilibrium.

For an economy to be in general equilibriumevery consumer~~s~~, every firm, every industry and every factor-service are in equilibrium simultaneously and they are interlinked through commodity and factor prices. Therefore, general equilibrium exists when all prices are stable; every consumer spend~~s~~ income in a manner that yields the maximum utility; every firm~~s~~ in every industry is in equilibrium at all prices and output while demand and supply for productive resources are synchronized.

Table 1: Difference between Partial and General Equilibrium

General Equilibrium	Partial Equilibrium
General equilibrium theory analyzes many markets.	Partial equilibrium theory only analyzes single markets.
Analyzes more than one variable	Analyzes single variable
All markets are cleared at a given price level in both product and factor markets	Only one market is cleared at a given price level.
There is an effect on other sectors due to change in one sector.	Other sectors of economy are not affected due to change in one sector.
Different sectors of the economy are jointly interdependent.	Based on assumption of *Ceteris Paribus*.
Prices of goods are determined simultaneously and mutually.	With *Ceteris Paribus* assumption, price of a good is determined.

Stable Equilibrium and Unstable equilibrium should be considered as well

SELF ASSESSMENT EXERCISE

Differentiate between partial and general equilibrium analysis

3.3 Importance of Equilibrium Analysis in Economic Analysis

(a) Useful in explaining functions of price system in the economy

(b) Useful in providing basis for the input-output analysis

(c) Useful for analyzing problems of the market together with the working of the economic system

(d) It provides basis for evaluating determinants of the patterns of the economy

SELF ASSESSMENT EXERCISE

Describe in your own words role of equilibrium in economic analysis

4.0 CONCLUSION

Economic equilibrium is a fundamentally theoretical construct that may never actually occur in an economy because the conditions underlying demand and supply are often dynamic and uncertain. Consequently, given that the state of all relevant economic variables changes constantly, it would be difficult for any economy to actually reach economic equilibrium in practice.

5.0 SUMMARY

In this unit, we have been able to discuss the concept of equilibrium in economics, types of equilibrium in economics as well as the importance of equilibrium in economic analysis.

6.0 TUTORED MARKED ASSIGNMENTS

1. Briefly give an overview equilibrium analysis in economics.
2. In what ways is partial equilibrium analysis different from general equilibrium analysis?
3. Describe the rationale for equilibrium analysis in economics

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MODULE 2: MATRIX ALGEBRA, SYSTEM OF LINEAR EQUATIONS AND MATRIX APPLICATION TO ECONOMICS: INPUT-OUTPUT ANALYSIS

UNIT 1: MATRIX-ALGEBRA

CONTENTS

1.0 INTRODUCTION

In this unit, we shall be discussing the meaning of matrix as well as the size of matrix, matrix representation of system of linear equations, minor, cofactors and the adjoint of a matrix as well as thebasic operations with matrix.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Understand the meaning of a matrix
- Demonstrate an understanding of what the size of a matrix is all about
- Represent a system of linear equations in matrix format
- Obtain the minors of a matrix and be able to calculate the cofactors and adjoint of a matrix
- Carry out basic operations with matrix.

3.0 MAIN CONTENT

3.1 Meaning of Matrix and Size of Matrix:A matrix is a quadrangular collection of numbers, parametersor constants for which mathematical summations and multiplications are performed. In particular therefore, a matrix is a set of figures arranged in rows and columns so as to form a rectangular array. The numbers are called the elements, or entries, of the matrix. In an (h × g) matrix, the h rows are horizontal and the g columns are vertical. Each element of a matrix is often symbolized by a variable with two subscripts. For example, z2,1 represents the element at the second row and first column of the matrix.

The matrix size is determined by the number of rows and columns therein. A matrix with m rows and n columns is called an (m × n) which reads, "m-by-n" matrix, with m and n called dimensions of the matrix.

SELF ASSESSMENT EXERCISE
Define a matrix. What do you understand as the size of a matrix?

3.2 Matrix Representation of System of Linear Equations: A system of linear n-equations can be symbolized in matrix form using a coefficient matrix, a variable matrix, and a constant matrix. The standard matrix representation of a system of linear equations is given thus:

$$Az = B$$

Where A is the coefficient matrix,
Z is the column vector of unknown variables,
B is the column vector of constant,
Consider the system of two equations in two variables,

$$6w + 12y = 32$$

$$5w - 3y = -6$$

The simultaneous system can be represented in matrix format as follows:

$$\begin{pmatrix} 6 & 12 \\ 15 & -3 \end{pmatrix}\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ -6 \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & 12 \\ 15 & -3 \end{pmatrix}, \qquad z = \begin{pmatrix} w \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 32 \\ -6 \end{pmatrix}$$

In a similar way, for a system of three equations in three variables,

$$3w + 6y + 9x = 15$$

$$6w + 12y + 15x = 12$$

$$9w + 15y + 18x = 21$$

Now, the simultaneous system can be represented in matrix format as follows:

$$\begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{pmatrix}\begin{pmatrix} w \\ y \\ x \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 21 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{pmatrix}, \qquad z = \begin{pmatrix} w \\ y \\ x \end{pmatrix}, \qquad B = \begin{pmatrix} 15 \\ 12 \\ 21 \end{pmatrix}$$

Types of Matrices: There are different types of matrices. The common types include the **Row and Column Vectors**: A row matrix is a matrix with one row and it is a (1 by n) matrix. This is given below:

$$\begin{bmatrix} 7 & 4 & 5 \end{bmatrix}$$

A column vector is a matrix with one column and it is denoted by (n by 1) matrix. Thi is given below:

$$\begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix}$$

Square matrix: A square matrix is a matrix with the similar number of rows and columns. Hence, it is popularly referred to as an n by n matrix of order n. Any two square matrices of the matching order can be added and multiplied.

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix}$$

Diagonal Matrix: The diagonal matrix is the type of matrix in which all entries outside the main diagonal are zero. An example is given below:

$$\begin{vmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & b \end{vmatrix}$$

Upper Diagonal Matrix: The Upper diagonal matrix is the type in whichall entries of the matrix below the main diagonal are zero. An example is given below:

$$\begin{vmatrix} a & c & f \\ 0 & d & h \\ 0 & 0 & b \end{vmatrix}$$

Lower Diagonal Matrix: The lower diagonal matrix is the type in which all entries of the matrix above the main diagonal are zero. An example is given below:

$$\begin{vmatrix} a & 0 & 0 \\ c & d & 0 \\ f & e & b \end{vmatrix}$$

Identity Matrix: The identity matrix is the n by n matrix in which all the elements along the principal diagonal are equal to 1 and all other elements are equal to 0. This is an example:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

3.3. Minors of a Matrix: The minor of a matrix A is the determinant of some minor square matrix, taken out from the original matrix by removing one or more of its rows and columns. Minors obtained by removing just one row and one column from square matrices (first minors) are required for calculating matrix cofactors, which in turn are useful for computing both the determinant and inverse of square matrices.

Let the matrix A be given by:

$$A = \begin{vmatrix} a & d & e \\ b & c & f \\ g & w & h \end{vmatrix}$$

The Matrix of Minors is generated thus: For each element of the matrix, ignore the values on the current row and column, calculate the determinant of the remaining values and obtain the "Matrix of Minors".

$$Matrix\ \ of\ Minors = \begin{bmatrix} \begin{vmatrix} c & f \\ w & h \end{vmatrix} & \begin{vmatrix} b & f \\ g & h \end{vmatrix} & \begin{vmatrix} b & c \\ g & w \end{vmatrix} \\ \begin{vmatrix} d & e \\ w & h \end{vmatrix} & \begin{vmatrix} a & e \\ g & h \end{vmatrix} & \begin{vmatrix} a & d \\ g & w \end{vmatrix} \\ \begin{vmatrix} d & e \\ c & f \end{vmatrix} & \begin{vmatrix} a & e \\ b & f \end{vmatrix} & \begin{vmatrix} a & d \\ b & c \end{vmatrix} \end{bmatrix}$$

3.4. Cofactors of a Matrix

A cofactor is the number obtained following the remove of the column and row of a chosen element in a matrix.The cofactors feature prominently in Laplace's formula for the expansion of determinants, which is a method of computing higher determinants in terms of lesser determinants.

In effect therefore, the cofactor matrix of A is the n × n matrix C whose (i, j) entry is the (i, j) cofactor of A, which is the (i, j)-minor times a sign factor as shown below:

$$Sign\ Matrix \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C_{ij} \equiv \begin{bmatrix} (+)C_{11} & (-)C_{12} & (+)C_{13} \\ (-)C_{21} & (+)C_{22} & (-)C_{23} \\ (+)C_{31} & (-)C_{32} & (+)C_{33} \end{bmatrix}^T$$

Algebraically, accepting the definition that $C_{ij} = (-1)^{i+j} S_{ij}$ it thus implies that both the cofactor expansion along the jthand the ithcolumns respectively are given by:

$$|A| = \varphi_{1j}C_{1j} + \varphi_{2j}C_{2j} + \varphi_{3j}C_{3j} + \varphi_{4j}C_{4j} + \ldots$$
$$+ \varphi_{nj}C_{nj} = \sum_{j=1}^{n} \varphi_{ij}C_{ij} = \sum_{j=1}^{n} \varphi_{ij}(-1)^{i+j} S_{ij}$$
$$|A| = \varphi_{i1}C_{i1} + \varphi_{i2}C_{i2} + \varphi_{i3}C_{i3} + \varphi_{i4}C_{i4} + \ldots$$
$$+ \varphi_{in}C_{in} = \sum_{j=1}^{n} \varphi_{ij}C_{ij} = \sum_{j=1}^{n} \varphi_{ij}(-1)^{i+j} S_{ij}$$

Given an n × n matrix $A = (\varphi_{ij})$, then $|A|$ equals the sum of the cofactors of any row or column of the matrix multiplied by the entries that generated them.

The Matrix of Minors is turned into the Matrix of cofactors by applying the checker sign matrix of minuses/pluses to the "Matrix of Minors". In other words, we need to change the sign of every element of the Matrix of Minor by multiplying the Matrix of Minors by each minus and plus element in the sign matrix below:

$$Sign\ Matrix \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Matrix\ \ of\ \ Cofactors = \begin{bmatrix} (+)\begin{vmatrix} c & f \\ w & h \end{vmatrix} & (-)\begin{vmatrix} b & f \\ g & h \end{vmatrix} & (+)\begin{vmatrix} b & c \\ g & w \end{vmatrix} \\ (-)\begin{vmatrix} d & e \\ w & h \end{vmatrix} & (+)\begin{vmatrix} a & e \\ g & h \end{vmatrix} & (-)\begin{vmatrix} a & d \\ g & w \end{vmatrix} \\ (+)\begin{vmatrix} d & e \\ c & f \end{vmatrix} & (-)\begin{vmatrix} a & e \\ b & f \end{vmatrix} & (+)\begin{vmatrix} a & d \\ b & c \end{vmatrix} \end{bmatrix}$$

3.5 Adjoint of a Matrix: The adjoint also called the adjugate, is the transpose of its cofactor matrix. In effect, the "adjoint" of a matrix denotes the equivalent adjoint operator, which is its conjugate transpose. Thus, for the matrix A, the adjoint is given by:

$$Adj(A) = \begin{bmatrix} (+)C_{11} & (-)C_{12} & (+)C_{13} \\ (-)C_{21} & (+)C_{22} & (-)C_{23} \\ (+)C_{31} & (-)C_{32} & (+)C_{33} \end{bmatrix}^T$$

$$Adj(A) = C^T = [(-1)^{i+j} S_{ij}]_{(1 \le i,\, j \le n)}$$

Where C is the cofactor matrix of A, that is, $C = [(-1)^{i+j} S_{ij}]_{(1 \le i,\, j \le n)}$

S_{ij} is the (i,j) minor of matrix A, and it is the determinant of the (n − 1) × (n − 1) matrix that results from deleting row i and column j of A.

Consequently, defining the adjoint of matrix A such that the product of A with its adjugate yields a diagonal matrix whose diagonal entries are the determinant det(A), we then have:

$$A adj(A) = adj(A)A \equiv |A| I$$

where I is the n×n identity matrix.

Given that matrix A is invertible provided the determinant of A is invertible, it then implies that:

$$adj(A) = |A| A^{-1}$$

3.2. Basic Operations with Matrix

The basic operations that can be carried out with matrices include matrix addition, scalar multiplication, transposition, matrix multiplication, and sub-matrix.

Matrix Addition: The sum A+B of two m-by-n matrices A and B is calculated entry wise:

$$A=\begin{bmatrix}1 & 6 & 9\\ 4 & 2 & 8\\ 3 & 5 & 7\end{bmatrix}, B=\begin{bmatrix}0 & 7 & 9\\ 3 & 1 & 2\\ 5 & 4 & 6\end{bmatrix}$$

$$A+B=\begin{bmatrix}1+0 & 6+7 & 9+9\\ 4+3 & 2+1 & 8+2\\ 3+5 & 5+4 & 7+6\end{bmatrix}=\begin{bmatrix}1 & 13 & 18\\ 7 & 3 & 10\\ 8 & 9 & 13\end{bmatrix}$$

Scalar Multiplication: Scalar multiplication of kA where k is a constant and A is an (n by n) matrix is calculate by multiplying every entry of A by k.Thus, (kA)i,j = k · Ai,j. This is shown below:

$$A=\begin{bmatrix}-3 & 4 & 8\\ 4 & 10 & 7\\ 5 & -9 & 2\end{bmatrix},$$

$$4A=\begin{bmatrix}-12 & 16 & 32\\ 16 & 40 & 21\\ 20 & 36 & 8\end{bmatrix}$$

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is an m-by-n matrix and B is an n-by-g matrix, then their matrix product AB is the m-by-g matrix whose entries are given by product of the corresponding row of A and the corresponding column of B:

$$A=\begin{bmatrix}3 & 4 & 5\\ 1 & 0 & 6\\ 5 & 1 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 5\\ 2 & -4 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}3(1)+4(0)+5(2) & 3(0)+4(3)+5(-4) & 3(2)+4(5)+5(0)\\ 1(1)+0(0)+6(2) & 1(0)+0(3)+6(-4) & 1(2)+0(5)+6(0)\\ 5(1)+1(0)+2(2) & 5(0)+1(3)+2(-4) & 5(2)+1(5)+2(0)\end{bmatrix}$$

$$=\begin{bmatrix}13 & 3 & 15\\ 13 & -24 & 2\\ 9 & -5 & 15\end{bmatrix}$$

Matrix Transposition: The transpose of a matrix is the carried out by turning columns into rows and rows into columns. For an m-by-n matrix A, the transposition is simply the n-by-m matrix symbolized by A^T.

$$A = \begin{bmatrix} 6 & 10 & 7 \\ 5 & -4 & 2 \end{bmatrix},$$

$$A^T = \begin{bmatrix} 6 & 5 \\ 10 & -4 \\ 7 & 2 \end{bmatrix}$$

Submatrix Operation: The submatrix operation of a matrix is obtained by cancelling any collection of rows and columns. For example, from the subsequent 3-by-4 matrix, we can construct a 2-by-3 submatrix by removing row 3 and column 2:

$$A = \begin{bmatrix} 0 & 4 & 8 & 1 \\ 4 & -5 & 7 & 4 \\ 2 & -9 & 2 & 6 \end{bmatrix},$$

$$\equiv \begin{bmatrix} 4 & -5 & 7 \\ 2 & -9 & 2 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 0 & 8 & 1 \\ 4 & 7 & 4 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 0 & 4 & 8 \\ 4 & -5 & 7 \end{bmatrix}$$

SELF ASSESSMENT EXERCISE

Describe the following: matrix addition, scalar multiplication, transposition, matrix multiplication, and sub-matrix.

4.0 CONCLUSION

Matrices have wide ranging applications in economics. We have different types of matrix, and these matrices all play significant roles in matrix application.The different types of row operations which include addition, subtraction and the interchange of two rows of a matrix all jointly play significant roles including finding solutions to linear equations and evaluating matrix inverses.

5.0 SUMMARY

In this unit, we have discussedconcept of matrix and size of matrix, evaluated basic operations with matrices.

7.0 TUTOR-MARKED ASSIGNMENT

1. Add up the following matrices

$$A=\begin{bmatrix} 9 & 6 & 8 \\ 3 & 5 & 7 \end{bmatrix}, B=\begin{bmatrix} 0 & 7 & -4 \\ 3 & 1 & 2 \end{bmatrix}$$

$$C=\begin{bmatrix} 2 & 6 & 9 \\ 4 & 7 & 8 \\ 3 & 5 & 7 \end{bmatrix}, D=\begin{bmatrix} 6 & 7 & 9 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

2 Form 3 sub matrices of order 2 by 3 from the following 3 by 4 matrix

$$A=\begin{bmatrix} 12 & 16 & 8 & 11 \\ 14 & -5 & 27 & 24 \\ 2 & 39 & 0 & 15 \end{bmatrix},$$

3. Multiply the following matrices:

$$A=\begin{bmatrix} 19 & 60 & 2 \\ 3 & 37 & 20 \\ 1 & 0 & 13 \end{bmatrix}, B=\begin{bmatrix} 0 & 7 & -4 \\ 3 & 1 & 2 \\ 4 & 5 & 9 \end{bmatrix}$$

$$C=\begin{bmatrix} 2 & 6 & 0 \\ 4 & 7 & 3 \\ 3 & 0 & 7 \end{bmatrix}, \quad D=\begin{bmatrix} 6 & 8 & 9 \\ 3 & 0 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

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UNIT 2: SYSTEM OF LINEAR EQUATIONS AND CRAMER'S RULE

CONTENTS

2.0 INTRODUCTION

In this unit, we shall be discussing the meaning of Cramer's rule, determinants and the features of determinants, sign rule, Cramer's rule technique and also solved some linear equations with Cramer's rule.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Understand what Cramer's rule is all about
- Demonstrate the application of Cramer's rule
- Solve simultaneous equation problems with Cramer's rule

3.0 MAIN CONTENT

3.1 Meaning of Cramer's Rule: The Cramer's rule due to Gabriel Cramer (1704–1752). Cramer's rule is a mathematical formulae for the finding solution of a system of linear equations by means of determinants. It expresses the solution in terms of the determinants of the (square) coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right-hand-sides of the equations.

Cramer's rule is computationally effectual for finding the solution of a system of two or three linear equations. In the case of n equations in n unknowns, it requires computation of n + 1 determinants.

SELF ASSESSMENT EXERCISE

What do you understand by Cramer's rule?

3.2 Determinants and the Features of Determinants: Thedeterminant is a scalar value that can be computed from the elements of a square matrix and translates properties of the linear transformation described by the matrix. The determinant of a matrix W is denoted det(W), or |W|. Mathematically, it is the volume scaling factor of the linear transformation defined by the matrix such that we obtained a positive determinant or a negative determinant depending on whether or not the linear mapping preserves or reverses the orientation of n-space.

The features of determinants are hereby summarized as follows:

1. Determinant of a matrix in upper triangular arrangement is equal to the product of entries along the main diagonal.
2. The interchange of two rows leaves the matrix determinant to change sign.
3. Determinant would be equal to zero when two rows are the same or two columns are the same.
4. Determinant of a matrix whose row or column has zero values is equal to zero.
5. The determinant of an inverse matrix $\left|A^{-1}\right|$ is the reciprocal of the determinant of the matrix, A.
6. Multiplying a column or a row of a matrix by a constant increases the determinant by same constant.

Considering a 2 × 2 matrix, the determinant may be computed as:

$$A = \begin{pmatrix} 30 & 0 \\ 12 & -3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 30 & 0 \\ 12 & -3 \end{vmatrix} = 30(-3) - 0(12) = -90$$

Correspondingly, for a 3 × 3 matrix *A*, the determinant is calculated thus:

$$A = \begin{pmatrix} 3 & 0 & 4 \\ 6 & -3 & 7 \\ 5 & 2 & 1 \end{pmatrix}$$

$$\begin{aligned} |A| = \begin{vmatrix} 3 & 0 & 4 \\ 6 & -3 & 7 \\ 5 & 2 & 1 \end{vmatrix} &= 3\begin{vmatrix} -3 & 7 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 6 & 7 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 6 & -3 \\ 5 & 2 \end{vmatrix} \\ &= -3[-3(1) - 2(7)] - 0 + 4[2(6) - 5(-3)] \\ &= -51 + 108 \\ &= 57 \end{aligned}$$

In the above evaluation of determinant, we have the Laplace formula given by:

$$|A| = \begin{vmatrix} 3 & 0 & 4 \\ 6 & -3 & 7 \\ 5 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} -3 & 7 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 6 & 7 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 6 & -3 \\ 5 & 2 \end{vmatrix}$$

Likewise manner, we have the Leibniz formula for the determinant given by:

$$|A| = \begin{vmatrix} 3 & 0 & 4 \\ 6 & -3 & 7 \\ 5 & 2 & 1 \end{vmatrix} = -3[-3(1) - 2(7)] - 0 + 4[2(6) - 5(-3)]$$

Higher determinants are evaluated following a stepwise procedure, expanding them into sums of terms, each the product of a coefficient and a smaller determinant. Any row or column of the matrix is selected, each of its elements arc is multiplied by the factor (−1)r + c and by the smaller determinant formed by deleting the ith row and jth column from the original array. Each of these products is expanded in the same way until the small determinants can be evaluated by checkup. At each stage, the process is facilitated by choosing the row or column containing the most zeros.

3.4 Cramer's Rule: Cramer's rule was invented by Gabriel Cramer (1704 – 1752). In linear algebra, Cramer's rule is an explicit formula for obtaining a unique solution of a system of linear equations by expressing the solution in terms of the determinants of the coefficient matrix and of matrices obtained from it by replacing one column by the column vector of right-hand-sides of the equations. Given the system of n linear equations for n unknowns, represented in matrix form as follows:

$$Az = B$$

A is $(n \times n)$ coefficient matrix *with* a non-zero determinant, z is the vector of variables and B is the column vector of the variables. The Cramer's rule is thus given by:

$$z_i = \frac{\det(A_i)}{\det(A)}$$

$where\ i = 1, 2, 3, \ldots, n$, A_i is the matrix formed by replacing the ith column of coefficient matrix A by the column vector B.

3.3.1. Cramer's Rule for Two Variable Matrix: Given a systems of Linear Equations with Two Variables as shown below:

$$\phi x + \beta y = b$$
$$\delta x + \alpha y = d$$
$$\begin{pmatrix} \phi & \beta \\ \delta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

Thus, we obtain the coefficient, x and y matrices as follows:

$$coefficient\ matrix,\ A = \begin{pmatrix} \phi & \beta \\ \delta & \alpha \end{pmatrix}$$
$$x\ \ matrix,\ D_x = \begin{pmatrix} b & \beta \\ d & \alpha \end{pmatrix}$$
$$y\ \ matrix,\ D_y = \begin{pmatrix} \phi & b \\ \delta & d \end{pmatrix}$$

Finding the solution for variable x, we have it as follows:

$$x = \frac{D_x}{A} = \frac{\begin{pmatrix} b & \beta \\ d & \alpha \end{pmatrix}}{\begin{pmatrix} \phi & \beta \\ \delta & \alpha \end{pmatrix}}$$

Finding the solution for variable y, we have it as follows:

$$y = \frac{D_y}{A} = \frac{\begin{pmatrix} \phi & b \\ \delta & d \end{pmatrix}}{\begin{pmatrix} \phi & \beta \\ \delta & \alpha \end{pmatrix}}$$

Observe that both denominators in finding the solutions of x and y are the same. They emanates from the columns of x and y respectively.

3.3.2. Cramer's Rule for Three Variable Matrix: Given a systems of Linear Equations with three Variables as shown below:

$$\begin{aligned} \phi x + \beta y + \varphi z &= b \\ \delta x + \alpha y + \theta z &= d \\ \eta x + \omega y + \gamma z &= c \end{aligned}$$

In matrix notation, we have it as:

$$\begin{pmatrix} \phi & \beta & \varphi \\ \delta & \alpha & \theta \\ \eta & \omega & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ d \\ c \end{pmatrix}$$

Thus, we obtain the coefficient, x and y matrices as follows:

$$coefficient\ matrix,\ A = \begin{pmatrix} \phi & \beta & \varphi \\ \delta & \alpha & \theta \\ \eta & \omega & \gamma \end{pmatrix}$$

$$x\ \ matrix,\ D_x = \begin{pmatrix} b & \beta & \varphi \\ d & \alpha & \theta \\ c & \omega & \gamma \end{pmatrix}$$

$$y\ \ matrix,\ D_y = \begin{pmatrix} \phi & b & \varphi \\ \delta & d & \theta \\ \eta & c & \gamma \end{pmatrix}$$

$$z\ matrix,\ D_z = \begin{pmatrix} \phi & \beta & b \\ \delta & \alpha & d \\ \eta & \omega & c \end{pmatrix}$$

Finding the solution for variable x, we have it as follows:

$$x = \frac{D_x}{A} = \frac{\begin{pmatrix} b & \beta & \varphi \\ d & \alpha & \theta \\ c & \omega & \gamma \end{pmatrix}}{\begin{pmatrix} \phi & \beta & \varphi \\ \delta & \alpha & \theta \\ \eta & \omega & \gamma \end{pmatrix}}$$

Finding the solution for variable y, we have it as follows:

$$y = \frac{D_y}{A} = y = \frac{D_y}{A} = \frac{\begin{pmatrix} \phi & b & \varphi \\ \delta & d & \theta \\ \eta & c & \gamma \end{pmatrix}}{\begin{pmatrix} \phi & \beta & \varphi \\ \delta & \alpha & \theta \\ \eta & \omega & \gamma \end{pmatrix}}$$

Finding the solution for variable z, we have it as follows:

$$z = \frac{D_z}{A} = z = \frac{D_z}{A} = \frac{\begin{pmatrix} \phi & \beta & b \\ \delta & \alpha & d \\ \eta & \omega & c \end{pmatrix}}{\begin{pmatrix} \phi & \beta & \varphi \\ \delta & \alpha & \theta \\ \eta & \omega & \gamma \end{pmatrix}}$$

From the above procedures, observe the following:

i. Both denominators in finding the solutions of x, yand z are the same. They emanates from the columns of x, yand z respectively.

ii. The numerator in finding the solution for x is such that the coefficients of x-column are replaced by the constant column

iii. Also, the numerator in finding the solution for y is such that the coefficients of y-column are replaced by the constant column

3.5. Solving Linear Equations with Cramer's Rule: Consider our system of two equations in two variables above,

$$\begin{pmatrix} 6 & 12 \\ 15 & -3 \end{pmatrix}\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ -6 \end{pmatrix}$$

The column vector of unknown, *w* and *y* can be calculated using Cramer's rule as:

$$w = \frac{\begin{vmatrix} 32 & 12 \\ -6 & -3 \end{vmatrix}}{\begin{vmatrix} 6 & 12 \\ 15 & -3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 6 & 32 \\ 15 & -6 \end{vmatrix}}{\begin{vmatrix} 6 & 12 \\ 15 & -3 \end{vmatrix}}$$

Evaluating the determinants for w and y respectively, we have it as follows:

$$w = \frac{\begin{vmatrix} 32 & 12 \\ -6 & -3 \end{vmatrix}}{\begin{vmatrix} 6 & 12 \\ 15 & -3 \end{vmatrix}} = \frac{32(-3) - 12(-6)}{6(-3) - 12(15)} = \frac{-24}{-198} = 0.12$$

$$y = \frac{\begin{vmatrix} 6 & 32 \\ 15 & -6 \end{vmatrix}}{\begin{vmatrix} 6 & 12 \\ 15 & -3 \end{vmatrix}} = \frac{6(-6) - 32(15)}{6(-3) - 12(15)} = \frac{-516}{-198} = 2.6$$

Consider our system of three equations in three variables above,

$$\begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{pmatrix}\begin{pmatrix} w \\ y \\ x \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 21 \end{pmatrix}$$

The column vector of unknown, *w*, *y* and *e* can be calculated using Cramer's rule as:

$$w = \frac{\begin{vmatrix} 15 & 6 & 9 \\ 12 & 12 & 15 \\ 21 & 15 & 18 \end{vmatrix}}{\begin{vmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 3 & 15 & 9 \\ 6 & 12 & 15 \\ 9 & 21 & 18 \end{vmatrix}}{\begin{vmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{vmatrix}}, \quad x = \frac{\begin{vmatrix} 3 & 6 & 15 \\ 6 & 12 & 12 \\ 9 & 15 & 21 \end{vmatrix}}{\begin{vmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{vmatrix}}$$

Numerical Example 3: Solve the system of equations using Cramer’s Rule.

$$36w + 9y = 45$$
$$6w - 9y = 39$$

Solution to Numerical Example 3: Representing the simultaneous system in matrix format, we have as follows:

$$\begin{bmatrix} 36 & 9 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} 45 \\ 39 \end{bmatrix}$$

$$\begin{vmatrix} 36 & 9 \\ 6 & -9 \end{vmatrix} = -324 - 54$$

$$= -378$$

$$w = \frac{\begin{vmatrix} 45 & 9 \\ 39 & -9 \end{vmatrix}}{\begin{vmatrix} 36 & 9 \\ 6 & -9 \end{vmatrix}}$$

$$= -405 - 351$$

$$= -756$$

$$w = \frac{-756}{-378}$$

$$= 2$$

$$y = \frac{\begin{vmatrix} 36 & 45 \\ 6 & 39 \end{vmatrix}}{\begin{vmatrix} 36 & 9 \\ 6 & -9 \end{vmatrix}}$$

$$= 1404 - 270$$

$$= 1134$$

$$y = \frac{1134}{-378}$$

$$= -3$$

Numerical Example 4: Find the solution to the following system of equations using Cramer's Rule.

$$3w + 3y - 3a = 18$$
$$9w - 6y + 3a = -15$$
$$3w + 9y - 6a = 42$$

Solution to Numerical Example 4: Representing the simultaneous system in matrix format, we have as follows:

$$\begin{bmatrix} 3 & 3 & -3 \\ 9 & -6 & 3 \\ 3 & 9 & -6 \end{bmatrix}\begin{bmatrix} w \\ y \\ a \end{bmatrix} = \begin{bmatrix} 18 \\ -15 \\ 42 \end{bmatrix}$$

$$\begin{vmatrix} 3 & 3 & -3 \\ 9 & -6 & 3 \\ 3 & 9 & -6 \end{vmatrix} = 3\begin{vmatrix} -6 & 3 \\ 9 & -6 \end{vmatrix} - 3\begin{vmatrix} 9 & 3 \\ 3 & -6 \end{vmatrix} - 3\begin{vmatrix} 9 & -6 \\ 3 & 9 \end{vmatrix}$$

$$\begin{aligned} &= 3(9) - 3(-63) - 3(99) \\ &= 27 + 189 - 297 \\ &= -81 \end{aligned}$$

$$D_w = \begin{vmatrix} 18 & 3 & -3 \\ -15 & -6 & 3 \\ 42 & 9 & -6 \end{vmatrix} = 18\begin{vmatrix} -6 & 3 \\ 9 & -6 \end{vmatrix} - 3\begin{vmatrix} -15 & 3 \\ 42 & -6 \end{vmatrix} - 3\begin{vmatrix} -15 & -6 \\ 42 & 9 \end{vmatrix}$$

$$\begin{aligned} &= 18(9) - 3(-36) - 3(117) \\ &= 162 + 108 - 351 \\ &= -81 \end{aligned}$$

$$w = \frac{\begin{vmatrix} 18 & 3 & 3 \\ 15 & 6 & 3 \\ 42 & 9 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 3 & 3 \\ 9 & 6 & 3 \\ 3 & 9 & 6 \end{vmatrix}}$$

$$\begin{aligned} w &= \frac{-81}{-81} \\ &= 1 \end{aligned}$$

$$D_y = \begin{vmatrix} 3 & 18 & -3 \\ 9 & -15 & 3 \\ 3 & 42 & -6 \end{vmatrix} = 3\begin{vmatrix} -15 & 3 \\ 42 & -6 \end{vmatrix} - 18\begin{vmatrix} 9 & 3 \\ 3 & -6 \end{vmatrix} - 3\begin{vmatrix} 9 & -15 \\ 3 & 42 \end{vmatrix}$$

$$\begin{aligned} &= 3(-36) - 18(-63) - 3(423) \\ &= -108 + 1134 - 1269 \\ &= -243 \end{aligned}$$

$$y=\frac{\begin{vmatrix}3 & 18 & -3\\ 9 & -15 & 3\\ 3 & 42 & -6\end{vmatrix}}{\begin{vmatrix}3 & 3 & -3\\ 9 & -6 & 3\\ 3 & 9 & -6\end{vmatrix}}$$

$$y=\frac{-243}{-81}$$
$$=3$$

$$D_a=\begin{vmatrix}3 & 3 & 18\\ 9 & -6 & -15\\ 3 & 9 & 42\end{vmatrix}=3\begin{vmatrix}-6 & -15\\ 9 & 42\end{vmatrix}-3\begin{vmatrix}9 & -15\\ 3 & 42\end{vmatrix}+18\begin{vmatrix}9 & -6\\ 3 & 9\end{vmatrix}$$
$$=3(-117)-3(423)+18(99)$$
$$=-351-1269+1782$$
$$=162$$

$$a=\frac{\begin{vmatrix}3 & 3 & 18\\ 9 & -6 & -15\\ 3 & 9 & 42\end{vmatrix}}{\begin{vmatrix}3 & 3 & 3\\ 9 & 6 & 3\\ 3 & 9 & 6\end{vmatrix}}$$

$$a=\frac{162}{-81}$$
$$=-2$$

SELF ASSESSMENT EXERCISE

Describe the relationship between Cramer's rule and matrix inversion.

4.0 CONCLUSION

Determinants can be used to solve system of linear equations, however, there are other methods for finding solution of linear equations. In linear algebra, a singular matrix is not invertible because determinant is zero. In that case, determinants can be used to typify the polynomial of a matrix, whose roots are the eigenvalues.

5.0 SUMMARY

In this unit, we have discussed concept of Cramer's rule, determinants and the features of determinants, sign rule, Cramer's rule technique and also solved some linear equations with Cramer's rule.

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve algebraically the following system of equations

$$64x - 4y - 4z + 48 = 0$$
$$-4x + 96y - 4z + 32 = 0$$
$$168x - 4y - 4z - 64 = 0$$

(a) Using Cramer's rule

2. Solve algebraically the following system of equations

$$24s - 36f + 48z + 144 = 0$$
$$-48s + 12f - 12z + 84 = 0$$
$$-36s - 12f + 12z = 0$$

(a) Using Cramer's rule

3. Solve the following sets of simultaneous equations

$$135z_1 + 27z_2 = 324$$
$$81z_1 + 54z_2 = 135$$

$$75z_1 + 90z_2 = -50$$
$$30z_1 + 120z_2 = 200$$

(a) Using Cramer's rule

4. Consider the data below on inflation (X1), money demand (X2) and Income (Y).

Z	10	14	2	2	5	3	8
D1	12	12	16	5	1	2	5
D2	24	22	25	6	8	5	7

(a) Formulate the basic model for money demand in matrix format.

(b) Calculate the solution using matrix cramer's rule.

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UNIT 3: SYSTEM OF LINEAR EQUATIONS AND MATRIX INVERSION

CONTENTS

3.0 INTRODUCTION

In this unit, we shall be discussing the meaning of matrix as well as the size of matrix, matrix representation of system of linear equations, system of linear equations and matrix inversion, system of linear equations and Cramer's rule and also we shall solve some numerical problems on matrix algebra.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Understand what matrix inversion is all about
- Carry out matrix inversion
- Solve simultaneous equation problems with matrix inversion

3.0 MAIN CONTENT

3.1 Meaning of Matrix Inversion:

Matrix inversion is the process of finding a unique solution to a system of linear equations with unknown variables. Thus, the inverse of a matrix is the transpose of the cofactor matrix multiplied by the reciprocal of the determinant of the matrix. A square matrix that is not invertible is called singular or degenerate matrix. A square matrix is singular once its determinant is zero.

3.2. Invertible Matrix Theorem

Let A be a square n by n matrix over a field R of real numbers, it thus holds that:

i. A is invertible. This implies that A has an inverse, it is nonsingular and as such it is non-degenerate.
ii. The columns of A are linearly independent.
iii. The columns of A span Kn, thus, Col A = Kn.
iv. The transpose AT is an invertible matrix (hence rows of A are linearly independent, span Kn, and form a basis of Kn).
v. The number 0 is not an eigenvalue of A.
vi. The matrix A is mathematically describable as a finite product of elementary matrices.
vii. The columns of A form a basis of Kn.
viii. The linear transformation mapping z to Az is a bijection from Kn to Kn.
ix. There is an n-by-n matrix B such that AB = In = BA.
x. A is row-equivalent to the n-by-n identity matrix In.
xi. A is column-equivalent to the n-by-n identity matrix In.
xii. A has n pivot positions.
xiii. Determinant of A ≠ 0. In general, a square matrix over a commutative ring is invertible if and only if its determinant is a unit in that ring.
xiv. A has full rank.
xv. The equation Az = 0 has only the trivial solution z = 0.
xvi. The kernel of A is trivial, that is, it contains only the null vector as an element, ker(A) = {0}.
xvii. The equation Az = d has exactly one solution for each d in Kn.
xviii. The matrix A exhibit a left inverse for which there exists a B such that BA = I.
xix. The matrix A exhibit a right inverse for which there exists a C such that AC = I.
xx. Both left and right inverses exist and B = C = A−1.

(Source: Wikipedia)

SELF ASSESSMENT EXERCISE

How would you describe the invertible theorem of matrix?

3.3. Properties of Invertible Matrix: Furthermore, the following properties hold for an invertible matrix A:

Given the matrix, $A = \begin{vmatrix} a & d & e \\ b & c & f \\ g & w & h \end{vmatrix}$

i. $(A^{-1})^{-1} = A$
ii. $(A^{T})^{-1} = (A^{-1})^{T}$
iii. $\left|A^{-1}\right| = \left|A\right|^{-1}$

iv. For nonzero scalar k; $(kA)^{-1} = k^{-1}A^{-1}$

v. For any invertible n-by-n matrices A and B, $(AB)^{-1} = A^{-1}B^{-1}$

vi. If A is an (n by n) invertible matrix, then $(A)^{-1} = \frac{1}{|A|} Adj(A)$

vii. Following the associativity of matrix multiplication for finite square matrices, A and B, then, $AB = I \Rightarrow BA = I$

viii. If A has orthonormal columns, where + denotes Moore-Penrose inverse and z is a vector; then, $(Az)^{+} = z^{+}A^{-1}$

ix. The rows of the inverse matrix B of a matrix A are orthonormal to the columns of A and vice versa interchanging rows for columns.

SELF ASSESSMENT EXERCISE

Describe 3 properties of an invertible matrix

3.4 Matrix Inversion Technique: A system of linear n-equations can be symbolized in matrix form using a coefficient matrix, a variable matrix, and a constant matrix. The standard matrix representation of a system of linear equations is given thus:

$$Az = B \qquad (1)$$

Where A is the coefficient matrix,

z is the column vector of unknown variables,

B is the column vector of constant,

Considering the existence of the inverse of A, we multiply both sides of equation (1) by it to obtain:

$$A^{-1}(Az) = A^{-1}(B)$$

$$whereas, A^{-1}(A) = I,$$

$$Iz = A^{-1}(B)$$

$$Iz = z$$

$$Thus,$$

$$z = A^{-1}B(B)$$

$$A^{-1} = \frac{Adjoint(A)}{|A|}$$

3.4.1. Matrix Inversion Technique for Two Variable Matrix: The two variable matrix is given by:

$$\begin{vmatrix} a & d \\ b & c \end{vmatrix}$$

The Inverse of a two variable Matrix can be calculated as follows:

Step 1: Generate the Matrix of Minors: This step is to create a "Matrix of Minors". For each element of the matrix, ignore the values on the current row and column, calculate the determinant of the remaining values and obtain the "Matrix of Minors".

Step 2: Generate the Matrix of Cofactors: The Matrix of Minors is turned into the Matrix of cofactors by applying the checker sign matrix of minuses/pluses to the "Matrix of Minors". In other words, we need to change the sign of every element of the Matrix of Minor by multiplying the Matrix of Minors by each minus and plus element in the sign matrix below:

$$Sign\ Matrix \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$Matrix\ \ of\ \ Cofactors = \begin{vmatrix} a & d \\ b & c \end{vmatrix} \square \quad A^T = \begin{vmatrix} (+)\,c & (-)\,b \\ (-)\,d & (+)\,a \end{vmatrix}$$

Step 3: Calculate the Adjoint: This entails a transpose all elements of the matrix of cofactors, that is, swapping theposition of each of the element in the Matrix of cofactors over the diagonal while the diagonal remains unchanged.

$$A = \begin{vmatrix} a & d \\ b & c \end{vmatrix} \square \quad A^T = \begin{vmatrix} a & b \\ d & c \end{vmatrix}$$

Step 4: Calculate the Determinant of the Coefficient Matrix: This involves a computation of the determinant of the original matrix.

$$|A| = \begin{vmatrix} a & d \\ b & c \end{vmatrix} = a(c) - b(d)$$

Step 5: Divide the Adjoint Matrixby Determinant: This entails a division of the adjoint matrix by the determinant of the original coefficient matrix to obtain the inverse of the matric. In other words, we multiply the adjoint matrix by the reciprocal of the determinant of the coefficient matrix to obtain the inverse of the matrix.

3.4.2. Matrix Inversion Technique for Three Variable Matrix: The three variable matrix is given by:

$$\begin{vmatrix} a & d & e \\ b & c & f \\ g & w & h \end{vmatrix}$$

The Inverse of a three variable Matrix can be calculated as follows:

Step 1: Generate the Matrix of Minors: This step is to create a "Matrix of Minors". For each element of the matrix, ignore the values on the current row and column, calculate the determinant of the remaining values and obtain the "Matrix of Minors".

Step 2: Generate the Matrix of Cofactors: The Matrix of Minors is turned into the Matrix of cofactors by applying the checker sign matrix of minuses/pluses to the "Matrix of Minors". In other words, we need to change the sign of every element of the Matrix of Minor by multiplying the Matrix of Minors by each minus and plus element in the sign matrix below:

$$Sign\ Matrix\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Matrix\ \ of\ \ Cofactors = \begin{bmatrix} (+)C_{11} & (-)C_{12} & (+)C_{13} \\ (-)C_{21} & (+)C_{22} & (-)C_{23} \\ (+)C_{31} & (-)C_{32} & (+)C_{33} \end{bmatrix}$$

Step 3: Calculate the Adjoint: This entails a transpose all elements of the matrix of cofactors, that is, swapping the position of each of the element in the Matrix of cofactors s over the diagonal while the diagonal remains unchanged.

$$A = \begin{vmatrix} a & d & e \\ b & c & f \\ g & w & h \end{vmatrix} \square \ A^T = \begin{vmatrix} a & b & g \\ d & c & w \\ e & f & h \end{vmatrix}$$

Step 4: Calculate the Determinant of the Coefficient Matrix: This involves a computation of the determinant of the original matrix.

$$|A| = \begin{vmatrix} a & d & e \\ b & c & f \\ g & w & h \end{vmatrix} = a\begin{vmatrix} c & f \\ w & h \end{vmatrix} - d\begin{vmatrix} b & f \\ g & h \end{vmatrix} + e\begin{vmatrix} b & c \\ g & w \end{vmatrix}$$

Step 5: Divide the Adjoint Matrix by Determinant: This entails a division of the adjoint matrix by the determinant of the original coefficient matrix to obtain the inverse of the matric. In other words, we multiply the adjoint matrix by the reciprocal of the determinant of the coefficient matrix to obtain the inverse of the matrix.

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} (+)C_{11} & (-)C_{12} & (+)C_{13} \\ (-)C_{21} & (+)C_{22} & (-)C_{23} \\ (+)C_{31} & (-)C_{32} & (+)C_{33} \end{bmatrix}^T$$

$$= \frac{1}{|A|}C^T$$

3.5Solving Linear Equations with Matrix Inversion

Solving system of simultaneous equations in matrix specification necessitates finding the inverse of the matrix as explained above.

Numerical Example 1: Solve the simultaneous equations

$$10z + 20y = 40$$
$$30z + 50y = 10$$

Solution to Numerical Example 1: Representing the simultaneous system in matrix format, we have as follows:

$$\begin{bmatrix} 10 & 20 \\ 30 & 50 \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 10 & 20 \\ 30 & 50 \end{bmatrix}, z = \begin{bmatrix} w \\ y \end{bmatrix}, B = \begin{bmatrix} 40 \\ 10 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\left(A\right)}{|A|}$$

$$where|A| = \begin{vmatrix} 10 & 20 \\ 30 & 50 \end{vmatrix}$$

$$= 10(50) - 20(30)$$
$$= 500 - 600$$
$$= -100$$

$$Sign\ Matrix \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$Cofactor\ Matrix\, C_A = \begin{bmatrix} 50 & -30 \\ -20 & 10 \end{bmatrix}$$

$$Adj_{[A]} = C^T_{[A]} = \begin{bmatrix} 50 & -30 \\ -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 50 & -20 \\ -30 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\,(A)}{|A|} = \frac{1}{-100}\begin{bmatrix} 50 & -20 \\ -30 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\,(A)}{|A|} = \frac{1}{-100}\begin{bmatrix} 50 & -20 \\ -30 & 10 \end{bmatrix}$$

$$z = \begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} -0.5 & 0.2 \\ 0.3 & -0.1 \end{bmatrix}\begin{bmatrix} 40 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -18 \\ 7 \end{bmatrix}$$

$$z = -18,$$
$$y = 7$$

Numerical Example 2: Solve the following sets of simultaneous equations using matrix inversion.

$$4y + 2z + 14x = 24$$
$$6y + 4z + 2x = 16$$
$$2y + 2z + 8x = 32$$

Solution to Numerical Example 2: Representing the simultaneous system in matrix format, we have as follows:

$$\begin{bmatrix} 4 & 2 & 14 \\ 6 & 4 & 2 \\ 2 & 2 & 8 \end{bmatrix}\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} 24 \\ 16 \\ 32 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 6 & 4 & 2 \\ 2 & 2 & 8 \end{bmatrix} Z = \begin{bmatrix} y \\ z \\ x \end{bmatrix}, B = \begin{bmatrix} 24 \\ 16 \\ 32 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\left(A\right)}{|A|}$$

$$where|A| = 4\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} - 2\begin{vmatrix} 6 & 2 \\ 2 & 8 \end{vmatrix} + 14\begin{vmatrix} 6 & 4 \\ 2 & 2 \end{vmatrix}$$

$$= 4(32-4) - 2(48-4) + 14(12-8)$$

$$= 112 - 88 + 56$$

$$= 80$$

$$Cofactors = \begin{bmatrix} (+)\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix} & (-)\begin{vmatrix} 6 & 2 \\ 2 & 8 \end{vmatrix} & (+)\begin{vmatrix} 6 & 4 \\ 2 & 2 \end{vmatrix} \\ (-)\begin{vmatrix} 2 & 14 \\ 2 & 8 \end{vmatrix} & (+)\begin{vmatrix} 4 & 14 \\ 2 & 8 \end{vmatrix} & (-)\begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} \\ (+)\begin{vmatrix} 2 & 14 \\ 4 & 2 \end{vmatrix} & (-)\begin{vmatrix} 4 & 14 \\ 6 & 2 \end{vmatrix} & (+)\begin{vmatrix} 4 & 2 \\ 6 & 4 \end{vmatrix} \end{bmatrix}$$

$$Cofactor\ Matrix\, C_A = \begin{bmatrix} 28 & -44 & 4 \\ 12 & 4 & -4 \\ -52 & 76 & 4 \end{bmatrix}$$

$$Adj_{[A]} = C^T_{[A]} = \begin{bmatrix} 28 & -44 & 4 \\ 12 & 4 & -4 \\ -52 & 76 & 4 \end{bmatrix}^T$$

$$= \begin{bmatrix} 28 & 12 & -52 \\ -44 & 4 & 76 \\ 4 & -4 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{Adj\left(A\right)}{|A|} = \frac{1}{80}\begin{bmatrix} 28 & 12 & -52 \\ -44 & 4 & 76 \\ 4 & -4 & 4 \end{bmatrix}$$

$$z = \begin{bmatrix} y \\ z \\ x \end{bmatrix} = A^{-1} = \frac{Adj\left(A\right)}{|A|} = \begin{bmatrix} 0.35 & 0.15 & -0.65 \\ -0.55 & 0.05 & 0.95 \\ 0.05 & -0.05 & 0.05 \end{bmatrix} \begin{bmatrix} 24 \\ 16 \\ 32 \end{bmatrix}$$

$$= \begin{bmatrix} -10 \\ 18 \\ 2 \end{bmatrix}$$

$$y = -10,$$
$$z = 18,$$
$$x = 2$$

4.0 CONCLUSION

Matrix inversion plays a significant role in physical simulation exercises, multiple-input-output technologies,as well as in computer graphics, especially in the areas of screen-to-world ray casting, world-to-subspace-to-world object transformations.

5.0 SUMMARY

In this unit, we have discussed concept Meaning of Matrix Inversion, treated the concepts of Cofactors of a Matrix, Minors of a Matrix, Matrix Inversion Technique, and solved some linear equation problems with matrix inversion.

6.0 TUTOR-MARKED ASSIGNMENT

1. Solve algebraically the following system of equations

$$32w - 2d - 2z = 0$$
$$-2w + 48d - 2z = 0$$
$$84 - 2w - 2d = 0$$

(b) Using Matrix inversion

2. Solve algebraically the following system of equations

$$12w - 18f + 24z + 72 = 0$$
$$-24w + 6f - 6z + 42 = 0$$
$$-18w - 6f + 6z = 0$$

(b) Using Matrix inversion

3. Solve the following sets of simultaneous equations

$$45z_1 + 9z_2 = 108$$
$$27z_1 + 18z_2 = 45$$

$$15z_1 + 18z_2 = -10$$
$$6z_1 + 24z_2 = 40$$

(a) Using Matrix inversion

4. Consider the system of two equations in two variables,

$$6w + 12y = 32$$
$$5w - 3y = -6$$

The simultaneous system can be represented in matrix format as follows:

$$\begin{pmatrix} 6 & 12 \\ 15 & -3 \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 32 \\ -6 \end{pmatrix}$$

$$A = \begin{pmatrix} 6 & 12 \\ 15 & -3 \end{pmatrix}, \qquad z = \begin{pmatrix} w \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 32 \\ -6 \end{pmatrix}$$

Find the solution values for the z column vector.

5. Consider the system of three equations in three variables,

$$3w + 6y + 9x = 15$$
$$6w + 12y + 15x = 12$$
$$9w + 15y + 18x = 21$$

Now, the simultaneous system can be represented in matrix format as follows:

$$\begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{pmatrix} \begin{pmatrix} w \\ y \\ x \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \\ 21 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 6 & 12 & 15 \\ 9 & 15 & 18 \end{pmatrix}, \quad z = \begin{pmatrix} w \\ y \\ x \end{pmatrix}, \quad B = \begin{pmatrix} 15 \\ 12 \\ 21 \end{pmatrix}$$

Find the solution values for the z column vector.

6. Consider the data below on inflation (X1), money demand (X2) and National Income (Y).

Z	145	146	40	20	52	53	58

D1	110	112	13	15	18	22	25
D2	114	22	20	36	48	56	87

(a) Formulate the basic model for national income in matrix format.
(b) Calculate the solution using matrix inversion.

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UNIT 4: MATRIX APPLICATION TO ECONOMICS: INPUT-OUTPUT ANALYSIS

CONTENTS

1. 0. INTRODUCTION

In this unit, we shall be discussing the meaning of input-output analysis, input-output matrix/table, deriving Leontief matrix/model, explaining some of the usefulness of the input-output table and solving numerical problems of input-output model.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Discuss the meaning of input-output Analysis
- Explain the usefulness of input-output matrix/table
- Solving numerical problems of input-output model base on the Leontief matrix

3.0 MAIN CONTENT

3.1 Meaning of Input Output Analysis: An input-output (I-O) model is a numerical economic model that exemplifies the interrelationships that drives diverse sectors of an economy or diverse regional economies. This analysis was developed by Wassily Leontief (1906 -1999).An I-O model is a quantifiable economic model that denotes the interdependencies between different sectors of country or different regional economies. There is a closed I-O system in which the output is observedfor purpose of making feedback to the system of production should the case the output produced is not within specification. For the open model, factor inputs are not changed to guarantee the level of output each of the industries in an economy produce for there to be sufficiency to meet total demand for the product.

3.2. The Input-Output Matrix/Table: The I-O table is an inter-industry matrix in which column entries typically represent inputs to an industrial sector, while row entries represent outputs from a given industrial sector. Considerably therefore, each sector is dependent on every other sector, both as a customer of outputs from other sectors and as a supplier of inputs.

The basic assumptions entail consideration of the fact that we have an economy with:
1. n sectors,
2. Each sector produces z units of a single homogeneous good,
3. j^{th} sector uses a_{ij} units from sector *i* to produce 1 unit,
4. Each sector sells some of its output to other sectors as intermediate output,
5. Each sector sells some of its output to consumers as final demand, d_i.

In line with the above axioms, we may consider dividing the overall economy into 3 sectors, namely, *services,manufacturing, and power*. The analysis is that the three sectors each use inputs from two sources: Locally made commodities from the three industries as well as other inputs, such as imports, labour, and capital.

The outputs of the industries serve as intermediate inputs to production of the three industries and also as final demand measured in terms of *consumption, investment, government expenditure, exports*. These are so summarized in the I-O table below.

Table 1: Input-Output Table

Economic activities	Output						Final Demand	Total Outputs
		2	3	.	.	z		
Inputs	1							
	2							
	3							
	.							
	.							
	z							
Value Added								

Considering a Leontief closed economy with two industries, namely agriculture and services. Each industry would directly needs the use of labour in its manufacturing process, and each needs in its productive process inputs which are output of the other industry.

This is tabulated in Table 2 below where agriculture and services are the first two entries, and is given to the primary factor, labour, of which the economy has surplus labor units of 6000 (labourers). These 600 units of labour in the country are allocated as inputs to agriculture and services industries in amounts 400 and 200 respectively.

Table 2: Input-Output Table

Industries	Amount of inputs donated to Agriculture	Amount of inputs donated to Power	Final Demand	Total Outputs
Agriculture	180	300	720	1200
Power	158	88	314	560
Labour man-hours provided	400	200	0	600

As shown in table 2, agricultural output totals 1200 units per annum. Out of which, 720 units is directly devoted to final consumption, that is demand by households, firms and government agencies. The outstanding 480 units of agricultural output used as inputs to make readily available both power supply and agricultural produce in the country. Thus, 300 units of agricultural output is required as material inputs in order to make possible power supply as shown in the table while 180 units is used by agriculture itself.

Similarly, row 2 shows the allocation of the total output of power sector, 560 units of megawatts annually among final consumption/demand and intermediate inputs needed in the two industries, namely; 158 units allocated to agriculture, 88 units allocated to power and 314 allocated to final demand respectively.

The column designatesthat the 1200 units of agricultural output was produced with the use of 180 units of agricultural materials, 158 units of power supply, and 400 units of labour man-hours.

Likewise, the second column details the input structure of the power industry in the sense that the 560 megawatts of power supplied was generated with the use of 300 units agricultural inputs, 88 units of power factor and 200 man-hours of labour provided. The 'final demand' column displays the commodity analysis of what is available for consumption and demand by households, firms and government while labour services are not directly consumed.

The economic analysis thus signifies that the sales of the two agricultural and power industries to themselves and to each other constitutes the non-Gross National Product items. The 'final demand' column constitutes the output item of GNP, and labour forms the factor-cost element of the Gross National Product. What this further implies in effect is that the overall economy is such that uses up labour and has 600 units of labour man-hours at the close of the year at its disposal while simultaneously produces final consumption. With its 600 units of labour, the country produces an annual flow of 720 units of agricultural commodities and 314 units of power.

SELF ASSESSEMENT EXERCISE

i. What are the basic assumptions of the I-O Matrix?
ii. What do you understand about input output analysis?

3.3 Leontief Model/Matrix: The Leontief model is a mathematical model that can accordingly be deduced as in equation (1):

$$Z_i = a_{i1}Z_1 + a_{i2}Z_2 + a_{i3}Z_3 + ... + a_{in}Z_n + D_i \qquad (1)$$

$$Z = AZ + D$$

$$(Z - AZ) = D$$

$$(I - A)Z = D \;\; (Leontoef \;\; matrix)$$

Where, A be the matrix of technical coefficients,

(a_{ij}) technical coefficients of the ith firm in the jth industry
Z be the vector of total output,
D be the vector of final demand

According to equation (1), total output of the economy equals intermediate output plus final output. Since the Leontief matrix is invertible, it becomes a linear system of equations with a unique solution, and so given some final demand vector the required output can be calculated mathematically.

The Usefulness of the Input-Output Table:

i. I-O models are useful for studying the economic impact of inter-regional trade, as well as public investments programs
ii. I-O models aid in the calculation of national income/output.
iii. I-O models are useful tools for national and regional economic planning of resource allocation.

SELF-ASSESSMENT EXERCISE

1. What is Leontief Matrix?
2. What do you understand about input output analysis?
3. How would you describe the input-output table?

3.5 Solving Numerical Problems of Input-Output Model

Numerical Example 1: Consider a developing economy that has 3 sectors namely, power, transport and services as shown in the table below. Calculate the output vector of this economy if final demand varies from 122 to 200 for the power sector, from 184 to 260 for the transport sector and from 58 to 160 for the services sector.

Producing sector	Using sectors			Final Demand	Total Output
	Power	Transport	Services		
Power	90	20	48	122	280
Transport	40	36	40	184	300
Services	30	52	60	58	200

Solution to Numerical Example 1: Recall that,

$$(I-A)Z = D \quad (Leontoef \;\; matrix)$$

$$where \;\; Z = (I-A)^{-1}D$$

Where, A be the matrix of technical coefficients,

(aij) technical coefficients of the i^{th} firm in the j^{th} industry

Z be the vector of total output,

D be the vector of final demand

(aij) technical coefficients of the i^{th} firm in the j^{th} industry are calculated as shown below:

$$A = \begin{bmatrix} 90/280 & 20/300 & 48/200 \\ 40/280 & 36/300 & 40/200 \\ 30/280 & 52/300 & 60/200 \end{bmatrix} = \begin{bmatrix} 0.32 & 0.07 & 0.24 \\ 0.14 & 0.12 & 0.20 \\ 0.11 & 0.17 & 0.30 \end{bmatrix}$$

$$[I-A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.32 & 0.07 & 0.24 \\ 0.14 & 0.12 & 0.20 \\ 0.11 & 0.17 & 0.30 \end{bmatrix} = \begin{bmatrix} 0.68 & -0.07 & -0.24 \\ -0.14 & 0.88 & -0.20 \\ -0.11 & -0.17 & 0.70 \end{bmatrix}$$

$$[I-A]^{-1} = \frac{Adj[I-A]}{|I-A|} = \frac{C^T_{[I-A]}}{|I-A|}$$

$$|I-A| = 0.68\begin{vmatrix} 0.88 & -0.2 \\ -0.17 & 0.7 \end{vmatrix} - 0.07\begin{vmatrix} -0.14 & -0.2 \\ -0.11 & 0.7 \end{vmatrix} - 0.24\begin{vmatrix} -0.14 & 0.88 \\ -0.11 & -0.317 \end{vmatrix}$$

$$= 0.3957 + 0.0084 + 0.03339$$

$$= 0.437$$

$$Cofactor\ matrix[I-A]=\begin{bmatrix} +\begin{vmatrix} 0.88 & -0.2 \\ -0.17 & 0.7 \end{vmatrix} & -\begin{vmatrix} -0.14 & -0.2 \\ -0.11 & 0.7 \end{vmatrix} & +\begin{vmatrix} -0.14 & 0.88 \\ -0.11 & -0.317 \end{vmatrix} \\ -\begin{vmatrix} -0.07 & -0.24 \\ -0.17 & 0.70 \end{vmatrix} & +\begin{vmatrix} 0.68 & -0.24 \\ -0.11 & 0.70 \end{vmatrix} & -\begin{vmatrix} 0.68 & -0.07 \\ -0.11 & -0.17 \end{vmatrix} \\ +\begin{vmatrix} -0.07 & -0.24 \\ 0.88 & -0.20 \end{vmatrix} & -\begin{vmatrix} 0.68 & -0.24 \\ -0.14 & -0.20 \end{vmatrix} & +\begin{vmatrix} 0.68 & -0.07 \\ -0.14 & 0.88 \end{vmatrix} \end{bmatrix}$$

$$Cofactor\ matrix[I-A]=\begin{bmatrix} 0.582 & 0.12 & 0.14 \\ 0.089 & 0.4496 & 0.1233 \\ 0.225 & 0.1696 & 0.5886 \end{bmatrix}$$

$$[I-A]^{-1}=\frac{1}{0.437}\begin{bmatrix} 0.582 & 0.089 & 0.225 \\ 0.12 & 0.4496 & 0.1696 \\ 0.14 & 0.1233 & 0.5886 \end{bmatrix}$$

$$=\begin{bmatrix} 1.33 & 0.20 & 0.51 \\ 0.27 & 1.029 & 0.39 \\ 0.32 & 0.28 & 1.35 \end{bmatrix}$$

Given that $D=\begin{bmatrix} 200 \\ 260 \\ 160 \end{bmatrix}$

$$X=[I-A]^{-1}D$$

$$=\begin{bmatrix} 1.33 & 0.20 & 0.51 \\ 0.27 & 1.029 & 0.39 \\ 0.32 & 0.28 & 1.35 \end{bmatrix}\begin{bmatrix} 200 \\ 260 \\ 160 \end{bmatrix}$$

$$=\begin{bmatrix} 399.6 \\ 384.2 \\ 352.8 \end{bmatrix}$$

$$= 266 + 52 + 81.6$$
$$= 399.6$$

$$= 54 + 267.8 + 62.4$$
$$= 384.2$$

$$= 64 + 72.8 + 216$$
$$= 352.8$$

The analysis is that output production of 399.6 units of power, 384.2 units of transportation and 352.8 units of services is required to meet the final demands of 200, 260 and 160 respectively.

Numerical Example 2: Let us suppose an advanced economy has 3 sectors namely, power, transport and services such that the generation of one unit of power requires 0.6 units of transport and 0.2 units of services. The production of one unit of transportation requires 0.5 units of power and 0.5 units of service sector. Lastly, the production of one unit of services requires 0.8 units of the power sector and 0.8 units of the transport sector.

(1) Formulate the input output matrix of the economy.
(2) If the economy supplies 80 units of power, 50 units of transport and 60 units of services, determine how much of each sector is used up in the production process of the economy.
(3) Calculate the amount of each sector that is not used up in the production process of the economy
(4) Suppose the demand for power supply increases to 25 units, 3000 units for transport and 60 for services, what would be the production output for each sector?

Solution to Numerical Example 2: I-O Table

		Industry/sector consuming Output		
		Power	Transport	services
Industry/sector supply Input	Power	0	0.5	0.8
	Transport	0.6	0	0.8
	Services	0.2	0.5	0

$$A = \begin{bmatrix} 0 & 0.5 & 0.8 \\ 0.6 & 0 & 0.8 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 80 \\ 50 \\ 60 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0.5 & 0.8 \\ 0.6 & 0 & 0.8 \\ 0.2 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 80 \\ 50 \\ 60 \end{bmatrix} = \begin{bmatrix} 73 \\ 96 \\ 41 \end{bmatrix}$$

Power supply	The amount of power supply used up by the transport sector is $0.5 \times 50 = 25$ The amount of power supply used up by the services sector is $0.8 \times 60 = 48$ Thus, 73 units of power is used up in the production process of the economy
Transport supply	The units of transport used up by the power sector is $0.6 \times 80 = 48$ The amount of transport used up by the services sector is $0.8 \times 60 = 48$ Thus, 96 units of transport is used up in the production process of the economy
Services supply	The amount of services used up by the power sector is $0.2 \times 80 = 16$ The amount of services used up by the transport sector is $0.5 \times 50 = 25$ Thus, 41 units of services sector is used up in the production process of the economy

$$\begin{aligned} D &= Z - AZ \\ &= \begin{bmatrix} 80 \\ 50 \\ 60 \end{bmatrix} - \begin{bmatrix} 73 \\ 96 \\ 41 \end{bmatrix} \\ &= \begin{bmatrix} 7 \\ -46 \\ 19 \end{bmatrix} \end{aligned}$$

The interpretation of the demand vector is that 80 units of power was supplied, 73 units was consumed, that is used up in the nation's production process while 7 units is not used up in the production process of the economy. Similarly, 60 units of services were rendered, 41 units was consumed in the nation's production process while 19 units is not used up in the production process of the economy. However, the 50 units of transport produced could not serve the economy as 96 units of transportation was used up in the nation's production.

Suppose the demand for power supply increases to 400 units, 300 units for transport and 220 for services, the final demand vector would be given by:

$$D = \begin{bmatrix} 400 \\ 300 \\ 250 \end{bmatrix}$$

4.0 CONCLUSION

Each column of the input-output matrix shows the monetary value of inputs to each sector and each row represents the value of each sector's outputs.There has been research on disaggregation to clustered inter-industry flows, and on the study of constellations of industries. A great deal of empirical work has been done to identify coefficients, and data has been published for the national economy as well as for regions. The I-O model can be extended to a model of Walrasian equilibrium analysis; whereby monetary value of inputs are contained in column while sectorial output are contained in the rows.

5.0 SUMMARY

In this unit, we have discussed the meaning of input-output analysis, the input-output table, usefulness of the input-output table and solved numerical problems of input-output model.

6.0 TUTOR-MARKED ASSIGNMENT

1. Consider that an economy has 3 sectors namely, power, transport and services such that the generation of one unit of power requires 0.3 units of transport and 0.7 units of services. The production of one unit of transportation requires 0.8 units of power and 0.8 units of service sector. Lastly, the production of one unit of services requires 0.2 units of the power sector and 0.4 units of the transport sector.

(i) Formulate the input output matrix of the economy.
(ii) If the economy supplies 300 units of power, 400 units of transport and 500 units of services, determine how much of each sector is used up in the production process of the economy.
(iii) Calculate the amount of each sector that is not used up in the production process of the economy
(iv) Suppose the demand for power supply increases to 600 units, 400 units for transport and 900 for services, what would be the production output for each sector?

2. If an advanced economy has 3 sectors namely, power, transport and services such that one unit of power requires 0.5 units of transport and 0.5 units of services. The production of one unit of

transportation requires 0.9 units of power and 0.9 units of service sector. Lastly, the production of one unit of services requires 0.3 units of the power sector and 0.3 units of the transport sector.
(i) Formulate the input output matrix of the economy.
(ii) If the economy supplies 270 units of power, 350 units of transport and 360 units of services, determine how much of each sector is used up in the production process of the economy.
(iii) Calculate the amount of each sector that is not used up in the production process of the economy
(iv) Suppose the demand for power supply increases to 490 units, 350 units for transport and 250 for services, what would be the production output for each sector?

3. Consider that an economy has 3 sectors namely, power, transport and services such that the generation of one unit of power requires 0.25 units of transport and 0.75 units of services. The production of one unit of transportation requires 0.36 units of power and 0.85 units of service sector. Lastly, the production of one unit of services requires 0.25 units of the power sector and 0.45 units of the transport sector.
(i) Formulate the input output matrix of the economy.
(ii) If the economy supplies 880 units of power, 850 units of transport and 860 units of services, determine how much of each sector is used up in the production process of the economy.
(iii) Calculate the amount of each sector that is not used up in the production process of the economy
(iv) Suppose the demand for power supply falls to 90 units, 100 units for transport and 150 for services, what would be the production output for each sector?

4. Consider a developing country that has 3 sectors as shown in the table below. Calculate the output vector of this economy if final demand increases to 190 for the services sector, to 400 for the transport sector and to 250 for the power sector.

Producing sector	Using sectors			Final Demand	Total Output
	Services	Transport	Power		
Services	290	220	250	170	930
Transport	240	236	240	180	896
Power	230	252	260	200	942

3. Consider a developing economy that has 3 sectors as shown in the table below. Calculate the output vector of this economy if final demand varies from 80 to 200 for the services sector, from 40 to 220 for the transport sector and from 70 to 300 for the power sector.

Producing sector	Using sectors			Final Demand	Total Output
	Services	Transport	Power		
Services	100	120	140	80	440

Transport	140	130	190	40	500
Power	130	150	160	70	510

4. Consider a developing economy that has 3 sectors as shown in the table below. Calculate the output vector of this economy if final demand changes to 700 for the service sector, 680 for transport and 800 for the power sector.

Producing sector	Using sectors			Final Demand	Total Output
	Services	Transport	Power		
Services	500	250	380	665	1300
Transport	500	350	400	500	1750
Power	600	200	200	600	1600

7.0. REFRENCES/FURTHER READING

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MODULE THREE: DIFFRENTIATIONS, INTEGRATION AND OPTIMIZATION TECHNIQUES

UNIT 1: DIFFERENTIAL CALCULUS AND SOME ECONOMIC APPLICATIONS

CONTENTS

1. 0. INTRODUCTION

This unit provides discussion on meaning of differential calculus, derivative of a function, derivative of implicit functions, economic applications of derivatives and solving numerical problems on differential calculus.

4.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Find the derivative of explicit and an implicit functions
- Solving Numerical Problems on Differential Calculus
- Carry some applications of differential calculus

3.0. MAIN CONTENT

3.1. Meaning of Differential Calculus

Differential calculus studies the rates at which quantities change. The entities involved in the study of differential calculus are the derivative of a function, or differentiation, and the

associated applications.

3.2 Derivative of a Function: The derivative of a function defines the rate of change of the function at given value. It is a sequence of differentiation. The geometrical explanation of derivative hinges on the slope of the line tangential to the graph of the function at the point of tangent.

Suppose that x and z are real numbers and that z is a function of x, that is, for every value of x, there is a corresponding value of y (*which - y or z?*). This mathematical link can be written as:

$$z = f(x) = c + bx$$

Thus, the slope of the function is obtained by obtaining the derivative of the function w.r.t. z. This is shown below:

$$b = \frac{change\ in\ z}{change\ in\ x}$$

$$b = \frac{\Delta z}{\Delta x}$$

Where the symbol Δ symbolizes "change in". It follows that $\Delta z = b\Delta x$. The geometry of the derivative of f at the point $z = b$ is the slope of the tangent line to the function f at the point b. This is shown in the figure below and is denoted using either Langrange or Leibniz's notation as shown below:

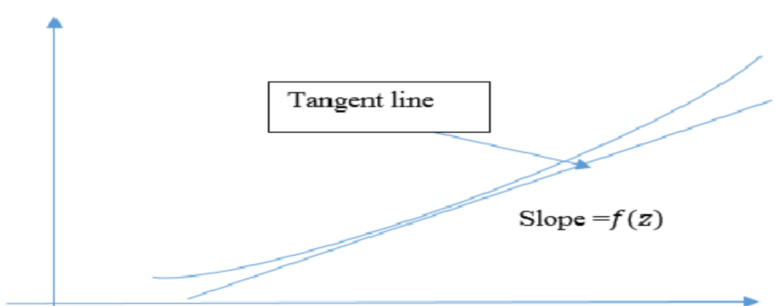

Figure 2: Derivative Function

Source: Wikipedia

$f'(x)$ (Langrange notation)

$\left.\frac{dy}{dx}\right|_{x=d}$ (Leibniz's notation)

3.3. Rules of Differentiation: These are rule for finding the derivative of a function. They include the constant rule, power rule, product rule, quotient rule, exponential rule, logarithmic rule, trigonometry rule etc. Nevertheless, the derivative $\frac{dy}{dx}$ is $\lim\limits_{\delta \to 0} \frac{\delta y}{\delta x}$ if the limit exists

i. Constant Rule of Differentiation: For any constant k, such that

$$f(x) = k$$

$$f'(x) = 0 \equiv \frac{d}{dx}(k) = 0$$

ii. Power Rule of Differentiation: The derivative of $f(x)$, from the first principles is as follows:

$$f(x) = x^n, \quad f'(x) \equiv \frac{dy}{dx} = nx^{n-1}$$

$$f(x) = x^4$$

$$f'(x) \equiv \frac{dy}{dx} = nx^{n-1} = 4x^3$$

$$f(x) = 3x^5$$

$$f'(x) \equiv \frac{dy}{dx} = nx^{n-1} = 15x^2$$

$$f(x) = 2x^6$$

$$f'(x) \equiv \frac{dy}{dx} = nx^{n-1} = 12x^5$$

For example, if $f(x) = x^6$, then the derivative function $f'(x)$ would be given as:

$$\cancel{f'(x) = \frac{\partial y}{\partial x} = 6x^5}$$

$$f'(x) \equiv \frac{dy}{dx} = 6x^5$$

iii. The derivatives of exponential and logarithmic functions obeys the following rules

$$\frac{d}{dx} e^{(ax)} = ae^{ax}$$
$$\frac{d}{dx} \log_e x = \frac{1}{x \ln k}, \quad where\ k > 0,\ k \neq 1$$
$$\frac{d}{dx} (\ln x) = \frac{1}{x}, \quad where\ x > 0$$
$$\frac{d}{dx} (\ln |x|) = \frac{1}{x}, \quad where\ x > 0$$
$$\frac{d}{dx} (x^x) = x^x (1 + \ln x), \quad where\ x > 0$$
$$\frac{d}{dx} [f(x)^{h(x)}] = h(x) f(x)^{h(x)-1} \frac{df}{dx} + f(x)^{h(x)} \ln[f(x)] \frac{dh}{dx} \quad provided\ \ f(x) > 0$$

iv. The derivatives of trigonometric functions obeys the following rules of engagement:

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\tan x)' = \sec^2 x \equiv 1 + \tan^2 x$$
$$(\sec x)' = \sec x \tan x$$
$$(\cot x)' = -(1 + \cot^2 x)$$

v. The derivatives of hyperbolic functions obeys the following rules of engagement:

$$(\sinh x)' = \cosh x \equiv \frac{e^x + e^{-x}}{2}$$
$$(\cosh x)' = \sinh x \equiv \frac{e^x - e^{-x}}{2}$$
$$(\tanh x)' = \sec h^2$$
$$(\sec hx)' = -\tanh x \sec hx$$
$$(\coth x)' = -\csc h^2 x$$

vi. The derivative of implicit functions: The derivative of an implicit function is a derivative of both sides of the function w.r.t. one of the variables while holding the other variables relatively constant. Examples of implicit function is: $g(z, y) = z^3 + y^7 + 2$, then the circle is the set of all pairs (x, y) such that g(z, y) = 0. So,

$$g(z,y)=0$$
$$g_z dz + g_y dy = 0$$
$$g_z dz = -g_y dy$$
$$\frac{dy}{dz} = -\frac{g_z}{g_y}$$

vii. The product rule: Given that a function $g(uv)$ where u and v are two separate functions of the same variable z, the derivative of $g(uv)$ is given as $g'(uv) = u\frac{dv}{dz} + v\frac{du}{dz}$

viii. Given that u and v are two separate functions of the same variable z, the derivative of the function $g(z) = \frac{u}{v}$

$$g'(z) = \frac{v\frac{du}{dz} - u\frac{dv}{dz}}{v^2}$$

SELF ASSESSMENT EXERCISE

Explain the meaning of an implicit function

How would you differentiate between Langrange notation and Leibniz's notation?

Differentiate between an exponential and a logarithmic functions

3.4Solving Numerical Problems on Differential Calculus

Numerical Example 1: Find the derivative of the given function.

$(i)\quad g(z) = 5z^3 - 4z + 2$

Solution to Numerical Example 1: Taking the derivative, we have that:

$$g'(z) = \frac{\partial g}{\partial z} = 15z^2 - 4$$

Numerical Example 2: Find the derivative of the given function.

$(i)\ \ g(z) = (5z-2)(9z+z^2)$

Solution to Numerical Example 2: Taking the derivative, we have that:

$$g(z) = (5z-2)(9z+z^2)$$
$$let\ \ u = (5z-2),\ v = (9z+z^2)$$
$$Utilizing\ \ the\ \ product\ \ riule, we\ \ have,$$
$$uv = u\frac{dv}{dz} + v\frac{du}{dz}$$
$$g'(z) = \frac{\partial g}{\partial z} = \left[(5z-2)\frac{d}{dv}(9z+z^2) + (9z+z^2)\frac{d}{dv}(5z-2)\right]$$

$$= (5z-2)(9+2z)+(9z+z^2)5$$
$$= 45z-18+10z^2-4z+45z+5z^2$$
$$= 15z^2+86z-18$$

Numerical Example 2

$$g(z) = ze^{5z}$$
$$g'(z) = u\frac{dv}{dz}+v\frac{du}{dz}$$
$$let \ \ u = z, \ v = e^{5z}$$
$$Utilizing \ the \ product \ riule, we \ have,$$
$$uv = u\frac{dv}{dz}+v\frac{du}{dz}$$
$$g(z) = \frac{\partial g}{\partial z} = \left[z\frac{d}{dv}(e^{5z})+(e^{5z})\frac{d}{dv}(z)\right]$$
$$= (5ze^{5z})+(e^{5z})(1)$$
$$= e^{5z}(5z+1)$$

Numerical Example 3: Find the derivative of the given functions.

$$(i) \quad g(z) = \frac{5z^2}{z+6}$$
$$(ii) \quad g(z) = \frac{5z^3-10z+2}{z}$$
$$(iii) \quad g(z) = \frac{2z^6-z^3+z}{z^3}$$

Solution to Numerical Example 3:

$$g(z) = \frac{5z^2}{z+6}$$
$$let \ \ u = 5z^2, \ v = z+6$$
$$Utilizing \ the \ quotient \ riule, we \ have,$$

$$\frac{\partial g}{\partial z} = \frac{v\frac{du}{dz} - u\frac{dv}{dz}}{v^2}$$

$$g(z) = \frac{\partial g}{\partial z} = \left[\frac{(z+6)\frac{d}{dv}(5z^2) - (5z^2)\frac{d}{dv}(z+6)}{(z+6)^2} \right]$$

$$= \frac{(z+6)10z - (5z^2)}{(z+6)^2}$$

$$= \frac{10z^2 + 60z - 5z^2}{(z+6)^2}$$

$$= \frac{5z^2 + 60z}{(z+6)^2}$$

$$g(z) = \frac{5z^3 - 10z + 2}{z}$$

$let \;\; u = (5z^3 - 10z + 2), \; v = z$

Utilizing the quotient riule, we have,

$$\frac{\partial g}{\partial z} = \frac{v\frac{du}{dz} - u\frac{dv}{dz}}{v^2}$$

$$g(z) = \frac{\partial g}{\partial z} = \left[\frac{z\frac{d}{dv}(5z^3 - 10z + 2) - (5z^3 - 10z + 2)\frac{d}{dv}(z)}{z^2} \right]$$

$$= \frac{z(15z^2 - 10) - (5z^3 - 10z + 2)}{z^2}$$

$$= \frac{15z^3 - 10z - 5z^3 + 10z - 2}{z^2}$$

$$= \frac{10z^3 - 2}{z^2}$$

$$= 10z - 2z^{-2}$$

$$g(z)=\frac{2z^6-z^3+z}{z^3}$$

$$let\ \ u=(2z^6-z^3+z),\ v=z^3$$

$$Utilizing\ the\ quotient\ riule, we\ have,$$

$$\frac{\partial g}{\partial z}=\frac{v\frac{du}{dz}-u\frac{dv}{dz}}{v^2}$$

$$g(z)=\frac{\partial g}{\partial z}=\left[\frac{z^3\frac{d}{dv}(2z^6-z^3+z)-(2z^6-z^3+z)\frac{d}{dv}(z^3)}{(z^3).(z^3)}\right]$$

$$=\frac{z^3(12z^5-3z^2+1)-3z^2(2z^6-z^3+z)}{z^6}$$

$$=\frac{12z^8-3z^5+z^3-6z^8+3z^5-3z^3}{z^6}$$

$$=\frac{6z^8-2z^3}{z^6}$$

$$=6z^2-2z^{-3}$$

Alternatively, we can divide the numerator of $g(z)$ by the denominator of each function respectively and apply the power function rule as follows:

$$\frac{5z^3-10z+2}{z}=5z^2-10+2z^{-1}$$

$$\frac{d}{dz}\left(5z^2-10+2z^{-1}\right)=10z-2z^{-2}$$

$$\frac{2z^6-2z^3+z}{z^3}=2z^3-2+z^{-2}$$

$$\frac{d}{dz}\left(2z^3-2+z^{-2}\right)=6z^2-2z^{-3}$$

Numerical Example 4: Consider the following function

$$32z^2-8y^3+29=0$$

Solution to Numerical Example 4: Carrying out an implicit differentiation of the function, we have as follows:

$$32z^2 - 8y^3 + 29 = 0$$

$$\frac{d}{dz}(32z^2 - 8y^3 + 29) = \frac{d}{dz}(0)$$

$$\frac{d}{dz}(32z^2) - \frac{d}{dz}(8y^3) + \frac{d}{dz}(29) = 0$$

$$64z - 24y^2\frac{dy}{dz} = 0$$

$$64z = 24y^2\frac{dy}{dz}$$

$$\frac{dy}{dz} = -\frac{64z}{24y^2}$$

Alternatively,

$$f(z, y) = 0$$

$$\frac{dy}{dz} = -\frac{f_z}{f_y}$$

$$\frac{dy}{dz} = -\frac{64z}{24y^2}$$

Numerical Example 5: Find the derivative of the following functions:

i. $y = \exp(x) \equiv e^x$

ii. $y = x^7 \ln x$

iii. $y = \ln(4x^2 - 5x + 6)$

Solutions: Given that $y = \exp(x) \equiv e^x$

$$where \quad e = 2.71828...$$

$$y' = \frac{dy}{dx} = \exp(x) \equiv e^x$$

Given that $y = x^7 \ln x$

$$
\begin{aligned}
& let\ u = x^7 \\
& v = \ln x \\
& Applying\ \ the\ \ product\ \ rule,\ we\ \ have\ \ as\ \ follows: \\
& \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \\
& \quad = x^7\frac{1}{x} + \ln x.7x^6 \\
& \quad = x^6 + 7x^6\ln x \\
& \quad = x^6(1+7\ln x)
\end{aligned}
$$

Given that $y = \ln(4x^2 - 5x + 6)$

$$
\begin{aligned}
& let\ u = (4x^2 - 5x + 6) \\
& y = \ln u \\
& Applying\ \ the\ \ chain\ \ rule,\ we\ \ have\ \ as\ \ follows: \\
& \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \\
& \quad = \frac{1}{4x^2 - 5x + 6}.(8x-5) \\
& \quad = \frac{8x-5}{4x^2 - 5x + 6}
\end{aligned}
$$

SELF ASSESSMENT EXERCISE

Describe the relationship between an explicit and an implicit derivative. Give examples

3.5. Application of Differentiation to Economic Problems: Recall that the slope of the graph of a function is called the derivative of the function such that:

$$
\begin{aligned}
& y = f(z) = z \\
& f'(z) = \frac{dy}{dz} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta z}
\end{aligned}
$$

3.4.1 Application I: Calculating Marginal Functions (Revenue, Costs and Profit)

Given a total revenue function as: $TR = PQ$

$$
MR = \frac{d(TR)}{dQ}
$$

So, if a firm faces thedemand curve $P = 8 - 2Q$, the TR and the corresponding MR functions of the firm becomes:

$$P = 8 - 2Q$$
$$TR = PQ = (8 - 2Q)Q$$
$$= 8Q - 2Q^2$$

$$MR = \frac{d(TR)}{dQ} = 8 - 4Q$$

The firm optimizing quantity and price levels are obtained as follows:

$$MR = \frac{d(TR)}{dQ} = 8 - 4Q \quad F.O.C$$
$$Setting\ MR = 0,$$
$$8 - 4Q = 0$$
$$Q = 8/4$$
$$= 2$$

$$P = 8 - 2(2)$$
$$= N4$$

$$Evaluating\ \ S.O.C$$
$$\frac{d^2(TR)}{dQ^2} = -4 < 0$$

The firm optimizing price and quantity levels are N4 and 2 respectively.

Given a total cost function as: $TC = PQ$

$$MC = \frac{d(TC)}{dQ}$$

A firms total cost curve is given by

$$TC=2Q^3- 8Q^2+24Q$$

(i) Obtain the AC function
(ii) Obtain the MC function
(iii) When does the slope of AC = 0?
(iv) What does the relationship of graphs of MC and AC curves?

$$TC = 2Q^3 - 8Q^2 + 24Q$$

$$\text{(i)}\quad \mathrm{A}C = \frac{TC}{Q} \equiv \frac{2Q^3 - 8Q^2 + 24Q}{Q} \;\square\; 2Q^2 - 8Q + 24$$

$$(ii)\quad MC = \frac{dTC}{dQ} = 6Q^2 - 16Q + 24$$

$$(iii)\quad AC = 0 \;\; when \;\; \frac{d(AC)}{dQ} = 0$$

$$\frac{d(AC)}{dQ} = 4Q - 8 = 0$$

$$Q = 2$$

$$The \;\; slope \;\; of \;\; AC = 0 \; when \; units \; of \;\; output = 2$$

(iv) The economic significance points to a minimum at the point whereby MC curve intersect AC curve.

3.5.2. Applications II: Calculating Price Elasticity of Demand: Recall that by definition, price elasticity of demand refers to the proportionate change in demand with respect to proportionate change in price level. Mathematically,

$$|e_d| = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q} \;\square\; \frac{dQ}{dP} \times \frac{P}{Q}$$

Thus, if $|e_d|$ <1, we have inelastic demand

if $|e_d|$ =1, we have *unit elasticity of* demand

if $|e_d|$ >1, we have elastic demand

Supposing the (inverse) Demand equation is P = 100 – 10ln(Q+1), calculate the price elasticity of demandwhen Q = 42

$$|e_d| = \frac{dQ}{dP} \times \frac{P}{Q}$$

$$\frac{dP}{dQ} = -\frac{10}{\text{Q+1}}$$

Applying the inverse rule, we have that:

$$\frac{dQ}{dP} = -\frac{\text{Q+1}}{10}$$

When $Q = 42$,

$P = 100 - 10\ln(43)$

$=62.4$

Thus, $|e_d| = \frac{dQ}{dP} \times \frac{P}{Q} = -\frac{43}{10} \times \frac{62.4}{42}$

$$|e_d| = -6.387$$

Demand is elastic

4.0 CONCLUSION

The fundamental theorem of calculus guides the mathematical processes involved in differentiation and integration. The driving principle in study of differential calculus is derivative of a function whose procedure is referred to as differentiation. Differentiation has wide ranging economic applications.

5.0 SUMMARY

In this unit, we have discussedthe meaning of differential calculus, appraised derivative of a function (both explicit and implicit functions). Also, we studied applications of derivatives to economic problems haven solved some numerical problems on differential calculus.

6.0 TUTOR-MARKED ASSIGNMENT

1. Find the derivative of the given function.

   (i) $g(z) = 52z^5 + 9z^2 + 4$

   (ii) $g(z) = (26z^2 - 6)(18z + z^3)$

2. Find the derivative of the given functions.

   (i) $g(w) = \frac{(5w+1)^3}{w^2 + 9}$

   (ii) $g(w) = \frac{13w^3 + 2}{7w^6 - 4z^3 + 3z}$

   (iii) $g(w) = \frac{6w^6 - wz + 1}{w}$

3. Consider the following functions

$$3z^6 - zy^3 + 1 = 0$$

$$z^6 + y^3 = 23$$

$$10z^2 - \cos(3z) = y^3 + 34$$

$$3z^6 - 5zy^3 + 40zy = 4y$$

Find the implicit derivative of the function

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UNIT 2: INTEGRAL CALCULUS AND SOME ECONOMIC APPLICATIONS

CONTENTS

1. 0. INTRODUCTION

This unit provides a discussion on the meaning of integral calculus, integration techniques, such as integration by part and integration by substitution as well as economic applications of integral calculus together with some solved numerical examples on integral calculus.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Find the integral of a function
- Find the integral of a function by using substitution method
- Find the integral of a function using integration by part technique
- Carry some economic applications of integral calculus

3.0. MAIN CONTENT

3.1 Meaning of Integral Calculus: Integration is the process of reversing differentiation. It is means of finding a primitive function from a derivative. Hence, it is also called anti-derivative. An integral of a function describes area or, volume. Given a function *f* of a real variable *x* and an interval [a, b] of the real line, the definite integral becomes:

$$\int_{t_1}^{t_2} f(x)dx$$

As it is, integration defines the area of the region in the xy-plane that is bounded by the graph of *f*, the *x*-axis and the vertical lines $x = t_1$ and $x = t_2$. in effect, the region above the x-axis increases total area and the region below the x-axis decreases the total area. Integral may also refer to antiderivative, a function F whose derivative is the given function f and the indefinite integral is written thus:

$$F(x) = \int_{t_1}^{t_2} f(x)dx$$

The definition of the definite integral begins with a function f(x), which is continuous on a closed interval [a, b]. The given interval is partitioned into “n” subintervals that can be taken to be of equal lengths (Δ x). Similar to differentiation, a significant relationship exists between continuity and integration such that if a function f(x) is continuous on a closed interval [a, b], then the definite integral of f(x) on [a, b] exists and f is said to be integrable on [a, b]. In other words, continuity guarantees that the definite integral exists, but the converse is not the case.

If *f* is a continuous real-valued function defined on a closed interval [t_1, t_2], then, once an anti-derivative F of *f* is identified, the definite integral of *f* over that interval is given by:

$$\int_{t_1}^{t_2} f(x)dx = [F(x)]_{t_1}^{t_2}$$

$$F(t_2) - F(t_1)$$

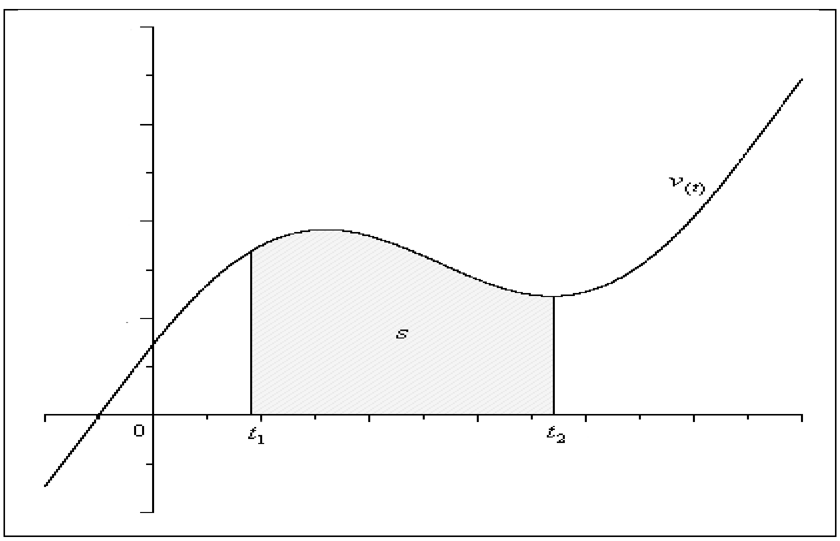

Figure 3: Area of Integration
Source: mathnotes.org, en.wikipedia.org

3.3. Fundamental Theorem of Calculus: The fundamental theorem of calculus inaugurates the link between indefinite and definite integrals such that if f(x) is continuous on the interval [a, b], and F(x) is any anti-derivative of f(x) on [a, b], it thus implies:

$$\int_b^d f(x)dx = F(x) + k$$
$$= F(d) - F(b)$$
$$\equiv F(x)\Big|_b^d$$
$$where\ k\ is\ the\ constant\ of\ integration$$
$$\int \quad is\ the\ integral\ sign$$

This goes to show that the value of the definite integral of a function on [a, b] is the difference of any antiderivative of the function evaluated at the upper limit of integration minus the same antiderivative evaluated at the lower limit of integration.

Basic rules of integration: These are rules solving definite integral problems. They include: the power function rule, sum-difference rule, integration of exponential and logarithmic functions etc.
Power Function Rule:

$$\int_b^d f(x)dx = -\int_d^b f(x)dx$$
$$\int_b^d kdx = k[d-b] \quad where\ k\ is\ a\ constant$$
$$\int_b^d kf(x)dx = k\int_b^d f(x)dx$$

Sum Rule: $\int_b^d [f(x)+g(x)]dx = \int_d^b f(x)dx + \int_d^b g(x)dx$

Difference Rule: $\int_b^d [f(x)-g(x)]dx = \int_d^b f(x)dx - \int_d^b g(x)dx$

3.2 Integration Techniques

3.2.1 Integration by Part: Integration by parts technique finds the integral of a product of functions in terms of integral of product of their derivative and anti-derivative. The technique of integration by part is the equivalence of product rule of differentiation.The theorem of integration by parts states that if the following conditions hold:

$$u = u(z)$$
$$du = u'(z)dz$$
$$w = w(z)$$
$$dw = w'(z)dz$$

$$\int_b^d u(x)w'(x)dx = [u(x)w'(x)]_b^d - \int_b^d u'(x)w(x)dx$$

□ □ □ $$\int udw = uw - \int wdu$$

3.2.2 Integration by Substitution: Integration by substitution as a method for solving integrals is the u-substitution technique. Using the fundamental theorem of calculus, it necessitates finding an antiderivative. Accordingly, integration by substitution is the equivalence of chain rule for differentiation.

For substitution for a single variable, consider that I ⊆ R be an interval and ψ : [b,d] → I be a continuous and differentiable function with integrable derivative, it thus follows that:

$$\int_{\psi(b)}^{\psi(d)} f(u)du = \int_{\psi(b)}^{\psi(d)} f(\psi(x))\psi'(x)dx$$

In Leibniz notation, the substitution $u = \psi(x)$ yields

$$\frac{\partial u}{\partial x} = \psi'(x)$$

□ □ □ $$\psi'(x)\partial x$$

The formula is used to transform one integral into another integral and by so doing, it simplifies a given integral.

3.3 Solving Numerical Problems on Integral Calculus

Numerical Example 1: Find the integral of the following functions

$$\int \frac{9z}{(2z^2 - 13)^7} dz$$

$$\int (5z - 13)^2 dz$$

$$\int \frac{192z^3}{3z^4 + 2} dz$$

Solution to Numerical Example 1:Using the substitution method

$$\int \frac{9z}{(2z^2-13)^7}dz$$

$$let\ u=(2z^2-13)$$

$$\frac{du}{dz}=4z$$

$$4zdz=du$$

$$dz=\frac{1}{4z}du$$

$Substitut\inf\ \ for\ u\ \&\ dz,\ we\ have,$

$$\int \frac{9z}{(2z^2-13)^7}dz=\int \frac{9z}{u^7}\frac{1}{4z}du$$

$$=\frac{9}{4}\int \frac{1}{u^7}du$$

$$=\frac{9}{4}\int u^{-7}du$$

$$=\frac{9}{4}\frac{u^{-6}}{-6}$$

$$=-\frac{9}{24(2z-13)^6}$$

$$u=2z^2-13$$

$$\int (5z-13)^2dz$$

$$let\ u=(5z-13)$$

$$\frac{du}{dz}=5$$

$$5dz=du$$

$$dz=\frac{1}{5}du$$

$Substitut\inf\ \ for\ u\ \ dz,\ we\ have,$

$$\int (5z-13)^2dz=\frac{1}{5}\int u^2du$$

$$=\frac{1}{15}u^3+k$$

$$=\frac{1}{15}(2z-13)^2+c$$

$$\int \frac{192z^3}{3z^4+2}dz$$

$$let\ u = (3z^4+2)$$

$$\frac{du}{dz} = 12z^3$$

$$12z^3 dz = du$$

$$dz = \frac{1}{12z^3}du$$

$$Substitut\inf\ \ for\ u\ \&\ dz,\ we\ have,$$

$$\int \frac{192z^3}{3z^4+2}dz = \int \frac{192z^3}{u}\left(\frac{1}{12z^3}du\right)$$

$$= 16\int \frac{1}{u}du$$

$$= 16\int \ln u + k$$

$$= 16\ln(3z^4+2)+k$$

Numerical Example 2: Find the integral of the following functions

$$\int z^3 e^z dz$$

$$\int z^4 \ln z dz$$

Solution to Numerical Example 2: Using the method of integration by parts

$$i.\int z^3 e^z dz$$

$$let\ u = z^3$$

$$w' = dw = e^z$$

$$u' = du = 3z^2$$

$$w = e^z$$

$$\int z^3 e^z dz = z^3 e^z - \int 3(z^2 e^z)dz$$

$$where \int 3(z^2 e^z)dz = 3[z^2 e^z - \int 2ze^z]$$

$$where \int 2ze^z = ze^z - \int e^z\ .1dz$$

$$= ze^z - e^z + k$$

$$Thus, \int z^3 e^z dz = z^3 e^z - 3z^2 e^z - 2(ze^z - e^z + k)$$

$$\int z^2 e^z dz = z^2 e^z - 2ze^z + k$$

$$ii. \int z^4 \ln z dz$$

$$\int z^4 \ln z dz$$

$$let\ u = \ln z$$

$$w' = dw = z^4$$

$$u' = du = \frac{1}{z}$$

$$w = \frac{z^5}{5}$$

$$\int z^4 \ln z dz = \frac{z^5}{5} \ln z - \int \frac{z^5}{5} \left(\frac{1}{z} \right) dz$$

$$= \frac{z^5}{5} \ln z - \int \frac{z^4}{5} dz$$

$$= \frac{z^5}{5} \ln z - \frac{z^5}{25} + k$$

3.4. Application of Integration in Economics:

Integral calculus can be applied to solve economic problems, business and commerce related problems. For example, integration helps us to find out the total cost function and total revenue function from the marginal cost and also obtain consumer's surplus and producer's surplus from the demand function and supply function respectively.

3.4.1. Cost and Revenue Functions: Here, total cost functions can be derived from marginal cost functions while total revenue functions can be obtained from Marginal revenue functions with the aid of integration.

Considering that C is the cost of producing an output of Q commodity, then marginal cost function MC is derived as follows:

$$MC = \frac{dC}{dQ}$$

Using integration, as the reverse process of differentiation, we obtain the total cost function as follows:

$$TC = \int (MC) dQ + k$$

Where k is the constant of integration which is to be evaluated,

Relatively, the Average cost function, AC is derivable as follows:

$$AC = \frac{C}{Q}, \quad \forall Q \neq 0$$

Example 1: Supposing the marginal cost function of producing Q eye shade is $12 + 20Q–12Q^2$. The cost of producing a duo of shades is N48. Find the total and average cost function.

$$MC = 12 + 20Q - 12Q^2$$
$$TC = \int (MC)dQ + k$$
$$TC = \int (12 + 20Q - 12Q^2)dQ + k$$
$$= 12Q + \frac{20Q^2}{2} - \frac{12Q^3}{3} + k$$
$$TC = 12Q + 10Q^2 - 4Q^3 + k$$

Since the cost of producing a duo of shades is N48, we have the TC as follows:

$$When\ Q = 4, k = 48$$

$$TC = 12Q + 10Q^2 - 4Q^3 + k$$
$$48 = 48 + 160 - 256 + k$$
$$k = 96$$
$$Thus,$$
$$TC = 12Q + 10Q^2 - 4Q^3 + 96$$

$$AC = \frac{C}{Q},$$
$$AC = \frac{12Q + 10Q^2 - 4Q^3 + 96}{Q}$$
$$AC = 12 + 10Q - 4Q^2 + \frac{96}{Q}$$

Example 2: A Nigerian firm has a marginal cost function given by MC = 375 + 30Q − Q2/3. If the fixed cost of production is N750, determine the cost of producing 45 units.

$$MC = 375 + 30Q - \frac{1}{3}Q^2$$
$$TC = \int (MC)dQ + k$$
$$TC = \int (375 + 30Q - \frac{1}{3}Q^2)dQ + k$$

$$= 375Q + 15Q^2 - \frac{1}{9}Q^3 + k$$

$$Fixed\ \cos t\ \ of\ \ production,\ k = 750$$

$$TC = 375Q + 15Q^2 - \frac{1}{9}Q^3 + 750$$

$$When\ Q = 45,$$

$$TC = 375(45) + 15(45)^2 - \frac{1}{9}(45)^3 + 750$$

$$= 37,875$$

Example 3: The saving cost of an electronic firm is given by the function, $f(t) = 30t$ .Supposing the price of an electric fan is 280,000. Determine the number of days necessary to recover the cost of the function.

$$S(t) = 30t$$

$$\int_0^t S(t)dt \equiv \int_0^t 30tdt$$

$$\equiv 15t^2$$

Cost saving function $= \mathrm{Re}\,coupin\ entails\ 15t^2 = 270,000$

$$t^2 = 18,000$$

$$t = \sqrt{18,000}$$

$$t \approx 134$$

The price can be recovered after 134 days.

Example 4: The marginal cost function of an oil producing firm in Nigeria is $MC = 18 + 45e^x$

(i) Calculate the value of TC supposing TC (0) = 900

(j) (ii) Generate the average cost function AC.

$$MC = 18 + 45e^x$$

$$TC = \int (MC)dQ + k$$

$$TC = \int (18 + 45e^Q)dQ + k$$

$$TC = 18Q + 45e^Q + k$$

$$Q = 0 \equiv TC = 60$$

$$60 = 18(0) + 45e^0 + k$$

$$60 = 45 + k$$

$$k = 15$$

$$TC = 18Q + 45e^Q + 15$$

$$AC = \frac{C}{Q}, \quad \forall Q \neq 0$$

$$AC = \frac{18Q + 45e^{Q} + 15}{Q}$$

$$AC = 18 + \frac{45e^{Q}}{Q} + 15Q^{-1}$$

Example 5: The rate of change of profits of an automobile advertisement is epitomized as, $f(t) = 5200e^{-0.4t}$ where t signifies the number of days after the advertisement.

i. Calculate the total cumulative sales after 10 days
ii. Calculate the total sales during the 12th days
iii. Calculate the total sales owing to the advertisement.

i. Calculate the total cumulative profits after 10 days

Re*call that the total profits after* $t-$*days would be given by*:

$$F(t) = \int_0^t f(t)dt, \qquad where\ profit\ rate\ f(t) = \frac{d}{dt}F(t)$$

$So,$

$$F(10) = \int_0^{10} (5200e^{-0.4t})dt$$

$$= 5200\left[\frac{e^{-0.4t}}{(-0.4)}\right]_0^{10}$$

$$= -13000[e^{-4} - e^{0}]$$

$$= -13000[0.0183 - 1]$$

$$= N12761.9$$

ii. Calculate the total profits during the 12th days

$$= \int_{10}^{12} (5200e^{-0.4t})dt$$

$$= 5200\left[\frac{e^{-0.4t}}{(-0.4)}\right]_{10}^{12}$$

$$= -13000[e^{-4.8} - e^{-4}]$$

$$= -13000[-0.01]$$

$$= N131.1$$

iii. Calculate the total profits owing to the advertisement

$$= \int_0^{\infty} (5200e^{-0.4t})dt$$

$$= \frac{5200}{-0.4}\left[e^{-0.4t}\right]_0^{\infty}$$

$$= -13000[e^{-4.8} - e^{-4}]$$

$$= -13000[0-1]$$

$$= N13,000$$

Example 6: The supply rate of a newly manufactured commodity is given by $f(Q) = 900 - 15700e^{-2Q}$ where Q is the number of days the item is brought to the market. Calculate total sale during the first 20 days of being made readily available in the market**.**

$$Total\ \ sales\ \ for\ \ first\ 20\ days\ in\ the\ market, TS = \int_0^{20} (900 - 15700e^{-2Q})dQ$$

$$TS = \left[900Q + 7,850e^{-2Q}\right]_0^{20}$$

$$= [(18,000 + 7850e^{-40}) - (7850e^{-0})]$$

$$= 18000 - 7850$$

$$TS = N10,150$$

3.4.2.Consumer and Producer Surpluses: Consumer surplus defines the difference between market price at consumer's willingness and the actual market price. . In the analysis of consumers' surplus, sales are established on basis of supply such that price-quantity is situated on supply curve.Producer surplus defines the difference between the acceptable lowest market price and actual market price received by the producer for a commodity.Correspondingly, in analysis of producers' surplus, price is fixed by demand while price-quantity is situated on demand curve. Hence, with numerous individuals acting independently, price-quantity jointly becomes equilibrium point and selling at that point maximizes total social gain. While a lower price implies greater quantity sold, and greater consumer surplus, a higher price implies greater quantity sold, and greater producer surplus.

Consumer's Surplus is the difference between the amount a consumer is willing to pay and the amount he actually pays for a basket of commodity. Considering the graph below, consumer's surplus is calculated as the area under the demand curve from the origin to the equilibrium price minus price multiplied by quantity. Algebraically, we have

$$\int_0^{Q^E} f(Q^D) - P^E Q^E \ .$$

Consumer's surplus is the area shaded pink while producer's surplus is the shaded light blue area.

Producer's (or Supplier's) surplus is the difference between the amount a producer (or a supplier) is willing to earn and the amount he actually earns from a basket of commodity. As shown in figure 4, producer's surplus is calculated as equilibrium price multiplied by equilibrium quantity minus area under the supply curve from the origin to the equilibrium quantity.
Mathematically,

$$P^E Q^E - \int_0^{Q^E} f(Q^S)$$

Producer's surplus is the area shaded yellow ( ).

Thus, while the amount that the consumer is willing to pay has to be greater, the amount that the producer receives should be greater (Mike May, S.J., & Anneke Bart; Suranovic Steve, 2004).

Largely, the lowest price producers are willing to accept is equal to their marginal cost of production. So, while the demand curve in a competitive free market represents the price consumers are willing to pay, supply curve represents the minimum price producers are willing to accept for different quantities produced. The sum of the consumer and producer surpluses is the total social gain or economic surplus. The consumer and producer surpluses are graphically represented in the figure below:

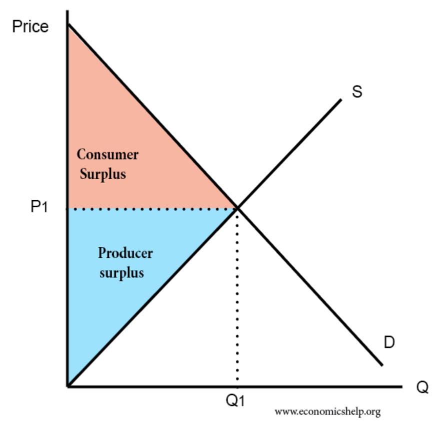

Figure 4: Consumer and Producer's surpluses
Sources:www.economicshelp.org,Economic-surpluses.svg

Numerical Example 3: Consider the following demand and supply functions

$$q^D = 1530 - q$$

$$q^S = 270 + q$$

Calculate the following at given market equilibrium,

a. Consumer surplus,
b. Producer surplus, and
c. Total collective gain.

Solution to Numerical Example 3: Proceed as follows:

$$Consumer\ \ surplus = \int_0^{q^e} (q^D)dq - D(q^e)q^e$$

$$Producer\, surplus = S(q^e)q^e - \int_0^{q^e} (q^S)dq$$

$$(q^D) = 1530 - q$$
$$(q^S) = 270 + q$$
$$Equating\ \ q^D = q^S,$$
$$1530 - q = 270 + q$$
$$1260 = 2q$$
$$q = 630$$
$$D(q^e) = 1530 - 630$$
$$= 900$$

$$Consumer\ \ surplus = \int_0^{q^e} D(q)dq - D(q^e)q^e$$
$$= \int_0^{630} (1530 - q)dq - 630(900)$$
$$= \left[1530q - \left(\frac{1}{2}\right)q^2\right]_0^{630} - 567000$$
$$= [1530(630) - (315)^2] - 567000$$
$$= [963900 - 198450] - 567000$$
$$= 198450$$

$$Producer\, surplus = S(q^e)q^e - \int_0^{q^e} S(q)dq$$
$$= 567000 - \int_0^{630} (270 + q)dq$$
$$= 567000 - \left[270q + \frac{q^2}{2}\right]_0^{630}$$
$$= 567000 - [270(630) + \frac{(630)^2}{2}]$$
$$= 567000 - [170100 + 198450]$$
$$= 198450$$

$$Total\ \ Collective\ \ Gains = Consumer\ \ surplus + Producer\, surplus$$
$$= 396900$$

$$Total\ \ Collective\ \ Gains = Consumer\ \ surplus + Producer\, surplus$$
$$= 947600$$

SELF-ASSESSEMENT EXERCISE

Describe the relationship between consumer surplus and producer surplus.

4.0 CONCLUSION

The rudimentary idea of integral calculus is calculating the area under a curve. This can be calculated by dividing the area into boundless rectangles of considerably small width and sum their areas. In effect, integral assigns numbers to functions in order to define shift, area, and volume. Integration is one of the two main operations of calculus.

5.0 SUMMARY

In this unit, we have discussed the meaning of integral calculus, integration techniques which include, integration by part and integration by substitution as well as economic applications of integral calculus while also solving numerical problems on integral calculus.

6.0 TUTOR-MARKED ASSIGNMENT

1. Consider these demand/supply equations in a free market equilibrium.

$$Q^s = 120 + 0.5q$$
$$Q^d = 175 - 0.2q$$

iv. Calculate consumer surplus,
v. Calculate producer surplus, and
vi. Calculate total collective gain.

2. Find the following integral, using integration by part

$$\int 108z^2(z^5 + 18)dz$$
$$\int (30x^2 e^{10x^3+5})dx$$

2. Find the following integral, using the technique of substitution

$$\int \left(\frac{160z}{z^2 - 16} \right) dz$$

$$\int \left(\frac{378z^6 - 2}{25z^4 + 25z} \right) dz$$

3. Consider the following demand/supply equations in a free market equilibrium.

$$Q^s = 180 + 4q$$
$$Q^d = 245 - 5q$$

(a) Calculate consumer surplus,

(b) Calculate producer surplus, and

(c) Calculate total social gain

5. Consider these demand/supply equations in a free market equilibrium.

$$Q^s = \ln(2q+20)$$

$$Q^d = 200-2q$$

i. Calculate consumer surplus,

ii. Calculate producer surplus, and

iii. Calculate total collective gain.

Consider the following demand and supply functions

$$D(q) = 64 - \frac{1}{24}q$$

$$S(q) = \frac{1}{48}q + 6q$$

$$D(q) = 100 - \frac{1}{50}p$$

$$S(q) = 30 + 10q$$

i. Calculate the following at the given market equilibrium,
   a. Consumer surplus,
   b. Producer surplus, and
   c. Total collective gain.

ii. If the producers can form a cartel and restrict the available quantity to 140, selling at the supply price for 70, what are the consumer surplus, producer surplus, and total social gain?

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UNIT 3: OPTIMIZATION TECHNIQUES

CONTENTS

1.0. INTRODUCTION

In this unit, we shall be focusing on the meaning of optimization, types of optimization, optimization techniques (substitution method and Lagrange multiplier) as well as the conditions for optimization.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Explain the meaning of optimization
- Solve free optimization problems
- Solve constrained optimization problems using both the substitution and the Lagrange Multiplier methods and be able to solveapplication problems to utility maximization

subject to budget constraint, output maximization in relation to cost constraint, cost minimization in relation to a fixed output target deadweight loss of taxation.

- Explain the different conditions for Optimization

3.0 MAIN CONTENT

3.1 Meaning of Optimization

Optimization in economics can be defined as the mathematical process of evaluating the maximum profits, revenue, output etc. or minimum losses, costs etc. Optimization in economic thus describes the best behavior of an individual economic agents. For example, consumers seek to maximize utility does that as by choosing consumption bundles which yield the highest level of utility. Similarly firms seek to maximize profit given production functions, prices of output and inputs by choosing input or factor levels which generate the highest level of profits.

SELF-ASSESSMENT EXERCISE

1. What is optimization in economic analysis?

3.2 Types of Optimization

3.1.1Free Optimization: Free optimization is an unconstrained optimization whereby the selectable variable assumes any value without restrictions. Example include, selecting optimal output level in order to maximize profits without being subjected to price constraint or cost level. Similarly, an economic agent, say the government, could be selecting quantities of ethanol and gas without paying attention to neither available income nor market prices. In the above circumstances, the selected output and quantities are optimal. Accordingly, the mathematical unconstrained problem to be solved is to numerically optimize an objective function without being subjected any constraint~~s~~.

3.1.2 Constrained Optimization: Constrained optimization is optimization of an objective function given relevant constraints on choice variables. Accordingly, choice variables cannot take on any value without boundaries. The objective function could be a profit/revenue/output/utility function, which is to be maximized or a price/cost function, which is to be minimized. Constraints are mostly established conditions for the variables that are required to be optimized.

SELF-ASSESSMENT EXERCISE

What is the difference between free and constrained optimization?

3.3 Optimization Techniques

3.3.1 Substitution Method: The substitution method entails substituting the constraint into the objective function in order to create a composite function that integrates the effect of the constraint.

3.3.2 Lagrange Multiplier: The Lagrange multiplier technique occasions a formulation of a composite function that integrates both the objective function and the constraints. The Lagrangian multiplier technique is mathematically demonstrated as follows:

$$L = g(z, y) + \lambda[d - h(z, y)]$$

Where $g(z, y)$ is the objective function

$d - h(z, y)$

λ (lamda), the Lagragian multiplier

Single Variable Functions: This is the optimization of a function with respect to one argument of the function.

$$y = f(z)$$

Where y is the dependent variable, z is the argument of the function. The domain of the function is the set of all possible values of z while the range of the function is the set of all possible values of y at z.Accordingly, economic optimization problems for single function entails finding maximum and/or minimum values of the function $y = f(z)$ at a point ϕ for which

$$f'(\phi) = 0$$

In effect, the critical point could be a localminimum, maximum point of inflection. Nevertheless, in the scenario whereby second derivative exists:

$$f''(\phi) > 0,\ \phi\ is\ a\ local\ \min imum$$

$$f''(\phi) < 0,\ \phi\ is\ a\ local\ \max imum$$

$$f''(\phi) = 0,\ \phi\ is\ a\ saddle\ po\text{int}\ (inconclusion)$$

Multivariable Functions: This is the optimization of a function with respect to more than one variable. Thus, economic optimization problems for multivariable functions entails finding maximum and/or minimum values of functions of several variables, as in the equation below over prescribed domains.

$$y = f(z, x)$$

Where y is the dependent variable, z and x are the argument of the function.

3.4 Conditions for Optimization

3.4.1 First Order Condition (F.O.C)/Necessary Condition: The First Order Condition (F.O.C) which is the necessary condition states that if g is a differentiable function on $\mathbb{R}$, the derivative of g at z is zero. The points where $g'(z) = 0$ are called critical points or stationary points and the value of *g* at z is called a critical value. The F.O.C necessary conditions for optimization are thus obtained as follows:

$$L_z = \frac{\partial L}{\partial z} = g_z - \lambda h(z) = 0$$

$$L_y = \frac{\partial L}{\partial y} = g_z - \lambda h(y) = 0$$

$$L_\lambda = \frac{\partial L}{\partial \lambda} = d - h(z, y) = 0$$

3.4.2 Second Order Condition (S.O.C)/Sufficient Condition: If *g* is twice differentiable, then conversely, a critical point z of *g* can be analyzed by considering the second derivative of *g* at z :

1. Suppose the 2nd derivative is positive, *z* is a local minimum;
2. Suppose the 2nd derivative is negative, *z* is a local maximum;
3. Suppose the 2nd derivative is zero, then *z* could be a local minimum, a local maximum, or neither.

Overall, by the extreme value theorem, a continuous function on a closed interval must attain its minimum or maximum values at least once at critical points.

The S.O.C sufficient conditions for optimization are thus obtained as follows:

$$L_z = \frac{\partial L}{\partial z} = g_z - \lambda h(z) = 0$$

$$(i)\ L_{zz} < 0\ (relative\ maximum\ point)$$

$$(ii)\ L_{zz} > 0\ (relative\ minimum\ point)$$

$$L_y = \frac{\partial L}{\partial y} = g_y - \lambda h(y) = 0$$

$$(i)\ L_{yy} > 0\,(relative\ minimum\ point)$$

$$(ii)\ L_{yy} < 0(relative\ maximum\ point)$$

$$L_z = \frac{\partial L}{\partial z} = g_z - \lambda h(z) = 0$$

$$(i)\ L_{zz} = 0\ (inflection\ point)$$

$$L_y = \frac{\partial L}{\partial y} = g_z - \lambda h(y) = 0$$

$$(ii)\ L_{yy} = 0\ (inflection\ point)$$

Hessian matrix: Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued objective function f(x,y) without constraints. It describes the local curvature of a function of many variables in relation to free optimization. These conditions are epitomized with the following table.

$$|H| = \begin{vmatrix} g_{zz} & g_{zy} \\ g_{yz} & g_{yy} \end{vmatrix}$$

$$= g_{zz}\, g_{yy} - (g_{zy})^2 < 0$$

Table 1: Hessian Conditions for Optimality

For a relatively maximum stationary value	For a relatively minimum stationary value	Saddle point
$\lvert H_1 \rvert < 0$ $\lvert H_2 \rvert > 0$ $\lvert H_3 \rvert < 0$	$\lvert H_1 \rvert > 0$ $\lvert H_2 \rvert > 0$ $\lvert H_3 \rvert > 0$	If the Hessian has both positive and negative eigenvalues, then z is a saddle point for f.
Hessian is negative definite	Hessian is positive definite	At a local minimum the Hessian is positive definite, and at a local maximum the Hessian is negative-definite.
If the Hessian is negative-definite at z, then f reaches an quarantined local maximum at z	If the Hessian is positive-definite at z, then f reaches an quarantined local minimum at z.	Test is inconclusive

Bordered Hessian: Bordered Hessian is a second-order derivative test whose matrix is a square matrix of second-order partial derivatives of a scalar-valued objective function f(x,y) in a constrained optimizationwith the constraint function given by g(x) = k. these conditions are explained in the following table.

Table 2: Bordered Hessian Conditions for Optimality

For a relatively maximum stationary value	For a relatively minimum stationary value	Saddle point
$\lvert \overline{H} \rvert > 0$	$\lvert \overline{H} \rvert < 0$	If the Bordered Hessian has both positive and negative eigenvalues, then z is a saddle point for f.
Bordered Hessian is positive	Bordered Hessian is negative	At a local minimum the Bordered Hessian is positive definite, and at a local maximum the Bordered Hessian is negative-definite.

If the Hessian is negative-definite at z, then f reaches an quarantined local maximum at z	If the Hessian is positive-definite at z, then f reaches an quarantined local minimum at z.	Test is inconclusive

3.5 Solving Numerical Problems on Optimization

Numerical Example 1: Using the substitution method, maximize the objective function subject to the constraint.

$$\begin{aligned} &Maximize\ q(z,y)=2zy \\ &subject\ to: z+y=32 \end{aligned}$$

Solution to Numerical Example 1: The solution is such that the constraints can be solved to arrive at: $y=32-z$. By substitution, the objective function becomes as follows:

$$\begin{aligned} q(z) &= 2z(32-z) \\ &= 64z-2z^2 \end{aligned}$$

$$F.O.C \Rightarrow \frac{\partial q}{\partial z}=64-4z$$

$$Equating\ \frac{\partial q}{\partial z}=0$$

$$z=\frac{64}{4}$$

$$z=16$$

$$Consequently,\ y=32-z \Rightarrow y=32-16=16$$

$$Thus,\ (z,y)\Rightarrow(16,16)$$

$$S.O.C \Rightarrow \frac{\partial q^2}{\partial z^2}=-4$$

$$\frac{\partial q^2}{\partial z^2}<0$$

Given that the S. O. C <0, it thus implies that the values are for maximum and the maximum value is 512.

Numerical Example 2: Find the critical points at which the following function may be optimized.

$$q=6z^2-2zy+4y^2-8z-14y+24$$

Establish if at those points the function q is maximized, minimized, an inflection point, or a saddle point.

Solution to Numerical Example 2: Find the first Order Condition as follows:

$$q = 6z^2 - 2zy + 4y^2 - 8z - 14y + 24$$

$$F.O.C = q_z = \frac{\partial q}{\partial z} = 12z - 2y - 8$$

$$\frac{\partial q}{\partial y} = -2z + 8y - 14$$

$$Setting\ \frac{\partial q}{\partial z} = \frac{\partial q}{\partial y} = 0$$

$$12z - 2y = 8$$

$$-12z + 48y = 84$$

$$46y = 92$$

$$y = 2$$

$$12z - 2(2) = 8$$

$$12z = 12$$

$$z = 1$$

The critical points are $(z^*, y^*) = (1,2)$

$S.O.C$

$$\frac{\partial^2 q}{\partial z^2} = 12 > 0$$

$$\frac{\partial^2 q}{\partial y^2} = 8 > 0$$

$$\frac{\partial^2 q}{\partial z \partial y} = -2$$

$$\frac{\partial^2 q}{\partial y \partial z} = -2$$

$$\frac{\partial^2 q}{\partial z^2}\left(\frac{\partial^2 q}{\partial y^2}\right) > \frac{\partial^2 q}{\partial z \partial y}\left(\frac{\partial^2 q}{\partial y \partial z}\right)$$

$$12(8) > 2(2)$$

$$96 > 4$$

Therefore, with the critical points $(z^*, y^*) = (1,2)$, the function q is at a minimum having successfully implemented the S.O.C for a minimum.

Numerical Example 3: Given the following function, optimize the following revenue function:

$$R = 2z_1 z_2 + 4z_1$$

$$s.t.\ 6z_1 + 8z_2 = 80$$

i. Optimize the function using the Lagrangian function

ii. State the nature of the critical values using the S.O.C

Solution to Numerical Example 3: Find the first Order Condition as follows:

$$L = (2z_1z_2 + 4z_1) + \lambda[80 - 6z_1 - 8z_2]$$

$$L_{z_1} = \frac{\partial L}{\partial z_1} = 2z_2 + 4 - 6\lambda = 0$$

$$L_{z_2} = \frac{\partial L}{\partial z_2} = 2z_1 - 8\lambda = 0$$

$$L_{\lambda} = \frac{\partial L}{\partial \lambda} = 80 - 6z_1 - 8z_2 = 0$$

Putting the equations in matrix format, we have as follows:

$$\begin{bmatrix} 0 & 2 & -6 \\ 2 & 0 & -8 \\ 6 & 8 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 80 \end{bmatrix}$$

The solution set can be obtained as $(z_1^*, z_2^*, \lambda^*) \Rightarrow (8,6,12)$

Revise please and note that $\bar{z}_1 = 8, \quad \bar{z}_2 = 4, \quad \bar{\lambda} = 2$

For the S. O. C. shows that $\begin{vmatrix} L_{z_1z_2} & L_{z_1z_2} \\ L_{z_2z_1} & L_{z_1z_2} \end{vmatrix} < 0$, therefore the solution set is a maximum

i. *How about examples in economics Utility maximization subject to budget constraint, cost minimization subject to output constraint, output maximization subject to cost constraint …?*

3.6. Economic Applications: Three different economic applications are carried out here. These include, applications to utility maximization subject to budget constraint, output maximization in relation to cost constraint, cost minimization in relation to a fixed output target deadweight loss of taxation.

3.6.1. Application 1: Consumer Problem/Utility Maximization: Recall the consumer's choice of utility maximization is stated thus:

$$Max\ U(z,y)$$
$$s.t.\ P_z z + P_y y \le I$$
$$where\ P_z\ is\ the\ price\ of\ commodity\ z$$
$$P_y\ is\ the\ price\ of\ commodity\ y$$
$$I\ is\ the\ consumer's\ income\ level\ (budeget)$$
$$U\ is\ the utiltity\ to\ be\ derived\ from\ the$$
$$consumsption\ of\ two\ commodities\ namely\ z\ \&\ y$$
$$L_z = U(z,y) + \lambda(I - P_z z + P_y y)$$
$$L_z = \frac{\partial L}{\partial z} = U_z - \lambda P_z = 0 \qquad 1$$
$$L_y = \frac{\partial L}{\partial y} = U_y - \lambda P_y = 0 \qquad 2$$
$$L_\lambda = \frac{\partial L}{\partial \lambda} = I - P_z z - P_y y = 0 \qquad 3$$

$$\text{Re}\,arranging,$$
$$\frac{U_z}{U_y} = \frac{P_z}{P_y} \quad 4 \qquad such\ that,\ \lambda = \frac{U_z}{P_z} = \frac{U_y}{P_y}$$
$$Equation\ 4\ sympolizes\ the\ equality$$
$$between\ psychic\ trade-off\ and$$
$$monetary\ trade-off\ between\ both\ commodities$$
$$\text{The value of } \lambda \text{ indicates that at the point of utility-maximization,}$$
$$\text{an additional naiar spent on each commodity bundle yields the same marginal utility.}$$

In effect, λ equals the "shadow price" of the budget constraint which defines the quantity of utils that could be obtained with additional naira income of the budget.

Numerical Example: Consider that the function $U(z,y) = 2z^{0.4}y^{0.6}$
The price of z is $N32$ and the price of y is $N16$
The individual agent has income of 160.

(a) Find the optimal consumption choice of this individual agent.

(b) Prove that at the optimum, the marginal rate of substitution equals the price ratio.

Solution:

$$L(z,y)=2z^{0.4}y^{0.6}+\lambda[160-32z-16y]$$

$$L_z=\frac{\partial L}{\partial z}=0.8z^{-0.6}y^{0.6}-32\lambda=0 \qquad 1$$

$$L_y=\frac{\partial L}{\partial y}=1.2z^{0.4}y^{-0.4}-16\lambda=0 \qquad 2$$

$$L_\lambda=\frac{\partial L}{\partial \lambda}=160-32z-16y=0 \qquad 3$$

Dividing eqns ((1) by (2), we have that:

$$\frac{0.8z^{-0.6}y^{0.6}}{1.2z^{0.4}y^{-0.4}}=2$$

$$\frac{0.8y}{1.2z}=2$$

$$2.4z=0.8y$$

$$y=3z$$

Substituting for y in eqn (3), we have that:

$$160-32z-16(3z)=0$$

$$160-32z-48z=0$$

$$z=2$$

$$y=6$$

$$(z^*,y^*)=(2,6)$$

checking for exterma values

$$\frac{\partial^2 L}{\partial z^2}<0$$

$$\frac{\partial^2 L}{\partial y^2}<0$$

Thus, (2, 6) *are relative* max*imum values*

$$\frac{MU_z}{MU_y}=MRS=\frac{0.8z^{-0.6}y^{0.6}}{1.2z^{0.4}y^{-0.4}}=\frac{0.8y}{1.2z}=\frac{0.8(6)}{1.2(2)}=2$$

$$\frac{P_z}{P_y}=\frac{32}{16}=2$$

Consequently, at the optimum, the marginal rate of substitution equals the price ratio. This is so because The MRS shows the maximum amount of z that could be traded for one unit of y, without losingutility. Accordingly, if it is lower than the relative price of y, the consumer would be better off consuming less of y and moreof z. on the other hand, supposing the price of z is higher than the relative price of y, the rational consumer would be better off consuming more of y and less of z. also, supposing MUz = MUy = 1 and the price of z is twice that of y, the consumer could give up one unit of y and get two units of z. relatively, should the price of y be twice the price of z, the consumer would be giving up one unit of y in order to get two units of z.

Numerical Example: A firm's utility function is given by $U(z, y) = 2z + 4\ln y$

Kezia's budget is given by: $I = P_z z + P_y y$

Where z is quantity of cakes, y is quantity of biscuits, P_z is the price of one unit of cake, P_y is the price of one unit of biscuit and I is Kezia's total income/budget

(a) Derive Kezia's demand equations for cake and biscuits

(b) Would Kezia rather decide to spend every additionalnaira income on cakes?

Solution to Numerical Example: Recall that:

$$\frac{MU_z}{MU_y} = \frac{P_z}{P_y} \quad \square \quad \frac{U_z}{U_y} = \frac{P_z}{P_y}$$

where,

$$U(z_1, z_2) = 2z_1 + 4\ln z_2$$

$$\frac{MU_z}{MU_y} = \frac{P_z}{P_y} \quad \square \quad \frac{U_z}{U_y} = \frac{P_z}{P_y}$$

where,

$$U(z, y) = 12z + 4\ln y$$

$$MU_z \equiv \frac{\partial U(z, y)}{z} = 12$$

$$MU_y \equiv \frac{\partial U(z, y)}{y} = \frac{4}{y}$$

$$\textit{Optimality is defined at } \frac{MU_z}{MU_y} = \frac{P_z}{P_y}$$

$$Thus,$$

$$\frac{12}{\frac{4}{y}} = \frac{P_z}{P_y} \Rightarrow 3y = \frac{P_z}{P_y}$$

$$I = P_z z + P_y y$$

$$Since\ 3y = \frac{P_z}{P_y}$$

$$y = \frac{3P_y}{P_z}$$

$$z = \frac{I - P_y y}{P_z}$$

$$= \frac{I - P_y\left(\frac{3P_y}{P_z}\right)}{P_z}$$

$$z = \frac{I}{P_z} - \frac{3}{P_z}$$

$$z = \frac{I-3}{P_z}$$

Kezia's demand equations for cake and biscuits are $z = \frac{I-3}{P_z}$, $y = \frac{3P_y}{P_z}$ respectively.

Kezia would of course rather spend everyadditional naira income on cakes provided her total budget exceeds the price of cake in the market.

Numerical Example: Kezia's utility function is given by $U(z, y) = z + 2\ln y$

Kezia's budget is given by: $24 = 4z + 2y$

Where z is quantity of cakes, y is quantity of biscuits, N4 is the price of one unit of cake, N2 is the price of one unit of biscuit and N24 is Kezia's total income/budget

(a) What is Kezia's demand for cake and biscuits?

(b) Is it true that Kezia would spend every naira in additional income on cakes?

$$\frac{MU_z}{MU_y} = \frac{P_z}{P_y} \quad \quad \frac{U_z}{U_y} = \frac{P_z}{P_y}$$

where,

$$U(z, y) = z + 2\ln y$$

$$MU_z \equiv \frac{\partial U(z, y)}{z} = 1$$

$$MU_y \equiv \frac{\partial U(z, y)}{y} = \frac{2}{y}$$

$$Optimality\ is\ defined\ at\ \frac{MU_z}{MU_y} = \frac{P_z}{P_y}$$

$$Given\ the\ budget\ constra\text{int}, 24 = 4z + 2y$$

$$\frac{1}{\frac{2}{y}} = \frac{P_z}{P_y} \quad \frac{y}{2} = \frac{4}{2}$$

$$2y = 8$$

$$y = 4$$

$$z = \frac{I - P_y y}{P_z}$$

$$= \frac{24 - 2\left(\frac{4}{2}\right)}{4}$$

$$z = 5$$

$$(z^*, y^*) = (5, 4)$$

$$S.O.C:$$

$$MU_{zz} \equiv \frac{\partial U^2(z, y)}{\partial z^2} = 0\ (inconclusive) \quad saddle\ po\text{int}$$

$$MU_{yy} \equiv \frac{\partial^2 U(z, y)}{\partial y^2} = \frac{2}{y^2}$$

$$At\ \ y = 4,\ MU_{yy} = -\frac{2}{16} < 0, \quad local\ \max imum\ po\text{int}$$

Thus, Kezia would be demanding and consuming 5 units of cakes and 4 units of biscuit.
It is true that Kezia would spend every additional naira income on cakes because her total income is greater than the price of cake (N4). In fact, she won’t buy any more biscuit which is less expensive compare to the price of cake once her total income/budget increases.

3.6.2. Application 2: Output-Maximization for a Given Cost and Cost Minimization for a Given Output: The firm also maximizes its profits by maximizing its output, given its cost outlay and the prices of the two factors. The firm can reach the optimal factor combination level of maximum output by moving along the isocost line.

In the theory of production, the profit maximization firm is in equilibrium when, given the cost-price function, it maximizes its profits on the basis of the least cost combination of factors. For this, it will choose that combination which minimizes its cost of production for a given output. This will be the optimal combination for it.

This analysis is based on the following assumptions: There are two factors, labour and capita; All units of labour and capital are homogeneous; prices of units of labour (w) and that of capital (r) are given and constant; cost outlay is given; firm produces a single product; price of the product is given and constant; firm aims at profit maximization and there is perfect competition in the factor market.

Given these assumptions, the position of least-cost combination of factors for a given level of output is where the isoquant curve is tangent to an isocost line. At optimal equilibrium,

$$\frac{MP_L}{MP_K} = \frac{r}{w}$$

$$\Rightarrow \frac{MP_L}{w} = \frac{MP_K}{r}$$

$$\Rightarrow \frac{w}{MP_L} = \frac{r}{MP_K}$$

$$where\ MRTS_{L,K} \equiv \frac{MP_L}{w}$$

The economic interpretation holds that if the firm is to maximize output subject to a cost constraint, the marginal product of money spent on each factor input should be equal, and the firm should distribute its money on purchasing the factors inputs accordingly. In effect, the firm hires one unit of labour provided it spends w amount more of money on labour and this in turn generates more additional output of MP_L.

Numerical Example: Consider that a firm has the following production function

$$Q = LK - 0.1K^2 - 0.2L^2$$

The prices per unit of L and K are N40 and N50 respectively. If the firm decided to maximize output in relation to a budget constraint of N1000.

i. Calculate the optimal values for K and L.

ii. Consider that the firm's budget was increased by 20%, calculate the increase in output given the new budget.

Solution to Numerical Example:

$$\Upsilon = LK - 0.1K^2 - 0.2L^2 + \lambda[1000 - 40L - 50K]$$

$$\frac{\partial \Upsilon}{\partial L} = K - 0.4L - 40\lambda$$

$$\frac{\partial \Upsilon}{\partial K} = L - 0.2K - 50\lambda$$

$$\frac{\partial \Upsilon}{\partial \lambda} = 1000 - 40L - 50K$$

$$Setting F.O.C = 0,\ we\ \ have\ that:$$

$$\frac{\partial \Upsilon}{\partial L} = 0 \quad K - 0.4L - 40\lambda = 0 \qquad 1$$

$$\frac{\partial \Upsilon}{\partial K} = 0 \quad L - 0.2K - 50\lambda = 0 \qquad 2$$

$$\frac{\partial \Upsilon}{\partial \lambda} = 0 \quad 1000 - 40L - 50K = 0 \qquad 3$$

$$Solving\ \ for\ \lambda\ in\ \ (1)\ \&\ (2),\ we\ have\ \ that:$$

$$\lambda = 0.025K - 0.01L$$

$$\lambda = 0.02L - 0.004K$$

$$Equating\ \lambda\ to\ \lambda,$$

$$0.025K - 0.01L = 0.02L - 0.004K$$

$$0.029K = 0.03L$$

$$K = 1.034L$$

$$Substituting the\ value\ of\ \ K\ \text{int}\, ooutput constra\,\text{int},\ we\ have\ as\, follows:$$

$$1000 - 40L - 50[1.034L] = 0$$

$$1000 = 91.7L$$

$$L = 10.9$$

$$K = 11.3$$

Considering that the firm's budget was increased by 20%, the increase in output given the new budget would be calculated as follows:

$$\lambda = 0.025(11.3) - 0.01(10.9)$$
$$= 0.173$$
$$20\%\ increase\ in\ budget\ of\ the\ firm\ is\ 200.$$
$$Therefore,\ the\ increase\ in\ the\ fim's\ output\ is\ 0.173(200) = 34.6$$

Consider the following short run production function:

$$Q = 108L^2 - 7.2L^3$$

i. Calculate the value of L that maximizes output
ii. Calculate the value of L that maximizes average product
iii. Calculate the value of L that maximizes marginal product

$$Q = 108L^2 - 7.2L^3$$
$$\frac{dQ}{dL} = 216L - 21.6L^2$$
$$Setting\ \frac{dQ}{dL} = 0,$$
$$216L - 21.6L^2 = 0$$
$$L(216 - 21.6L) = 0$$
$$L = \frac{216}{21.6}$$
$$= 10$$

$$Checking\ for\ reltave\ \max imum\ po\text{int}$$

$$\frac{d^2Q}{dL^2} = 216 - 43.2L < 0$$
$$At\ L = 10, \frac{d^2Q}{dL^2} = -216 < 0$$
$$Therefore,\ L = 10\ is\ a\ reletive\ \max imum\ po\text{int}$$

The value of L that maximizes average product

$$AP = \frac{Q}{L}$$
$$AP = \frac{108L^2 - 7.2L^3}{L}$$
$$AP = 108L - 7.2L^2$$

$$\frac{dAP}{dL} = 108 - 14.4L$$

$$Setting\ \frac{dAP}{dL} = 0,$$

$$108 - 14.4L = 0$$

$$L = \frac{108}{14.4}$$

$$= 7.5$$

The value of L that maximizes marginal product

$$Q = 108L^2 - 7.2L^3$$

$$MP = \frac{dQ}{dL} = 216L - 21.6L^2$$

$$Setting\ \frac{dMP}{dL} = \frac{d^2Q}{dL^2} = 216 - 43.2L$$

$$Setting\ \frac{dMP}{dL} = 0$$

$$216 - 43.2L = 0$$

$$L = \frac{216}{43.2}$$

$$= 5$$

$$Checking\ \ for\ \ reltave\ \ \max imum\ \ po\text{int}$$

$$\frac{d^2MP}{dL^2} = -43.2 < 0$$

$$Therefore,\ L = 5\ is\ a\ reletive\ \max imum\ \ po\text{int}$$

3.6.3. Application 3: Deadweight Loss of Taxation: The deadweight loss of taxation is the damage caused to economic efficiency and production by a tax. In other words, the deadweight loss of taxation is a measurement of how far taxes reduce the standard of living among the tax payers in the society. It is indeed, an excess burden pass on to the consumer, as the socially optimal quantity of a good or a service is not produced due to monopoly pricing in the case of fake scarcity, a positive or negative externality, a tax or subsidy, or a binding price ceiling or price floor such as a minimum wage.

Taxation has the opposite effect of a subsidy. Taxation dissuades consumers from a purchase while a subsidy induces consumers to purchase a product that would otherwise be too expensive for them in light of their marginal benefit. This excess burden of taxation therefore epitomizes the lost utility for the consumer. A common example of this is the tax levied against goods harmful to society and individuals. For example, taxes levied against tobacco. The indirect tax (VAT), weighs on the consumer affects the utility of the rational consumer.

When a tax is levied on consumers, the demand curve shifts downward in relation to the tax size. Similarly, when tax is levied on producer/sellers, the supply curve shifts upward by the tax size. When the tax is imposed, the goods' price rise and the consumer is made to pay more, and the price received by seller falls. Consequently, consumers and sellers share the burden of the tax, notwithstandingthe way and manner of imposition. Since price is the determining factor in the market, the quantity sold is reduced below what it could have been supposing there was no taxation. In effect, taxation drops the market size for that given commodity. Accordingly,the general market size declines below the optimum equilibrium.

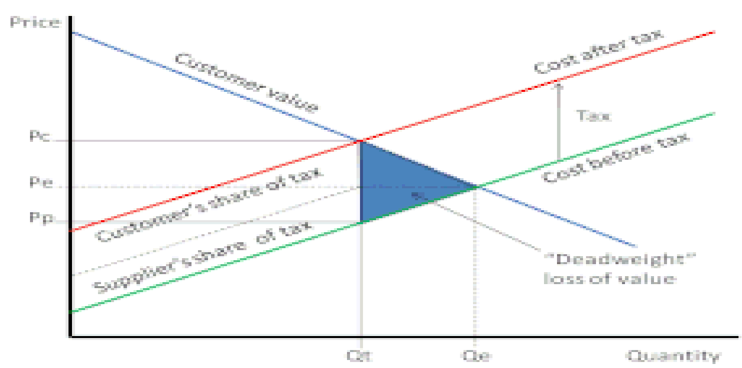

Figure 1: Deadweight Loss of Taxation
Source: Wikipedia

Shown in the figure above, is the deadweight loss of taxation whereby the tax increases the price paid by consumers to *Pc* and decreases price received by sellers to *Pp* and the quantity sold reduces from *Qe* to *Qt.*

SELF ASSESSMENT EXERCISE
Define the deadweight loss of taxation

The demand curve is $Q(D) = 200 - 10p$ and the supply curve is $Q(S) = 10p$

a. A quantity tax of N4 per unit is retained on the good. Calculate the dead-weight loss of the tax.
b. A value (ad valorem) tax of 20% is retained on the good. Calculate the dead-weight loss of the tax.

$$At\ equilibrium,\ Q(D) = Q(S)$$
$$without\ taxation:$$
$$200 - 10p = 10p$$
$$p = 10$$
$$Q = 100$$

$$At\ equilibrium,\ Q(D) = Q(S)$$
$$with\ taxation:$$
$$200 - 10p = 10p$$
$$p^S = \frac{Q(S)}{10} + 4$$
$$10P^S = Q(S) + 40$$
$$Q(S) = 10p - 40$$

$$At\ equilibrium,\ 200 - 10p = 10p - 40$$
$$p = 12$$
$$Q = 80$$

Deadweight loss = Quantity without taxes – Quantity with taxes

$$= 100 - 80$$
$$Deadweigh\ loss = 20$$

$Deadweight\ loss\ to\ the\ consumer\ due\ to\ tax\ imposition\ of\ N4\ per\ unit\ of\ consumption = 20$.
The selling price was N10 without taxes but rose to N12 after the imposition of taxes.

$$p^S = \frac{Q(S)}{10}$$
$$= \frac{80}{10}$$
$$= 8$$

It can be observed that the price increase (N2) is not equal to the tax (N4).

4.0 CONCLUSION

In the theory of economic optimization, maximizing or minimizing some functions is relative to some range of choices available in a certain circumstances. The function allows comparison of the different selection available for the single-mindedness of determining which selection is the best. The major economic applications is in cost/losses minimization and profit/revenue maximization.

5.0 SUMMARY

In this unit, we have discussed meaning of economic optimization, types of optimization, free optimization, constrained optimization as well as optimization techniques. Also, we solved some problems of unconstrained and constrained optimization including applications to utility maximization subject to budget constraint, output maximization in relation to cost constraint, cost minimization in relation to a fixed output target deadweight loss of taxation.

6.0 TUTOR-MARKED ASSIGNMENT

Kezia's utility function is given by $U(z, y) = 6z + 4\ln y^2$

Kezia's budget is given by: $48 = 8z + 4y$

Where z is quantity of cakes, y is quantity of biscuits, N8 is the price of one unit of cake, N4 is the price of one unit of biscuit and N48 is Kezia's total income/budget

(a) What is Kezia's demand for cake and biscuits?

(b) Is it true that Kezia would spend every naira in additional income on cakes?

(c) What happens to Keiza's consumption or demand when Kezia's income changes

(d) What happens to Kezia's consumption demand when the prices of both goods increase?

Considering the present economic hardship as induced by the corona virus pandemic, Miss Winifred is confronted with hard times. Winifred's income per day is N200, she spends N100 on food stuffs and N100 on local transportation despite the lockdown. Though, she is been given a social allowance in the form of 10 food tickets per day. The tickets can be exchanged for N10 worth of food, and she only has to pay N5 for such tickets. Show the budget line with and without the food ticket. Assuming Winifred exhibits homothetic consumption preferences, how much more food will she bargain for once she receives the food tickets?

A firm spends total of N3000 every year in the production of plastic tables and chairs. A unit of plastic chair costs N75 and a plastic table costs N125.

(a) Write the equation for firm cost constraint and draw it in a diagram.

(b) Consider that the firm never produces plastic chair without the production of plastic table and never produces a plastic table without the production of a plastic chair, calculate the units of each products the firm produces. How much of each will she consume?

Consider that the function $U(z, y) = 4z^{0.2}y^{0.8}$

The price of z is $N64$ and the price of y is $N16$

The individual agent has income of 264.

(a) Find the optimal consumption choice of this individual agent.

(b) Verify that at the optimum, the marginal rate of substitution equals the price ratio. Discuss briefly the economics of this equality.

(c) Assume that the price of z falls to 8, find the optimal consumption choice of this individual agent.

A University student plans to supplement the monthly allowances of N1000 given to him by his parent. The student then choose to take up a part time teaching job at the home of a certain rich man. The part time teaching wage is N200 every month. His utility function is:

$$U(c,l) = cl$$

where Q is units of consumption and l is leisure measured in hours.
The amount ofleisure time that he has left after allowing for necessary activities is 50 hours a week.

i. Determine the monetary value of the student's endowment?
ii. Set up the maximization problem and decide optimal consumption and leisure.
iii.Derive the demand equation for consumption.
iv. Calculate the number of hours the student would be engaged in a work activity if the student did not receive any study allowance?

The inverse demand curve (the demand curve but with p instead of q on the left hand side) is given by $Q(Q) = 1000 - 100p$ The consumer consumes ten units of the good (Q).

(a) Determine the amount of money needed to compensate the consumer reducing her consumption to zero?
(b) Suppose now that the consumer is buying the goods at a price of 300 per unit. Now, it is expected of the consumer to reduce purchases to zero, how much does the consumer needs to getcompensated?

Bridget has a demand function $Q = 20 - 4p$

a. Calculate Bridget's price elasticity of demand when the price is 6?
b. At what price is the elasticity of demand equal to -1?
c. Suppose Bridget's demand function takes the general form $Q = \delta - \alpha P$. Derive Bridget's algebraicexpression for elasticity of demand at an arbitrary price p.

Consider the demand function of a firm is $Q(P) = (4p + 4) - 8$

a. Calculate the price elasticity of demand?
b. At what price is the price elasticity of demand equal to minus one?

Suppose we have the following demand and supply equations

$$D(p) = 460 - 2p$$

$$S(p) = 300 + 2p$$

i. Calculate the equilibrium price and quantity?

ii. The government decides to restrict the industry to selling only 160 units by imposing a maximumprice and rationing the good. What maximum price should the government impose?
iii. The government choose that the firms in the industry need not receive more than the minimum pricethat it would take to have them supply 180 units of the good. In line with this, the government issued 180allocationslips. If the allocationslips were freely bought and sold on the open market, what would be theequilibrium price of these slips?
iv Calculate the dead-weight loss from restricting the supply of the goods. Will the dead-weightloss rise or fall if the government choose not allow the slips to be sold on the openmarket?

Given the function,

$$R = 130w_1 - 15w_1^2 + 25w_1w_2 - 30w_2^2 + 60w_2$$

$$s.t.\ \ 15w_1 + 5w_2 = 850$$

(a) Optimize the function

(b) State the nature of the critical values using the S.O.C

Consider an entrepreneur's short-run total cost function is $C = \frac{1}{2}Q^3 - 5Q^2 + 8.5Q + 33$ Determine the output level at which the entrepreneur maximizes profit if p = 10. Determine output elasticity of cost at maximum output level.

Consider that a firm has two production functions for two different goods, namely Q1 & Q2. Supposing Q1 has a higher elasticity of substitution and a lower value for the parameter $\boldsymbol{\alpha}$ than Q2. Decide the input price ratio at which the input use ratio would be the equal for both goods. Decide the good would have the higher input ratio if the input price ratio were lower? Decide the good that would have the higher use ratio if the price ratio were higher?

The long-run cost functions for every firm that supplies Q is $C = (1/Q^3) - 2Q^2 + 4Q$. Businesses will move into the industry if profits are positive and move out of the industry if profits are negative. Describe the industry's long-run supply function. If the demand function is $Q = 1000 - 10P$. Determine equilibrium price, aggregate quantity, and number of firms.

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UNIT 4: DIFFERENTIAL EQUATIONS

CONTENTS

1. 0. INTRODUCTION

In this unit, we focused on the concept of differential equations, order and degree of differential equations, general and particular solutions of differential equations, solving differential equations.

2.0. OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Evaluate integral of differential equations
- Evaluate order and degree of differential equations
- Evaluate general and particular solutions of differential equations
- Solve differential equations
- Apply differential equations to economic problems

3 MAIN CONTENT

3.1 Definition of Differential Equation

A differential equation (DE) is an equation of functions and their derivatives. In other words, such equations have derivatives contained in them. Examples of Differential Equations are these two:

$$\frac{dy}{dz} = z^9 - 6$$
$$dy = (z^9 - 6)dz$$

A standard first order linear difference equation is of the form.

$$y_{t+1} + by_t = c$$

On the other hand a standard first order linear differential equation is of the form

$$\frac{dy}{dt} + by = d\ .$$

Only dy/dx can enter a 1st order differential equation and it takes various powers to enter $dy/dx, (dy/dx)^2, (dy/dx)^3, (dy/dx)^4, (dy/dx)^5\ etc$. Examples of linear differential equations are the following:

$$\frac{d^3y}{dx^3} - 7\left(\frac{dy}{dx}\right) + 2y = 0$$
$$\frac{d^5y}{dx^5} + x^3\left(\frac{d^2y}{dx^2}\right) - x^2\left(\frac{dy}{dx}\right) = 2xe^x$$

A simple dynamic market model for the (discrete case) will appear as

$$Q_t^D = b - dp_t \quad b, d > 0$$
$$Q_t^S = -c + fp_{t-1} \quad c, f > 0$$

The demand equation simply states that quantity demanded at the current time t is a function of price at the current time t. The supply equation states indicates that quantity supplied at the current time t is a function of the price in the previous time t-1. This is in contrast to the static model where the issue of time is not taken into thought. Equating and rearranging will yield the standard first order linear difference equation.

$$Q_t^D = b - dp \quad b, d > 0$$
$$Q_t^S = -c + fp \quad c, f > 0$$

A differential equation will consider the speed of adjustment to equilibrium in a situation of excess demand

$$\frac{dp}{dt} = \phi(Q_t^D - Q_t^S) \text{ where } \phi \text{ is the speed of adjustment .}$$

Substituting the demand and supply functions into the differential equation yields

$$\frac{dp}{dt} = \phi\left[b - dp - (-cfp)\right]$$

Opening the brackets on the right hand side gives

$$\frac{dp}{dt} = \phi(b - dp + c - fp)$$

Re-arranging, it gives

$$\frac{dp}{dt} = \phi\left(d + f\right)p = \alpha\left(b + c\right)$$

This is a market first order linear differential equation.There are two common types of differential equation, namely, ordinary and partial DE.

3.1.1. Ordinary Differential Equation: An ordinary differential equation relates functions of one variable to the derivatives of the variable. Thus, OED is the type of DE that differential equation involving total derivatives of dependent variables with respect to only one independent variable. An example of a linear ordinary differential equation of order n is given by:

$$\alpha_0(x)\frac{d^n y}{dx^n} + \alpha_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \alpha_2(x)\frac{d^{n-2} y}{dx^{n-2}} + ... + \alpha_{n-1}(x)\frac{dy}{dx} + \alpha_n(x)y = f(x)$$

3.1.2. Partial Differential Equation:A partial differential equation relates functions of more than one variable to the partial derivatives of the variables.Accordingly, it is the differential equation involving total derivatives of dependent variables with respect to more than one independent variables.

3.1.3. Homogenous Linear Differentil Equations: An homogenous differential equation is an equation with constant coefficients such that α_n are independent of x. this is given below:

$$\alpha_0 \frac{d^n y}{dx^n} + \alpha_1 \frac{d^{n-1} y}{dx^{n-1}} + \alpha_2 \frac{d^{n-2} y}{dx^{n-2}} + ... + \alpha_{n-1} \frac{dy}{dx} + \alpha_n y = 0$$

3.2 Order and Degree of Differential Equations: The order of a differential equation is the highest derivative in the equation. In other words, it is the order of the highest differential coefficient that it holds. The degree of differential equations is the highest power of the highest derivative in the differential equation.In effect, it is the highest power of the highest order differential coefficient that the equation holds. So, there are first order, second order differential equations. First order Differential Equations contains only first derivatives while the Second order Differential Equations contains second derivatives and also first derivatives**.**

Consider the following examples:

$$\frac{d^3 y}{\partial z^3} + \left(\frac{dy}{\partial z}\right)^6 + 4z - 6y = 90 \; (1)$$

$$5z + \left(\frac{dy}{\partial z}\right)^9 = 6\cos(z) - \cos(y) \; (2)$$

$$(y''')^3 - 18(y')^2 + 10y = 5 \; (3)$$

i. The first differential equation has order 3 (the highest derivative appearing is the second derivative) and degree 1 (the power of the highest derivative is 1.)

ii. The second differential equation has order 1 (the highest derivative appearing is the first derivative) and degree 9 (the power of the highest derivative is 9.).

iii. The third differential equation has order 3 (the highest derivative appearing is the third derivative) and degree 3 (the power of the highest derivative is 3.)

3.3 General and Particular Solutions: When solving differential equations, we obtain the general solution first, before proceeding to obtain the particular solution. The general solution encompasses a constant, k while the particular solution is obtained by substituting known values called the initial value conditions of the variables say, z and y into the general solution in order to evaluate the value of the constant of integration.

In effect, the general solution is the sum of the complementary function and a particular integral and this can be mathematically expressed as:

$$y(x) = y_c + y_p = Ae^{-ax} + y(0)e^{-ax}$$

Where,

$$y_c = Ae^{-ax} \qquad (complementary\ function) \qquad (1)$$

$$y(0) = y(0)e^{-ax} \qquad (particular\ \ solution) \qquad (2)$$

The complementary function (yc) is the solution of the homogenous ODE. By equivalence, what this means is that whenever any particular value is substituted for A, the solution becomes a particular solution and it is such that y(0) is the only value that can make the solution fulfill the initial condition. In effect, the result of definitizing the arbitrary constant is christened the particluar/definite solution.

3.4 Solving Differential Equations

It is worthy of note to mention that the solution to a differential equation is not numerical, it is a function y(x) and this very solution is free to exhibits any derivative or differential notations.The solution to a DE will always involve integration at some point.

Numerical Example 1: Consider the differential equation below:

$$dy - 12zdz = 0$$

Obtain both general and particular solutions given that $y(0) = 6$.

Solution to Numerical Example 1: General solution for the differential equation is obtained by integration as follows:

$$dy - 12zdz = 0$$

$$dy = 12zdz$$

$$\int dy = \int 12zdz$$

$$y = \frac{12z^2}{2} + k$$

$$y = 6z^2 + k$$

Particular solution for the differential equation is obtained by substituting y(0) = 6 to the general solution. Thus, at z = 0, y = 6 such that:

$$y = 6z^2 + k$$

$$6 = 6(0)^2 + k$$

$$k = 6$$

Numerical Example 2: Find the particular solution of

$$y' = 10$$

Given that when z = 0, y = 5.

Solution to Numerical Example 1:

$$
\begin{aligned}
&y' = 10 \\
&dy = 10dz \\
&\int dy = \int 10dz \\
&y = 10z + k \\
&At\ \ z = 0,\ \ y = 5 \\
&k = 5
\end{aligned}
$$

Numerical Example 3:Solve the initial value problem

$$2x(y+1)dx - 2ydy = 0$$

$$where\ \ x = 0, y = -2$$

Solution to Example 3:

$$2xydx + 4xdx = ydy$$

$$\frac{dy}{dx} = \frac{2x(y+1)}{y}$$

$$ydy = 2x(y+1)dx$$

$$\int \left[1 - \left(\frac{1}{y+1}\right)\right] dy = 2xdx$$

$$x^2 = y - \ln|y+1| + k$$

We now use the initial conditions y(0) = - 4.

$$(0)^2 = -4 - \ln|-3| + k$$

$$Thus,$$

$$x^2 = y - \ln|y+1| + 5$$

SELF-ASSESSMENT EXERCISE

Describe the relationship between a difference equation and a differential equation.

3.5. Application of Differential Equations to Economics:The mathematical theory of differential equations developed with the sciences where the equations had originated together with the results founding application.In economics, differential equations are used to model equilibria, the time path analysis, economic growth etc.

3.5.1. Application I (Economic Equilibria Analysis): Equilibrium is a state of a system which does not change. If the dynamics of a system is described by a system of differential equations, then equilibria can be estimated by setting all derivatives equl to zero. An equilibrium is asymptotically stable if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable.

In mathematical economics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems with respect to any slight perturbations of initial conditions. The economic example is the price equilibrium points. For a given price trajectory it can be ascertained if a small change in the initial condition will lead to comparable behavior. Therefore, stability theory addresses the question of will a tracjectory converge to the given trajectory. For example, will the general price level converge to the equilibrium price level?

An equilibrium solution to an autonomous system of first order ordinary differential equations is stable iff: for every (small) $\varepsilon > 0$, there exists an $\alpha > 0$ such that every solution having initial conditions within the distance $|f(t_0) - f_e| < \alpha$ of the equilibrium stays within the distance $|f(t) - f_e| < \varepsilon \quad \forall t > t_0$. It is asymptotically stable if it is stable and there exists $\alpha_0 > 0$ such that whenever $|f(t_0) - f(0)| < \alpha_0,\ then\ f(t) \Rightarrow f_e\ as\ t \to \infty$. In effect, stability requires that the trajectories do not change significantly under slight perturbations.

Mathematically, let the autonomuous differential equation be given by the following equation,

$$\frac{dx}{dt} = f(x)$$

The stability theorem holds that if $x(t) = x^*$ is an equilibrium such that $f(x^*) = 0$, then,

i. Then, the equilibrium $x(t) = x^*$ is stable $f^!(x^*) < 0$

ii. the equilibrium $x(t) = x^*$ is unstable $f^!(x^*) > 0$

Stable equilibria are described by a negative slope whereas unstable equilibria are described by a positive slope.

3.5.2. Application 2 (Time Path Analysis): Consider the following dynamic model of demand and supply,

$$Q^D = \phi - \delta P \qquad (\phi, \delta > 0)$$
$$Q^S = -\gamma + \beta P \qquad (\gamma, \beta > 0)$$

The equilibrium price if obtained by equating demand to supply such that:

$$P^E = \frac{\phi + \gamma}{\delta + \beta}$$

We can determine the time path of the price level $P(t)$ if the initial conditions are such that

$P^E \neq P(0)$. Given that the rate of change of price level is proportional to excess demand, we have that:

$$\frac{dP}{dt} = \alpha(Q^D - Q^S) \qquad (\alpha > 0)$$

Where α is the adjustment coefficient of the market model. By substitution for both demand and supply, we have:

$$\frac{dP}{dt} = [\alpha(\phi - \delta P) - (-\gamma + \beta P)]$$
$$\frac{dP}{dt} = \alpha(\phi + \gamma) - \alpha(\delta + \beta)P$$
$$\frac{dP}{dt} + \alpha(\delta + \beta)P = \alpha(\phi + \gamma)$$

$$P(t) = \left(P(0) - \frac{\phi + \gamma}{\delta + \beta}\right)e^{-\alpha(\phi+\gamma)t} + \frac{\phi + \gamma}{\delta + \beta}$$
$$Thus,\ P(t) = \left(P(0) - P^E\right)e^{-kt} + P^E$$
$$where\ k = \alpha(\phi + \gamma)$$

Accordingly, while the complementary function is given by $(P(0) - P^E)e^{-kt}$, the particular solution is given by P^E. Thus, $P(t) = \left(P(0) - P^E\right)e^{-kt} + P^E$ (general Solution). The intertemporal equilibrium price level is denoted by P^E and the deviation from equilibrium is $(P(0) - P^E)e^{-kt}$.

The dynamic stability of the equilibrium of the model can be deduced on the ground that if $k > 0$, the time path of price level leads to equilibrium price level, that is, $P(t)$ leads to P^E and the model becomes dynamically stable.

3.5.3. Application 3 (Modelling Economic Growth): Recall the rudimentary tenets of the Solow growth model which says that the key to short-run growth is increased investments, while technology and efficiency improve long-run growth. Accordingly, the Solow's growth equation is given by:

$$\frac{dk}{dt} = g(k) = sAk^{\phi} - \beta k$$
$$\frac{dk}{dt} = sAk^{\phi} - \beta k$$
$$g'(k) = sA\phi k^{\phi-1} - \beta$$
$$\text{Re}\,call,\ y = Ak^{1-\phi}$$
$$\frac{dy}{dt} = (1-\phi)Ak^{-\phi}\frac{dk}{dt}$$

Re*writing the equation, we have that*:

$$\frac{dk}{dt} = \frac{1}{1-\phi}\left(\frac{k^{\phi}}{A}\right)\frac{dy}{dt}$$

substituting int*o the Solow's growth model, we have*:

$$\frac{1}{1-\phi}\left(\frac{k^{\phi}}{A}\right)\frac{dy}{dt} = sAk^{\phi} - \beta k$$

Dividing both sides by k^{ϕ} and multiplying by A yields:

$$\frac{1}{1-\phi}\frac{dy}{dt} = [sAk^{\phi} - \delta k]\left(\frac{A}{k^{\phi}}\right)$$
$$\frac{1}{1-\phi}\frac{dy}{dt} = sA^2 - \delta k^{1-\phi}$$
$$\frac{dy}{dt} = (1-\phi)sA^2 - \delta k^{1-\phi}$$
$$\frac{dy}{dt} = (1-\phi)sA^2 - \delta y$$

By separating variables, we have that:

$$\frac{dy}{dt} = (1-\phi)sA^2 - \beta y$$
$$\frac{dy}{sA^2 - \beta y} = (1-\phi)dt$$

Integrating both sides of the equation with respect to t, we have that:

$$\int \frac{dy}{sA^2-\beta y}=\int (1-\phi)dt$$

$$-\frac{1}{\beta}\ln(sA^2-\beta y)=(1-\phi)t+K$$

$$\ln(sA^2-\beta y)=-\beta(1-\phi)t+K$$

$$sA^2-\beta y=Ke^{-\beta(1-\phi)t}$$

$$y=\frac{sA^2}{\beta}+Ke^{-\beta(1-\phi)t}$$

where$K$ is an arbitrary constant of integration. Indeed, differential equations can be applied to analyze the Solow's economic growth model.

4.0 CONCLUSION

A differential equation contains differentials whose solution necessitates integration at some point in time.The theory of differential equations and dynamical systems deal with asymptotic properties of solutions and the trajectories.

5.0 SUMMARY

In this unit, we have discussed the meaning of differential equations, order and degree of differential equations, general and particular solutions of differential equations, and also gave some numerical examples on how differential equations are solved.

6.0 TUTOR-MARKED ASSIGNMENT

1. Determine which of the following equations are ordinary differential equations and which are partial differential equations:

$$\left(\frac{d^5z}{dt^5}\right)+9\left(\frac{d3z}{dt^3}\right)+2z=\sin t$$

$$\left(\frac{d^2y}{dz^2}\right)+[yz\left(\frac{dy}{dz}\right)^2]=0$$

$$\left(\frac{d^2y}{dz^2}\right)+[28\left(\frac{dy}{dz}\right)]+4y=0$$

2. What are the orders of the following differential equations:

$$z'+zy=z^4$$

$$(z')^6=\sin z$$

$$\left(\frac{\partial^6 y}{\partial^6 z}\right)+2[\left(\frac{\partial^5 y}{\partial^5 z}\right)]+y=e^{5z}$$

$$\left(\frac{\partial^2 w}{\partial^2 z}\right)+\left(\frac{\partial^2 w}{\partial z\partial y}\right)+[5\left(\frac{\partial^2 w}{\partial^2 y}\right)]=\cos zy$$

3. Differentiate between linear and non-linear ordinary differential equations.

4. What is homogeneity all about in the discussion of differential equation?
5. Give 6 examples of a non-linear differential equation.
6. Give 6 examples of a non-linear differential equation that can be approximated by a linear differential equation.
7. Define a system of differential equations

7. Explain the dissimilarity between the complementary function and the particular integral of a differential equation?
8. Why is the solution to a differential equation the complementary function plus the particular integral?
9. Expalin the rationale behiod using a complementary function to evaluate the solution of linear differential equations?
10. Solve the following differential equation, if the initial conditions are given use to definitize the arbitrary constants:

$$4t^2 y = 20t,\ y(0) = 12$$

11. Consider the following market and supply functions,

$$Q^D = 80 - 20P_t$$
$$Q^S = 4 + 18P_{t-1}$$

$$Q^D = 90 - 15P_t$$
$$Q^S = 60 + 3P_{t-1}$$

Determine equilibrium price and quantity for each market. Supposing there was an initial price 30% below the equilibrium price for each market, determine the number of periods necessary for each price to adjust to within 3 percent of equilibrium.

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MODULE 4: LINEAR/NON-LINEAR PROGRAMMING, AND GAME THEORY

UNIT 1: LINEAR PROGRAMMING

CONTENTS

6.0. Tutor-Marked Assignment
7.0. References/Further Readings

1. 0. INTRODUCTION

Under this unit one, we shall be discussing the meaning of linear programming, carry out a mathematical representation of linear programming problems, evaluate some linear programming techniques and also solve specific numerical problems of linear programming.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Provide mathematical expression of linear programming problems
- Solve linear programming problems via simplex algorithm/graphical method
- Discuss linear programming models/techniques

3.0 MAIN CONTENT

3.1 Meaning of Linear Programming

Linear programming (LP) is a mathematical programming technique for the optimization of a linear objective function, subject to both linear equality and inequality constraints. The objective function of an LP is a real-valued function such that a linear programming algorithm situates a point on the graph of the function where the function has the minimum or maximum values.

3.2. Fundamental Theorem of Linear Programming

Given that a bounded LPP has optimal solutions, then at least one of the solutions must occurs at a corner point of the feasible region. Also, if a feasible region is bounded, then there exits both a maximum value and a minimum value for the objective function. Relatively, if the feasible region is unbounded, only a minimum value exists for the objective function. In the absence of a feasible region, there is neither a maximum nor a minimum for the objective function. Consequently, while some corner points are established on either the x-axis or the y-axis, other corner points are found at the point of intersection of binding constraints. The origin (0, 0) is not a corner point for a minimization problem but a corner point for the maximization problem. To find the exact coordinates of a corner point at the intersection of two or more binding constraints, we ought to solve a system of linear equations.

For example, consider that an automobile company manufactures four different brands of cars, Toyota (T), Nissan (N), and Mercedes (M) and the selling price per unit of car manufactured in dollar are given as 10, 14, and 13, respectively. The company having employed 10 trained hands, 20 untrained hands and work 2 hours per day desires to maximize sales. The time to manufacture one unit of each brand of car when utilizing both the trained and untrained workers is given below

Product	Toyota (T)	Nissan (N)	Mercedes (M)
Trained hours	2	4	6
Untrained hours	6	4	2

In standard form, the LP can be represented as follows:

Let the unit of Toyota (T) be Z_1 while the selling price per Z_1 is 50
Let the unit of Nissan (N) be Z_2 while the selling price per Z_2 is 20
Let the unit of Mercedes (M) be Z_3 while the selling price per Z_3 is 30

Therefore, the objective function is to maximize and it is written as:

$$\Pi = 50Z_1 + 20Z_2 + 30Z_3$$

The total hours for trained personnel is 10 x 2 = 20
The total hours for unskilled personnel is 20 x 2 = 40

The 20 hours and 40 hours are to be shared among the various brands of cars (T, N, & M) as follows:

$2 Z_1 + 4 Z_2 + 6 Z_3 \leq 20$ (limit/constraint on trained hands)

$6Z_1 + 4Z_2 + 2 Z_3 \leq 40$ (limit/constraint on untrained hands)

Canonically, we have it as:

$$\textit{Maximize}: 50\mathrm{Z}_1 + 20\mathrm{Z}_2 + 30\mathrm{Z}_3$$

Subject to:

$$2\mathrm{Z}_1 + 4\mathrm{Z}_2 + 6\mathrm{Z}_3 \leq 20$$
$$6\mathrm{Z}_1 + 4\mathrm{Z}_2 + 2\mathrm{Z}_3 \leq 40$$

In matrix form this becomes, we have:

$$\textit{Maximize}: [\mathrm{T} \quad \mathrm{N} \quad \mathrm{M}] \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$$

Subject to:

$$\begin{bmatrix} 2\mathrm{T1} + 4\mathrm{N2} + 6\mathrm{M3} \\ 6\mathrm{T1} + 4\mathrm{N2} + 2\mathrm{M3} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} \leq \begin{bmatrix} T \\ N \\ M \end{bmatrix}$$

In effect, Z_1, Z_2, &Z_3 are the decision variables of the linear programming problem.

3.3 Linear Programming Techniques

3.3.1. Graphical Method of Linear Programming: The graphical method solves LP problems by constructing a feasible solution at a vertex of the polygon and then walking along a path on the edges of the polygon to vertices with non-decreasing values of the objective function until an optimum is attained. In which case, the LPP must embrace~~s~~ two decision variables such as X_1 and X_2. The graphical method necessitates the determination of the solution space that defines the feasible solution as well as the optimal solution from the feasible region as shown in the figure below.

Solving a LLP requires the following steps:

1. Listing the objective function
2. Listing the problem constraints
3. Listing the non-negative constraints.
4. Graph the constraints as equations, by ignoring the inequality sign in order to find the feasible region
5. Identify all the corner points of the feasible region
6. Substitute the corner points back into the objective function.

Geometrically, the linear constraints define the convex feasible region, of possible values for the variables being described. The figure below presents the two-variable case feasible region.

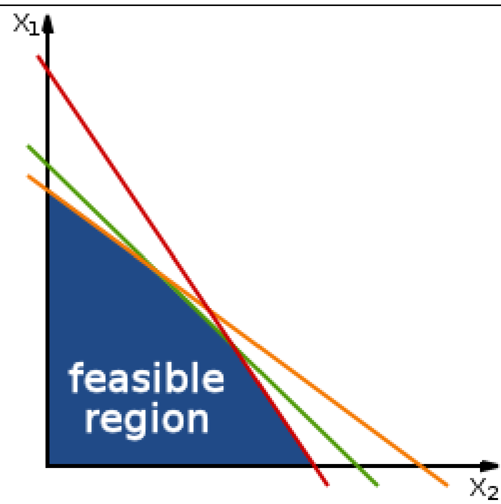


Figure 1: Feasible Region of LPP

Source: Wikipedia

3.3.2. Simplex Algorithm of Linear Programming: The simplex algorithm necessitates performing successive pivot operations which yields enhanced feasible solution. Consequently, the choice of pivot entry at each step is basically ascertained on the basis of the fact that the pivot advances the solution. In effect, there is the procedure of entering and leaving variable selection respectively.

Given that entering variable increases from zero to a positive number, the value of the objective function decreases because the derivative of the objective function with respect to this variable is negative and also the pivot column is selected in such a manner that guarantee the equivalent entry in the objective row of the tableau to be positive. Therefore, changing choice of entering variable in such a way that selects a column where the entry in the objective row is negative, the algorithm finds the maximum of the objective function

Conversely, with a pivot column selected, the choice of pivot row becomes ascertained on the requirement that the resulting solution be feasible. Hence, only positive entries in the pivot column are considered as it guarantees value of the entering variable to be nonnegative. Given that the entering variable can assume nonnegative values in the absence of no positive entries in the pivot column, the pivot row is then selected such that other basic variables remain positive. This guarantees that the value of the entering variable is at a minimum.

3.3 . Applications of Linear Programming

1. Many practical problems in operations research can be expressed as linear programming problems.
2. It is currently utilized in company management, such as planning, production, transportation, technology and other issues.
3. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing.
4. LP is worthwhile in modeling economic planning and assignment problems.

SELF-ASSESSMENT QUESTION

Linear programming is a widely used field of optimization for several reasons. Why?

Solving Numerical Problems of Linear Programming

Solving linear programming problems requires solving system of linear inequalities with two variables along with linear optimization.

Numerical Example 1: A restaurant offers two types of dishes, namely, A & B. For sake of making profits, the restaurant must sell a minimum of 80 dishes of type A and a minimum of 50 dishes of type B. Sales record available show that the restaurant makes a profit of N450 for each dish A and N400 for each dish B. At best, the restaurant accommodates a total of 300 customers. Determine the number of dish A & B that must be sold in order to maximize profits?

Solution 1: Let x = no of dish A, and y = no of dish B, minimum of 50 dish A implies 50 or more dishes A should be sold such that is, $y \geq 50$; minimum of 80 dish B implies 80 or more dishes B should be sold. That is, $x \geq 80$; the sum of dishes A & B should be 300 or fewer, that is, $x + y \leq 300$.

Thus, the objective function along with the three mathematical constraints is:

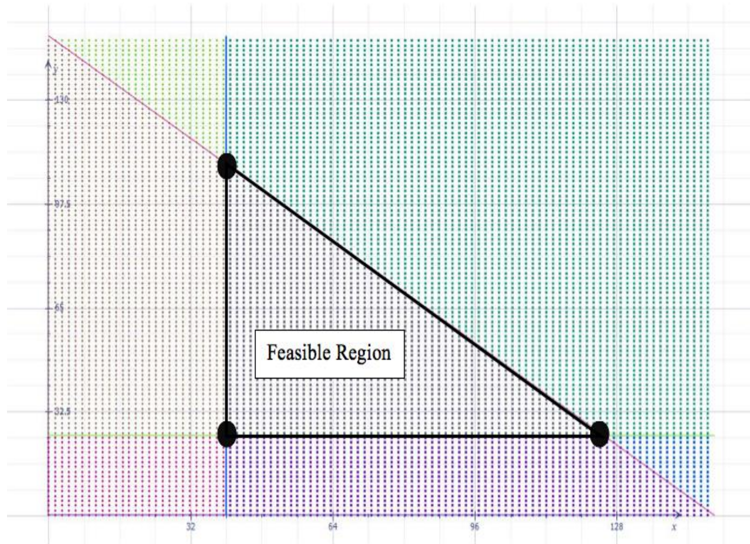
$$\pi_{(\text{Profit})} = 450x + 400y$$
$$\textit{Constraints}:,\ x \geq 80$$
$$y \geq 50$$
$$x + y \leq 300$$

Plotting each of the inequalities as equations, we ignore the inequality sign in order to find the corner solutions required to obtain the feasible region.

$$x = 80, y = 50$$
$$\Rightarrow (80, 50)$$
$$\textit{when}\ \ x = 80,$$
$$80 + y = 300$$
$$y = 220$$
$$\Rightarrow (80, 220)$$

$$\textit{when}\ \ y = 50,$$
$$x + 50 = 300$$
$$x = 250$$
$$\Rightarrow (250, 50)$$

Boundary	Intercepts	Test (0,0)	Corners	Revenue
$x = 80$	$(80, 50)$	$0 \geq 96,\ \textit{false}$	$(0, 0)$	$\pi_{(\text{Profit})} = 0$
$y = 50$	$(80, 220)$	$0 \geq 50,\ \textit{false}$	$(80, 50)$	$\pi_{(\text{Profit})} : 450(80) + 400(50) = 56,000$
$x + y = 300$	$(250, 50)$		$(80, 220)$	$\pi_{(\text{Profit})} : 450(80) + 400(220) = 124,000$
The maximum profit is N132,00 and it occurs at the corner where x =250 and y = 50 respectively.			$(250, 50)$	$\pi_{(\text{Profit})} : 450(250) + 400(50) = 132,000$



The corner, (125, 25) maximizes profit. Therefore, we conclude that the airline should sell 125 coach tickets and 25 first-class tickets in order to maximize profits.

Numerical Example 2: A bakery firm decides to use 80mg of flower and 100mg yeast to produce soft bread and hard bread. Flower cost N5 and yeast cost N4. The bakery decides to bake bread that would have at least the recommended daily flower intake of about 2200mg, but would like to maintain a double of the daily intake. In order to minimize cost, how many kg of flower and yeast should be used in the production of soft bread and hard bread?

Solution 2: Let x = no of kg of flower and y = no of kg of yeast so that the cost for x flower would be 5x and the cost for y yeast would be 3y.

Bread must contain at least 2200mg of flower and not more than 2200 × 2 = 4400mg of yeast respectively.

Mathematically, we have 80x mg of potassium in x servings of apricots and 100y mg of potassium in y servings of dates, that is,

$$80x + 100y \geq 2200$$

The same sum should be less than or equal to 4400 mg of potassium, that is,

$$80x + 100y \leq 4400 .$$

Thus, the objective function along with the mathematical constraints are given by:

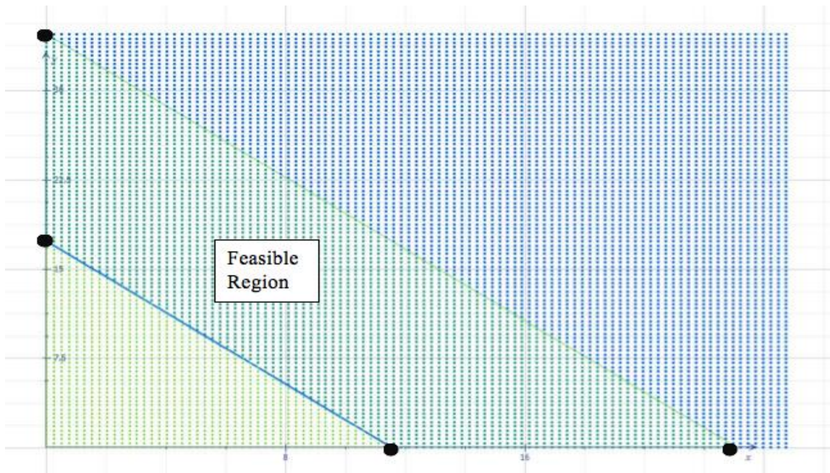
$$\begin{aligned} C_{(\cos t)} &= 5x + 3y \\ Constraints:\ & 80x + 100y \geq 2200 \\ & 80x + 100y \leq 4400 \\ & x \geq 0 \\ & y \geq 0 \end{aligned}$$

Graphing the constraints as equations, we ignore the inequality sign in order to solve for the corner solutions from bottom-to-top and left-to-right that could be used to form the feasible region:

$$
\begin{aligned}
&x = 0, y = 0 \\
&when\ \ x = 0, \\
&80x + 100y = 2200 \\
&100y = 2200 \\
&y = 22 \\
&\Rightarrow (0, 22) \\
&when\ \ y = 0, \\
&80x = 2200 \\
&x = 27.5 \\
&\Rightarrow (27.5, 0)
\end{aligned}
$$

$$
\begin{aligned}
&x = 0, y = 0 \\
&when\ \ x = 0, \\
&80x + 100y = 4400 \\
&100y = 4400 \\
&y = 44 \\
&\Rightarrow (0, 44) \\
&when\ y = 0 \\
&80x = 4400 \\
&x = 55 \\
&\Rightarrow (55, 0)
\end{aligned}
$$

<table>
<tr><td>Boundary</td><td>Intercepts</td><td>Test (0,0)</td><td>Corners</td><td>Cost</td></tr>
<tr><td>$80x + 100y = 2200$</td><td>$(0, 22)$,
$(27.5, 0)$</td><td>$0 \geq 2200$, false</td><td>$(0, 22)$</td><td>$C_{(\cos t)} : 5(0) + 3(22) = N66$</td></tr>
<tr><td>$80x + 100y = 4400$</td><td>$(0, 44)$,
$(55, 0)$</td><td>$0 \leq 4400$, true</td><td>$(27.5, 0)$</td><td>$C_{(\cos t)} : 5(27.5) + 3(0) = N137.5$</td></tr>
<tr><td colspan="3" rowspan="2">The minimum cost is N66 and it occurs at the corner where x =0 and y = 22 respectively.</td><td>$(0, 44)$</td><td>$C_{(\cos t)} : 5(0) + 3(44) = N132$</td></tr>
<tr><td>$(55, 0)$</td><td>$C_{(\cos t)} : 5(55) + 3(0) = N275$</td></tr>
</table>



The point, (0, 22) minimizes cost. Therefore, we conclude that the company create bars that contain no dried apricots and 22 servings of dried dates in order to minimize costs.

Numerical Example 5: ShaibuIyora limited produces three brands of cars, namely Toyota, Nissan and Mercedes. The production generates profit of 20, 60, and 30 respectively. Each unit of car uses 2, 4 and 6 hours of trained personnel and 6, 4 and 2 hours of untrained personnel respectively. There are 20 hours of trained and 40 hours of untrained personnel that are available. Find the production plan that maximizes Shaibu's profits using the Simplex Algorithm.

Solution 5: Proceed in the following steps.

Step 1: Let unit of Toyota be Z_1, unit of Nissan be Z_2, unit of Mercedes be Z_3, profit per Z_1 is 50, profit per Z_2 is 20 & profit per Z_3 is 30.
Step 2: Sate Shaibu's objective function and constraints knowing fully well that 20hrs of machine capacity and 40hrs of labour are to be shared among the products as follows:

Products	Toyota	Nissan	Mercedes	Constraints
Trained hours	2	4	6	20
Untrained hours	6	4	2	40

$$Maximize\ \pi = 50Z_1 + 20Z_2 + 30Z_3$$

$$\begin{aligned} subject\ to: \quad & 2Z_1 + 4Z_2 + 6Z_3 \le 20 \\ & 6Z_1 + 4Z_2 + 2Z_3 \le 40 \end{aligned}$$

Step 3: Set up the initial table by arranging the objective function and equalized constraints from Step 4 as follows: *(How do the slacks come in and what do they stand for?)*

Table 1: Simplex tableau

Solution variable	Products			Slack Variables		Solution
	Z_1	Z_2	Z_3	S_1	S_2	
S_1	2	4	6	1	0	20
S_2	6	4	2	0	1	40
π	20	60	30	0	0	0

Step 4: What is the highest profit amount? 60 and what product or column has the highest amount? Z_2. Divide the solution column by the values of the column with the highest amount.

20/4	5
40/4	10

Step 5: Select the row with the lowest value (S1 row) and obtain the entry which appear in both the identified column (Z_2) and row **(S_1) = 4.** This is the ***pivot element***

Step 6: Divide all entries in the identified row S_1 by the value of the pivot element **4** and change the solution variable to the heading of the identified column, Z_2.

Solution Variable	Products			Slack Variables		Solution
	Z_1	Z_2	Z_3	S_1	S_2	
New Row 1 (Z_2)	2/4	1	6/4	1/4	0	5
Old Row 2 (S_2)	6	4	2	0	1	40
Old Row 3 π	20	60	30	0	0	0

Step7: We commence row by row operation using newly identified row 1 by making all entries in the pivot element column to become zero.

Step 8: To change row 2, multiply the new row 1 (Z_2) by 4 and subtract it from row 2 (S2) as follows:

Solution Variable	Products			Slack Variables		Solution
	Z_1	Z_2	Z_3	S_1	S_2	
Old Row 2 (S_2)	6	4	2	0	1	40
4 x New Row 1 (Z_2)	2/4 x 4 = 2	1 x 4 = 4	6/4 x 4 = 6	1/4 x 4 = 4	0 x 4 = 0	5 x 4 = 20
New Row 2 = S_2- 4(Z_2)	4	0	-4	-1	1	20

<u>Step 9</u>: To change row 3, multiply New Row 1(Z2) by 60 and subtract it from row 3

Solution Variable	Products			Slack Variables		Solution
	Z_1	Z_2	Z_3	S1	S2	
Old Row 3 (Z)	20	60	30	0	0	0
60 x New Row 1 (Z_2)	2/4 x 60 = 30	1x 60 = 60	6/4 x 60 = 90	¼ x 60 = 15	0 x 60 = 0	5 x 60 = 300
New Row 3 = π - 60(Z_2)	-10	0	-60	-15	0	-300

<u>Step10:</u> Put together all new rows i.e. New Row 1, New Row 2 and New Row 3 to check for optimality

Solution Variable	Products			Slack Variables		Solution
	Z_1	Z_2	Z_3	S1	S2	

New Row 1 (Z_2)	2/4	1	6/4	1/4	0	5
New Row 2 (S2)	4	0	-4	-4	1	20
New Row 3 π	-10	0	-60	-15	0	-300

Step 11: Check for optimality. To test for optimality in maximization case, all entries in the profit row must either be zero or negative. In this case, the profit is at optimum.

i. *How is the table interpreted? What is the profit? What are the optimal values of the variables? What determines when not to continue with the process? Is there any excess capacity etc?*

ii. *How is the case of minimization treated?*

4.0 CONCLUSION

The signficance of linear programming techniques derived from it optimization principle as well as it foundation of microeconomics.*Revise please*

5.0 SUMMARY

In this unit, we have discussed the meaning of linear programming, provided a mathematical representation of linear programming problems, graphical solution to a LLP as well as the simplex algorithm of linear programming and solved some numerical questions on linear programming.

TUTOR-MARKED ASSIGNMENT

1. Consider the following LPP,

$$\begin{aligned} &Maximize\ 100Z_1 + 40Z_2 + 60Z_3 \\ &subject\ to:\ 4Z_1 + 8Z_2 + 12Z_3 \leq 40 \\ &\qquad 12Z_1 + 8Z_2 + 4Z_3 \leq 80 \\ &\qquad\qquad Z_1, Z_2, Z_3 \geq 0 \end{aligned}$$

$$
\begin{aligned}
&\textit{Minimize } 25Z_1 + 10Z_2 + 15Z_3 \\
&\textit{subject to}: \ Z_1 + 2Z_2 + 3Z_3 \geq 10 \\
&\qquad\qquad 3Z_1 + 2Z_2 + Z_3 \geq 20 \\
&\qquad\qquad\qquad\qquad Z_1, Z_2, Z_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\textit{Maximize} - 12Z_1 - 18Z_2 - 24Z_3 \\
&\textit{subject to}: \ 18Z_1 + 12Z_2 + 6Z_3 \leq 60 \\
&\qquad\qquad 12Z_1 + 30Z_2 + 18Z_3 \leq 90 \\
&\qquad\qquad\qquad\qquad Z_1, Z_2, Z_3 \geq 0
\end{aligned}
$$

(a) Solve the LPP using the simplex algorithm
(b) Solve the LPP using the graphical method

2. An automobile manufactures two types of vehicles, Toyota (T) and Benz (B). At the end of every month a minimum of 6 of each brand of car are manufactured. It takes 9 hours to manufacture Toyota brand and 8 hours to manufacture Benz in a 1000 working hours of the month. A minimum of 4 workers are required to manufacture Toyota while 2 workers are required to manufacture Benz. The profit on Toyota is ₦2000 and on Benz is ₦2900

(a) Represent the above information as a system of inequalities.

(b) Prepare the relevant graph of the system and indicate the feasible region.

(c) Determine the number of each type that must be produced each week to make a maximum profit. Determine the maximum profit.

3. A farmer cultivates Maize and Yam. To cultivate maize requires 5 hours of cutting and 6 hours of stitching. To cultivate yam requires 4 hours of cutting and 3 hours of stitching. The profit on ~~a~~ maize is ₦300 and on a tuber of yam is ₦200. The farmer works for a maximum of 10 hours a day. How many maize and yams should be cultivated in order to maximize profit and what is the maximum profit.

4. A gardener has 20 hectares of his smallholding available for planting type A and type B of a flower. He must farm at least 2 hectares of type A maize, 5 hectares of type B in order to satisfy final demands. The gardener wishes to cultivate more of type A than type B but the labour available only allows the cultivation of a maximum of 7 times the quantity of type A compared to type B.

(a) Present the information as a system of inequalities.
(b) Outline the graph of these inequalities.
(c) If the profit on type A of the flower is N5000 and on type B is N6000, determine the combination of the two types of flower that can guarantee a maximum profit. Calculate the profit at this level.

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UNIT 2: NON-LINEAR PROGRAMMING

CONTENTS

1. 0. INTRODUCTION

Unit 2 of module 4 discusses the meaning of nonlinear programming, two-dimensional nonlinear programming, three-dimensional nonlinear programming, solving numerical problems of nonlinear programming using the substitution method of solution.

2.0 OBJECTIVES

After a successful study of this unit, students should be able to do the following:

- Discuss the meaning of nonlinear programming
- Mathematically define both the two-dimensional and three-dimensional nonlinear programming
- Solving numerical problems of nonlinear programming using thesubstitution method of solution

3.0 MAIN CONTENT

3.1. Meaning of Nonlinear Programming

Nonlinear programming (NLP) is a mathematical optimization problem in which either the objective function or the constraints are nonlinear.

3.2. Two-dimensional Nonlinear Programming: The Two-dimensional nonlinear programming problem can be defined as follows:

$$Maximize \quad f(z) = z_1 + z_2$$

$$where \;\; z = (z_1,\; z_2)$$

$$z_1^2 + z_2^2 \geq 3$$
$$z_1^2 + z_2^2 \geq 5$$
$$where\,(z_1 \geq 0,\ z_2 \geq 0)$$

3.3 Three-dimensional Nonlinear Programming: The three-dimensional nonlinear programming problem can be defined as follows:

$$Maximize \quad f(z) = z_1 z_2 + z_2 z_3$$
$$where \quad z = (z_1,\ z_2, z_3)$$
$$7z_1^2 + 6z_2^2 - z_3^2 \leq 4$$
$$z_1^2 - 6z_2^2 + z_3^2 \leq 2$$
$$where\,(z_1 \geq 0,\ z_2 \geq, 0 z_3 \geq 0)$$

SELF-ASSESSMENT EXERCISE

How will you differentiate between a two dimensional and a three dimensional non-linear programming problems?

3.4. Solving Numerical Problems of Nonlinear Programming

In this section, we used the substitution and the Langrangian multiplier methods to solve some numerical nonlinear programming problems.

Numerical Example 1: Consider that a manufacturing firm incurs an annual fixed cost of ₦96, 000 and variable cost per unit of output ₦50. The profit function and the demand constraint of the firm are given by:

$$Maximize \;\; \pi = qp - 96,000 - 50q$$
$$Subject \;\; to: q = 1600 - 49p$$

Solution to Numerical Example 1: Calculate the optimal price level of the firm using the method of substitution.

$$Substituting\ the\ constraint\ into\ the\ profit\ function,$$
$$\pi = (1600-49p)p - 96,000 - 50(1600-49p)$$
$$\pi = 1600p - 49p^2 - 96,000 - 80,000 + 2450p$$
$$\pi = 4050p - 176,000 - 49p^2$$
$$\frac{\partial \pi}{\partial p} = 4050 - 98p$$
$$Setting\ \frac{\partial \pi}{\partial p} = 0,$$
$$98p = 4050$$
$$p = 41.3$$

Numerical Example 2: Consider the following nonlinear programming problem

$$Maximize \quad R = 8z_1 - 0.3z_1^2 + 4z_2 - 0.2z_2^2$$
$$Subject\ to:\ 2z_1 + 4z_2 = 60$$

Determine the optimum solution to the nonlinear programming problem using the

(a) Method of substitution.
(b) Langrangian multiplier method

Solution to Numerical Example 2: Using the substitution method, we solve as follows,

$$Maximize \quad R = 8z_1 - 0.3z_1^2 + 4z_2 - 0.2z_2^2$$
$$Subject\ to:\ 2z_1 + 4z_2 = 60$$
$$From\ the\ constraint$$
$$z_1 = \frac{60 - 4z_2}{2}$$
$$z_1 = 30 - 2z_2$$

Substituting the value of z_1 into the objective function, we have as follows:

$$R = 8(30-2z_2) - 0.3(30-2z_2)^2 + 4z_2 - 0.2z_2^2$$
$$R = 240 - 16z_2 - 0.3(900 - 120z_2 + 4z_2^2) + 4z_2 - 0.2z_2^2$$
$$R = 240 - 16z_2 - 270 + 36z_2 - 1.2z_2^2 + 4z_2 - 0.2z_2^2$$
$$R = -30 + 24z_2 - 1.4z_2^2$$

$$R = -30 + 24z_2 - 1.4z_2^2$$

$$\frac{\partial R}{\partial z_2} = 24 - 2.8z_2$$

$$Setting\ \frac{\partial R}{\partial z_2} = 0,$$

$$2.8z_2 = 24$$

$$z_2 = \frac{24}{2.8}$$

$$z_2 = 8.6$$

Substituting the value of z_2 into the constraint equation, we have that:

$$2z_1 + 4z_2 = 60$$

$$2z_1 + 4(8.6) = 60$$

$$z_1 = \frac{60 - 34.4}{2}$$

$$z_1 = 12.8$$

Therefore, Total Revenue becomes:

$$R = 8z_1 - 0.3z_1^2 + 4z_2 - 0.2z_2^2$$

$$= 8(12.8) - 0.3(12.8)^2 + 4(8.6) - 0.2(8.6)^2$$

$$= N85.9$$

Numerical Example 3: A shopping mall developed the following nonlinear programming model for the purpose of ascertaining the optimal number of shirts (z_1) and trousers (z_2) that are sold every day of sales

$$Maximize \quad S = 5z_1 - 0.2z_1^2 + 10z_2 - 0.3z_2^2$$

$$Subject\ to:\ 6z_1 + 25z_2 = 200$$

Calculate the optimal number of shirts (z_1) and trousers (z_2) in order to maximize sale (S) using the

(a) Method of substitution.
(b) Langrangian multiplier method

Solution to Numerical Example 3:Using the substitution method, we solve as follows:

$$Maximize \quad S = 9z_1 - 0.2z_1^2 + 10z_2 - 0.3z_2^2$$

$$Subject\ to:\ 6z_1 + 24z_2 = 300$$

From the constraint

$$z_1 = \frac{300 - 24z_2}{6}$$

$$z_1 = 50 - 4z_2$$

Substituting the value of z_1 into the objective function, we have as follows:

$$S = 9(50-4z_2) - 0.2(50-4z_2)^2 + 10z_2 - 0.3z_2^2$$
$$S = 450 - 36z_2 - 0.2(2500 - 400z_2 + 16z_2^2 + 10z_2 - 0.3z_2^2$$
$$S = 450 - 36z_2 - 500 + 80z_2 - 3.2z_2^2 + 10z_2 - 0.3z_2^2$$
$$S = -50 + 54z_2 - 3.5z_2^2$$

$$S = -50 + 54z_2 - 3.5z_2^2$$
$$\frac{\partial S}{\partial z_2} = 54 - 7z_2$$
$$Setting\ \frac{\partial S}{\partial z_2} = 0,$$
$$7z_2 = 54$$
$$z_2 = \frac{54}{7}$$
$$z_2 = 7.7$$

Substituting the value of z_2 into the constraint equation, we have that:

$$z_1 = 50 - 4z_2$$
$$z_1 = 50 - 4(7.7)$$
$$z_1 = 19.2$$

Therefore, Total Revenue becomes:

$$S = 9z_1 - 0.2z_1^2 + 10z_2 - 0.3z_2^2$$
$$= 9(19.2) - 0.2(19.2)^2 + 10(7.7) - 0.3(7.7)^2$$
$$= N158.3$$

Using the Langrangian multiplier method, we solve as follows,

$$L = 9z_1 - 0.2z_1^2 + 10z_2 - 0.3z_2^2 + \lambda(300 - 6z_1 - 24z_2)$$
$$\frac{\partial L}{\partial z_1} = 9 - 0.4z_1 - 6\lambda$$
$$\frac{\partial L}{\partial z_2} = 10 - 0.6z_2 - 24\lambda$$
$$\frac{\partial L}{\partial \lambda} = 300 - 6z_1 - 24z_2$$
$$Setting\ \frac{\partial L}{\partial z_1} = 0, \quad \frac{\partial L}{\partial z_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

$$9 - 0.4z_1 - 6\lambda = 0$$
$$10 - 0.6z_2 - 24\lambda = 0$$
$$300 - 6z_1 - 24z_2 = 0$$

We eliminate in order to solve simultaneously as follows

$$9 - 0.4z_1 - 6\lambda = 0 \quad \times \ (-4)$$
$$-36 + 1.6z_1 + 24\lambda = 0$$

Combining with $\frac{\partial L}{\partial z_2}$ equation we have as follows:

$$-36 + 1.6z_1 + 24\lambda = 0$$
$$10 - 0.6z_2 - 24\lambda = 0$$
$$\Rightarrow\Rightarrow\Rightarrow \quad -26 + 1.6z_1 - 0.6z_2 = 0$$

Combining with $\frac{\partial L}{\partial \lambda}$ equation, we have as follows:

$$-26 + 1.6z_1 - 0.6z_2 = 0$$
$$300 - 6z_1 - 24z_2 = 0$$
$$-26 + 1.6z_1 - 0.6z_2 \quad = 0 \ \ (-40)$$
$$1040 - 64z_1 + 24z_2 \quad = 0$$
$$300 - 6z_1 - 24z_2 = 0$$
$$1340 - 70z_1 = 0$$
$$z_1 = \frac{1340}{70}$$
$$z_1 = 19$$

$$300 - 6z_1 - 24z_2 = 0$$
$$300 - 6(19) - 24z_2 = 0$$
$$z_2 = \frac{186}{24}$$
$$z_2 = 7.7$$

Therefore, total sales becomes:

$$S = 9z_1 - 0.2z_1^2 + 10z_2 - 0.3z_2^2$$
$$= 9(19.2) - 0.2(19.2)^2 + 10(7.7) - 0.3(7.7)^2$$
$$= N158.3$$

SELF ASSESSMENT EXERCISE

Describe the relationship between a linear and a non-linear programming problem

4.0 CONCLUSION

Nonlinear programming problems are similar to linear programming problems except that the NLP*... incomplete sentence*

5.0 SUMMARY

In this unit, we have discussed the meaning of nonlinear programming, two-dimensional nonlinear programming, two-dimensional nonlinear programming, estimation of non-linear least squares and solved numerical problems of nonlinear programming using the substitution method of solution and verified with the langragean method.

6.0 TUTOR-MARKED ASSIGNMENT

Consider that a manufacturing firm incurs an annual fixed cost of N36,000 and variable cost per unit of output N45. The profit function and the demand constraint of the firm are given by:

$$Maximize\ \ \pi = 3qp - 36,000 - 51q$$
$$Subject\ to: q = 2400 - 45p$$

Find the optimal solution of the firm using the method of substitution.

Suppose the initial investment for machineriesand related technologies of a growing business wasN100,000 while labour and materials costed N60. Given the following demand function of the business,

$$q = 12,000 - 140p$$

i. Formulate the non-linear profit function of the business
ii. Find the optimal price of the growing business
iii. Find the optimal price of the growing business
iv. Find the optimal profit of the growing business

A motorbike company has the following non-linear programming model

$$Maximize\ \ \ R = 20z_1 - 0.05z_1^2 + 13z_2 - 0.05z_2^2$$
$$Subject\ to:\ 1.4z_1 + 3.5z_2 = 50$$

$where\, z_1\ is\ no.\ of\ \ yamaha\ bikes,\ z_2\ \ is\ no.\ of\ \ suzuki\ bikes$

Calculate the optimal combination of bikes (z_1) and (z_2) to be manufactured by the company using the:

(a) Method of substitution.
(b) Langrangian multiplier method

A Poultry farm is characterized by the following model of nonlinear programming model for the purpose of ascertain the optimal number of shirt (z_1) and trousers (z_2) that are sold very day of sales

$$Maximize \quad S = 40z_1 - 0.1z_1^2 + 26z_2 - 0.4z_2^2$$
$$Subject\ to: \ 4z_1 + 7z_2 = 80$$

$where\, z_1\ is\ no.\ of\ \ old\ \ layers,\ z_2\ \ is\ no.\ of\ \ boilers$

Calculate the optimal number of old layers (z_1) and boilers (z_2) to be farmed to maximize sale (S) using the:

(c) Method of substitution.

(d) Langrangian multiplier method

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UNIT 3: GAME THEORY

CONTENTS

1. 0. INTRODUCTION

Under this unit, we shall be discussing the concept of game vs. game theory, types of games and various forms of game representation. Also, we shall be solving numerical problem on game theory.

2.0 OBJECTIVES

After a successful study of this unit, students would be able to do the following:

- Discuss the meaning of game vs. game theory
- Explain the different types of games
- Carry out and extensive form of game representation
- Carry out a strategic form of game representation
- Solving numerical problems on game theory

3.0 MAIN CONTENT

3.1 Concept of Game vs. Game Theory

A game is a process of making decisions in circumstances where both conflict and cooperation exits. It thus implies a competitive circumstance where individuals, firms, institutions, governments, nations etc., as the case may be, pursue their own interest and no one party can command the outcome of another.

3.2 Types of Games

3. 2.1 Cooperative vs. Non-Cooperative Games: A game is cooperative if the players enters into some binding obligations externally enforced through contract law. A game is non-cooperative if players cannot form agreements.

Table 1: Difference between Cooperative vs. Non-Cooperative Game

Cooperative Game	**Non-cooperative Game**
All agreements need to be executed on basis	All agreements need to be self-enforcing by

of binding commitment by all players	individual players
Describes the strategies, and payoffs of alliances	Determines impact of bargaining on payoffs within each alliance
Can be analyzed through the approach of non-cooperative game theory	Cannot be analyzed through the approach of non-cooperative game theory
Allows analysis of the game without having to make any assumption about strategic bargaining powers.	Do not allow analysis of the game without having to make any assumption about strategic bargaining powers

3. 2.2 Symmetric vs. Asymmetric Games: A symmetric game is a game where the payoffs depends on strategies and not on players. That is, the identities of the players can be changed without changing the payoff to the strategies. Examples of symmetric games include chicken, the prisoner's dilemma etc.

3. 2.3 Zero-sum vs. Non-Zero-Sum Games: Zero-sum games are games whereby the total benefit to all players in the game, for every combination of strategies, always adds to zero. In other words, a player benefits only at the equal expense of others.

Zero-sum Game

	Player C	Player D
Player C	-2, 2	3,-3
Player D	0,0	-1, 1

Suppose that player C takes evens and player D takes odds. Then, each player simultaneously shows either ODD number or EVEN number. If the number shown is even, player C wins the game while if the number shown is odd, player D wins the game. Each player has two possible strategies: show an even or an odd number.

3. 2.4 Simultaneous vs. Sequential Games: Simultaneous games are games where both players decide on the move to make at the same time or one player is uninformed of actions of the other. Sequential games are dynamic games where one player is partly informed about the other player's actions/move to take.

Table 2: Difference between Simultaneous and Sequential Games

Simultaneous Games	**Sequential Games**
Games are represented with normal form of representation is used to represent	Games are represented with extensive form of representation
Games are strategy types of game	Games are mainly extensive-form game
Games are symbolized by Payoff matrices	Games are symbolized by Decision trees
Games do not provide for time.	Games provide for timing
Players of simultaneous games do not have	Players of sequential games take into

prior knowledge of opponent's move	cognizance prior knowledge of opponent's move

3. 2.5 Perfect Information vs. Imperfect InformationGames: In perfect information game, all players are fully informed of actions earlier taken by all other players. Conversely, the imperfect information game is game in which players do not have complete information set about the moves previously made by some other players.

3.3 Representation of Games

3.3.1 Extensive form of Game Representation: The extensive form can be used to formalize games with a time sequencing of moves whereby the node denotes a player's choice. The player is specified by a number listed by vertex with promising move of a player symbolized by line out of the vertex.

3.3.2 Strategic form of Game Representation: The strategic form is the matrix form of game representation which shows players, strategies, and payoffs as shown in matrix below. As shown in table 2, 16 is the payoff received by the row player (player C) while 12 is the payoff for the column player (Player D).

Table 2: Payoff Matrix of 2-Player, 2-Strategy Game

	Player D	Player D
Player C	16, 12	-3,-3
Player C	0,0	-12, -16

SELF ASSESSEMENT EXERCISE

Discuss the different types of games, discuss their similarities and dissimilarities

3.4 Applications of Game Theory:

i. Game theory is a major method used in mathematical economics and business for modeling competing behaviors of interacting agents.
ii. Game theoretic analysis aid bargaining, oligopolies and mergers & acquisitions pricing, fair division, duopolies,
iii. Game theoretic analysis aid information economics and industrial organization, and political economy.

SELF ASSESSEMENT EXERCISE

Discuss the various areas of applications of game theory

3.4. Solving Numerical Problems of Game Theory

Numerical Example 1: Consider a market whose inverse demand function is given by:

$$p = 60 - 2Q \quad \cancel{60 - 2Q}$$

There are two firms in the market facing the Cournot type of competition. The cost functions of the firms are given by:

$$C_1 = 30q_1$$

$$C_2 = 2q_2^2$$

(a) Find the firm whose marginal cost is relatively constant and the firm whose marginal cost is relatively rising

(b) Derive the reaction functions of both firms.

(c) What are the Cournot equilibrium quantities and price?

Solution to Numerical Example 1: Obtain the marginal cost function of both firms as follows:
Firm 1 marginal cost is relatively constant because,

$$\frac{dC_1}{dq_1} = 30$$

Firm 2 marginal cost is relatively rising in q_2 because,

$$\frac{dC_1}{dq_2} = 4q_2$$

The reaction functions are derived as follows:

$$\begin{aligned}
\pi_1 &= P(Q)q_1 - c(q_1) \\
&= (60 - 2Q)q_1 - 30q_1 \\
&= [60 - 2(q_1 + q_2)]q_1 - 30q_1 \\
&= 60 - 2q_1^2 - 2q_1q_2 - 30
\end{aligned}$$

$$\frac{\partial \pi_1}{\partial q_1} = 60 - 4q_1 - 2q_2 - 30$$

$$Setting\ \frac{\partial \pi_1}{\partial q_1} = 0$$

$$60 - 4q_1 - 2q_2 - 30 = 0$$

$$4q_1 = 30 - 2q_2$$

$$q_1 = \frac{30 - 2q_2}{4}$$

$$\pi_2 = P(Q)q_2 - c(q_2)$$
$$= (60 - 2Q)q_2 - 2q_2^2$$
$$= [60 - 2(q_1 + q_2)]q_2 - 2q_2^2$$
$$= 60 - 2q_1q_2 - q_2^2 - 2q_2^2$$
$$\frac{\partial \pi_2}{\partial q_2} = 60 - 2q_1 - 2q_2 - 4q_2$$
$$Setting \ \frac{\partial \pi_2}{\partial q_2} = 0$$
$$60 - 2q_1 - 2q_2 - 4q_2 = 0$$
$$60 - 2q_1 - 6q_2 = 0$$
$$q_2 = \frac{60 - 2q_1}{6}$$

the reaction functions of both firms are thus given by: $q_1 = \dfrac{30 - 2q_2}{4}$ & $q_2 = \dfrac{60 - 2q_1}{6}$ respectively.

Substituting the RF q_1 into RF q_2, we have:

$$30 - 4q_1 - 2q_2 = 0$$
$$60 - 2q_1 - 6q_2 = 0 \ (by \ elimination)$$
$$30 - 4q_1 - 2q_2 = 0$$
$$-120 + 4q_1 + 12q_2 = 0$$
$$-90 + 10q_2 = 0$$
$$q_2 = 9$$

$$q_1 = \frac{30 - 2(9)}{4}$$
$$q_1 = 3$$
$$Q^* = (q_1^* + q_2^*) = 12$$
$$P = 60 - 2Q^*$$
$$= 60 - 2(12)$$
$$P = 36$$

The Cournot equilibrium quantities and price are, $q_1 = 3$, $q_2 = 9$ & $P = 36$

SELF ASSESSMENT EXERCISE

Describe the relationship between an explicit and an implicit function.

4.0 CONCLUSION

Game theory is a theory of tactical relations among rational decision-makers. The application of game theory is all encompassing in social sciences. When a game is presented in normal form, it is presumed that each player acts simultaneously, that is, without knowing the move/actions of the other player. On the other hand, extensive form representation shows players have prior knowledge about each other's strategy.

5.0 SUMMARY

In this unit, we have discussed the game vs. game theory, explain the different types of games, provided extensive form of game representation, strategic form of game representation.

6.0 TUTOR-MARKED ASSIGNMENT

1. Consider the following pay-off matrix:

Player I	Player II		
	A	B	C
A	12	20	32
B	12	4	56
C	4	12	-3

Determine the strategy that each of the two players should play.

2. Consider the following pay-off matrix of a new contract for academic staff whereby university management and ASU entered into negotiation that resulted in ASU making 3 different proposals and management making 3 different proposals.

ASU Proposals	Management Proposals		
	G	Q	W
V	19	24	14
S	14	15.6	12.5
U	12	17	18

(a) Determine if there is an agreement between ASU and Management. Hint, find the point of equilibrium if there is

(b) Calculate the mixed strategies for management and ASU

(c) What is the optimum strategy for ASU?

(d) What is the optimum strategy for management?

3. Consider a game between propsective university students namely, Yinka (Y) and her father Momodu (M) such that Yinka A has to choose whether to seek admission into a univefrsity that cost N2,000 per semester or not. Momodu has to decide whether to pay get Yinka educated with a fee of N20,000 or both Yinka and her father share the family income equally. Yinka's

education and family income sharing has some impact on family's productivity. With no education and sharing of family's income, the productivity of the family would be N30,000, while if either education or family income is shared the family's productivity would rise and become N40,000. If both education and family income is shared, the producitity of the family would be N48,000.

(a) Construct the pay-off matrix for the game
(b) Is there any equilibrium in dominant strategies?
(c) Can you find the solution of the game with Iterated Elimination of Dominated Strategies?
(d) Is there any Nash equilibrium?

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